Midterm Solutions
Math 181, Fall 2001

1. The Black-Scholes equation is

$$-f_t = \tfrac{1}{2}\sigma^2 S^2 f_{SS} + rSf_S - rf$$

The value for a stock is $f(S,t) = S$.

Then $f_t = 0$

$f_S = 1$

$f_{SS} = 0$

So in B-S this gives

$$0 = \tfrac{1}{2}\sigma^2 S^2 \cdot 0 + rS \cdot 1 - rS$$
$$= 0 \quad \checkmark$$

The value for a cash account is $f = A = A_0 e^{rt}$

$$f_t = A_t = rA_0 e^{rt} = rA$$

$f_S = f_{SS} = A_S = A_{SS} = 0$

So in B-S this gives

$$-rA = 0 + 0 - rA$$

2. Suppose that S_0 and R_0 are the values at $t=0$. Form the portfolio

$$P = R_0 S - S_0 R$$

At $t=0$,

$$P = P_0 = R_0 S_0 - S_0 R_0 = 0$$

At $t = dt$, there are 4 possibilities

$$P = \begin{cases} R_0(uS_0) - S_0(uR_0) = 0 & \text{if } S \text{ and } R \text{ go up} \\ R_0(dS_0) - S_0(dR_0) = 0 & \text{if } S \text{ and } R \text{ go down} \\ R_0(uS_0) - S_0(dR_0) & \\ \quad = (u-d)R_0S_0 > 0 & \text{if } S \text{ goes up and } R \text{ goes down} \\ R_0(dS_0) - S_0(dR_0 & \\ \quad = -(u-d)S_0R_0 < 0 & \text{if } S \text{ goes down and } R \text{ goes up} \end{cases}$$

By assumption, the third possibility never happens, i.e. if S goes up, then R goes up. It follows that $P \leq 0$ at dt. No-arbitrage then implies that $P = 0$ for all possibilities at dt. But then the 4th possibility does not happen. So S and R either both go up or both go down. This implies that $p' = q'$.

3. The risk neutral probability is

$$p = \frac{e^{r\,dt} - d}{u - d} = \frac{e^{r} - .9}{1.2 - .9} = \frac{1.1 - .9}{1.2 - .9} = \frac{2}{3}$$

The payout at $T = 1.0 = dt$ is

$$C_1 = \begin{cases} C_u = \begin{cases} \max(uS_0 - X, 0) & \text{up step} \\ \max(dS_0 - X, 0) & \text{down step} \end{cases} \\ C_d \end{cases}$$

$$= \begin{cases} \max(120 - 100, 0) \\ \max(90 - 100, 0) \end{cases}$$

$$= \begin{cases} 20 \\ 0 \end{cases}$$

S_0

$$C_0 = e^{-r\,dt} (p C_u + (1-p) C_d)$$

$$= \frac{1}{1.1} \cdot \frac{2}{3} \cdot 20$$

$$= \frac{40}{3.3}$$

4. By a telescoping sum

$$x_2 = (x_2 - x_1) + (x_1 - x_0) + x_0$$
$$= \sqrt{dt}\,\omega_2 + \sqrt{dt}\,\omega_1 + x_0$$
$$= \sigma\,\omega + x_0$$

with

$$\sigma^2 = (\sqrt{dt})^2 + (\sqrt{dt})^2 = 2\,dt = 4\,\delta t, \text{ i.e. } \sigma = 2\sqrt{\delta t}$$

in which ω is $N(0,1)$

Also

$$y_4 = (y_4 - y_3) + (y_3 - y_2) + (y_2 - y_1) + (y_1 - y_0) + y_0$$
$$= \sqrt{\delta t}\,\nu_4 + \sqrt{\delta t}\,\nu_3 + \sqrt{\delta t}\,\nu_2 + \sqrt{\delta t}\,\nu_1 + x_0$$
$$= \gamma\,\nu + x_0$$

with

$$\gamma^2 = (\sqrt{\delta t})^2 + (\sqrt{\delta t})^2 + (\sqrt{\delta t})^2 + (\sqrt{\delta t})^2$$
$$= 4\,\delta t$$

i.e. $\gamma = 2\sqrt{\delta t}$

and ν is $N(0,1)$

So x_2 and y_4 have the same statistics.