Math 181: Final Exam
March 17, 2003

NOTATION. Use the following notation:

For an option, T is exercise time (in years) and K is strike price.

For the Black-Scholes model, S is equity price; t is time; σ, μ and r are the volatility, average growth rate and (continuously compounded) risk-free rate of return.

For the CRR model, S_0 is the initial stock price, S_n is the stock price after n steps, u and d denote the factors for increase and decrease of the equity price, dt is the time step, the real probabilities are p and q, and the risk-neutral probabilities are p^* and q^*. Use the continuously compounded interest rate r for the CRR model, as well as for the Black-Scholes model.

You may use the facts that $\log(100/90) = .105$ and $\exp(.05) = 1.05$.

1. Consider a call option, for an equity following the Black-Scholes model, with $T = 1$ and $K = 90$ on an equity with initial price $S(0) = 100$, and with $\sigma = 0.2$, $\mu = 0.1$ and $r = 0.05$.

(a) What is the value $c(0)$ of the call option at $t = 0$?

(b) What is the value of Δ for this option at $t = 0$?

2. Consider a forward contract to purchase an equity at strike price K and at time T.

(a) Use a no arbitrage argument to show that the price of the forward contract is $F(S,t) = S - Ke^{-r(T-t)}$.

(b) Show that F solves the Black-Scholes PDE.

3. Define a digital call d_c and a digital put d_p with strike price K and exercise time T to have payouts

$$\begin{aligned} d_c(S,T) &= \begin{cases} 1 & \text{if} \quad S \geq K \\ 0 & \text{if} \quad S < K \end{cases} \\ d_p(S,T) &= \begin{cases} 0 & \text{if} \quad S \geq K \\ 1 & \text{if} \quad S < K \end{cases} . \end{aligned}$$

(a) Use a no arbitrage argument to show that $d_c + d_p = e^{-r(T-t)}$.

(b) Find the initial price of d_p and d_c on a two step CRR model with $u = 1.1$, $d = 0.9$, $S_0 = K = 100$, $r = 0.05$, $dt = .5$ and $p = 0.5$.

(c) Show that the result from (b) satisfies the "digitial put-call parity" relation in (a).

4. Consider a call option for a two-step CRR model. Define the probability for exerercise of the option to be p_c in the real world and p_c^* risk-neutral world.

(a) Find a formula for p_c.

(b) Find a formula for p_c^*.

(c) Assuming that the CRR model has a risk premium (corresponding to risk aversion), show that $p_c^* < p_c$ for all values of the strike price K.

(c) (Bonus Problem) For a three-step CRR model, show that the inequality in (c) for some of values of K and p.