# Math 181 Lecture 9

## Pricing Options for the Random Walk with Discrete Steps

Consider a single step in a random walk

$$S_0 \;\text{ given at }\; t = 0$$
$$S_1 = \begin{cases} uS_0 & \text{probability} \;\; \text{p'} \\ dS_0 & \text{probability} \;\; \text{q' = 1-p'} \end{cases}$$

in which

$$\begin{aligned} u &= e^{\mu dt + \sigma\sqrt{dt}} \\ d &= e^{\mu dt - \sigma\sqrt{dt}}. \end{aligned}$$

Also consider an option where value is

$$\begin{aligned} &f_0 && \text{at } t = 0 \\ &f_u && \text{at } t = 1, \text{ if step is up} \\ &f_d && \text{at } t = 1, \text{ if step is down.} \end{aligned}$$

This is summarized in the following graph

For example, if the option is a cell with a strike price $X$ such that $dS_0 < X < uS_0$, and if $t_1$ is the expiration time, then

$$\begin{aligned} f_u &= uS_0 - X \\ f_d &= 0. \end{aligned}$$

It is tempting to think that the option price $f_0$ at $t_0$ is the expected value of the future option price, suitably discounted. Discounting the expected value $f_1$ back to $t = 0$ gives

$$\tilde{f}_0 = e^{-rdt} E(f_1) = e^{-rdt}(p' f_u + (1 - p') f_d). \tag{1}$$

As shown below, however, this price for $f$ would not be arbitrage free!

In order to correctly price $f$, we perform an arbitrage-free analysis by considering two portfolios

**A:** long $\Delta$ shares of stock, each with value $S_0$; short one option of value $f_0$

**B:** cash $K$

at time $t_0$. We will choose $\Delta$ so that there is no risk in the portfolio A. Then we'll choose $K$ to make $B$ equivalent to $A$. The value of $A$ at $t_0$ and $t_1$ is

$$\begin{aligned} A(t_0) &= \Delta S_0 - f_0 \\ A(t_1) &= \begin{cases} \Delta u S_0 - f_u & \text{if up} \\ \Delta d S_0 - f_d & \text{if down} \end{cases}. \end{aligned}$$

The profile will be riskless if the two outcomes are identical, i.e. if

$$\Delta u S_0 - f_u = \Delta d S_0 - f_d$$

i.e. if

$$\Delta = \frac{f_u - f_d}{u S_0 - d S_0}. \tag{2}$$

Now for portfolio B, take an amount of cash $K$ equal to the initial value of $A$, i.e. at $t = t_0$,

$$B(t_0) = K = \Delta S_0 - f_0.$$

Then at $t = t_1$

$$B(t_1) = K e^{rdt}.$$

Since portfolio A is riskless and its initial value is the same as B's, then its value at $t_1$ is also the same; otherwise there would be an arbitrage opportunity. This shows that

$$\begin{aligned} A(t_1) &= B(t_1) \\ &= e^{rdt} B(t_0) \\ &= e^{rdt} A(t_0). \end{aligned}$$

In the future, we will dispense with the second portfolio. We will summarize this argument by saying that since $A$ is riskfree, its value must growth at the riskfree rate $r$.

Now insert the values of $A(t_0)$, $A(t_1)$ and $\Delta$ to obtain

$$(\Delta u S_0 - f_u) = e^{rdt}(\Delta S_0 - f_0)$$

which leads to

$$f_0 = e^{-rdt}(pf_u + (1-p)f_d) \tag{3}$$

in which

$$p = \frac{e^{rdt} - d}{u - d}. \tag{4}$$

The formula (3) is a simplified version of the Black-Scholes equation for pricing as option. It has some surprising properties:

**(i)** The formula (3) for $f_0$ looks like a discounted average, as in (1). But the probability $p$ in (3) is not the probability $p'$ in for the the random walk to move up. In fact $p$ is determined just from $u, d, r$ and $dt$.

**(ii)** The formula is the same no what type of option is involved. It could be a call or put. The values of $f_u$ and $f_d$ would be different, but the relation between $f_0$ and these values would be the same.

**(iii)** The probabilities $p'$ and $q'$, for the stock to move up or down, do not influence the option price! They do not appear in (3) or (4). This says that the likelihood of the stock to move up or down does not affect the value of the option.

Although this is surprising, it has an explanation. The argument above shows that the risk in the option value is entirely due to risk in the stock, since a portfolio of stock and options can be riskless. Since the risks in the stock are already accounted for in its price, they do not enter into the price of the option.

This pricing method for the option is called "risk-neutral valuation".

# Math 181 Lecture 10

## Pricing Options on a Binomial Tree

The price of an option on a binomial tree can be determined from its values at expiration, by repeated application of the single step formulas from Lecture 9. First we perform this for a 2 step tree, then we generalize to an arbitrary sized tree.

Consider the following two step tree

In this formulation, the values $S, u, d$ are known. They are the initial stock value, and the proportional increase or decrease in its value. We take the final time to be the exercise time for the option; so that $f_{uu}, f_{ud}, f_{dd}$ are also known. We wish to compute $f$, and along the way we will compute $f_u$ and $f_d$.

Computation of $f_u$ is performed on a one step tree

The value $f_u$ is computed as a suitable discounted average of $f_{uu}$ and $f_{du}$ as in (3), (4) by the substitution $(f_{uu}, f_{du}, f_n)$ for $(f_u, f_d, f_0)$. The result is

$$\begin{aligned} f_u &= e^{-rdt}(pf_{uu} + (1-p)f_{du}) \\ p &= \frac{e^{rdt} - d}{u - d}. \end{aligned}$$

In a similar way

$$f_d = e^{-rdt}(pf_{ud} + (1-p)f_{dd})$$

with the same $p$.

Now we compute $f$ using the tree

$$\begin{aligned} f &= e^{-rdt}(pf_u + (1-p)f_d) \\ &= e^{-r2dt}\left\{p(pf_{uu} + (1-p)f_{ud}) + (1-p)(pf_{ud} + (1-p)f_{dd})\right\} \\ &= e^{-r2dt}\left\{p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\right\}. \end{aligned}$$

Note that

$$\begin{aligned} p^2 + 2p(1-p) + (1-p)^2 &= (p + (1-p))^2 \\ &= 1 \end{aligned}$$

so that this formula is still a discounted average.

## Risk-Neutral Valuation

We now give a tree-interpretation of these formulas. The real random walk has probabilities $p', q' = (1-p')$ of stepping up or down. We replace these probabilities with the probabilities that occur in the option pricing formula

The resulting random walk is called the "risk-neutral process". This is not the real process. It has the same states $S, uS, dS$, etc., but they occur with different probabilities.

The usefulness of the risk-neutral process is that it can be directly used to calculate the option price as an average (with discounting).

For a single step the tree is

and the price is

$$\begin{aligned} f_0 &= e^{-rdt}\bar{E}(f_1) \\ &= e^{-rdt}(pf_u + (1-p)f_d) \end{aligned}$$

in which $\bar{E}$ denotes average with respect to the risk-neutral process.

For the two step process the same formulas work

$$
\begin{aligned}
f_0 &= e^{-r2dt}\bar{E}(f_2) \\
&= e^{-r2dt}(p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}).
\end{aligned}
$$

More generally its true that for $m < n$

$$f_m = e^{-r(t_n - t_m)}\bar{E}(f_n). \tag{5}$$

This is the risk-neutral valuation formula.