Polynomial Functions on a Lattice

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Abstract

We present two characterizations of polynomial functions on a lattice $\mathbb{Z}^n$.

1 Finite Differences in One Variable

The main result of this section is a well-known characterization of polynomial functions on the integers. For any complex-valued function f on $\mathbb{Z}$ we define

$$(Df)(x) = f(x) - f(x-1).$$

Lemma 1. *If f is a polynomial of degree n and leading coefficient a_n then Df is a polynomial of degree $n-1$ and leading coefficient na_n.*

Proof. Easy. □

As an immediate consequence we have the forward direction of the main theorem.

Theorem 1. *Let D^n denote the n-th iterate of D. Specifically, $D^0 f = f$ and $D^{n+1} f = D(D^n f)$ for $n \geq 0$. Then f is a polynomial of degree strictly less than n if and only if $D^n f = 0$. (In particular, f has degree exactly n if and only if $D^n f = 0$ but $D^{n-1} f \neq 0$.)*

Proof. The forward direction follows easily from the Lemma.

We prove the reverse direction by induction. If $h(x) = x^n$ then $D^n h(x) = n!$. Now suppose f is a function with $D^{n+1} f = 0$. Subtracting a multiple of x^n, we obtain a function g such that $D^n g = 0$. Now apply the inductive hypothesis to g and use $f(x) = g(x) + a_n x^n$. □

This characterization of polynomials has an easy corollary.

Corollary 1. *For every integer $n \geq 0$ there is a polynomial p of degree $n+1$ such that*

$$\sum_{i=1}^{x} i^n = p(x)$$

for all positive integers x.

2 Finite Differences in Multiple Variables

We need some notation for multiindices. Let I denote the tuple $(i_1, \ldots, i_n)$, with each i_k a nonnegative integer. Define

$$x^I = x_1^{i_1} \cdots x_n^{i_n}$$

and

$$D_I f = D_{x_1}^{i_1} \cdots D_{x_n}^{i_n} f.$$

We use $|I|$ to denote $\sum_k i_k$. We say that a polynomial f has degree at most n if it can be expressed as $\sum_J a_J x^J$ with $|J| \leq n$.

Theorem 2. *A function $f : \mathbb{Z}^n \to \mathbb{C}$ is a polynomial of degree at most d if and only if $D^I f = 0$ for all $|I| > d$.*

Proof. The forward direction is obvious, since each D_{x_i} decreases the degree by one.

Now suppose given some f with $D^I f = 0$, all $|I| > d$. For every index with $|I| = d$, let $h_I(x) = x^I$. Then, if J is another index with $|J| = d$, we see that $D_J h_I$ is a constant, nonzero if and only if $I = J$. Thus we can find coefficients a_I such that $f - \sum a_I h_I$ is killed by all D_J with $|J| = d$. The theorem now follows by induction, as above. □

3 Polynomials on Lines

Theorem 3. *A function $f : \mathbb{Z}^n \to \mathbb{C}$ is a polynomial of degree at most d if and only if "the restriction of f to any line" is a polynomial of degree at most d, that is, for every $a, b \in \mathbb{Z}^n$, the function $\mathbb{Z} \to \mathbb{C}$ given by $t \mapsto f(a + tb)$ is a polynomial of degree at most d.*

Proof. One direction is trivial.

Suppose the restriction of f in any line is a polynomial of degree at most d. We will show by induction on k that the restriction to any affine k-space is a polynomial of degree at most d.

It is obviously enough to prove the special case where the restriction of f to any affine $(n-1)$-space is polynomial of degree at most d. So, writing $\overline{x}$ for the tuple $(x_1, \ldots, x_{n-1})$, we have that once we fix $x_n = t$,

$$f(\overline{x}, t) = \sum_I a_I(t) \overline{x}^I.$$

On the other hand, for any fixed $\overline{x}$, we know that

$$\sum_I a_I(t) \overline{x}^I$$

is a polynomial in t. But one sees easily that the values $a_I(t)$ are determined linearly by the values $\sum_I a_I(t) \overline{x}^I$. Hence the values $a_I(t)$ are themselves polynomials of degree at most d in t, and the result is proved. □