Introduction to Heights

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This talk is an introduction to the theory of heights on projective varieties over local and global fields. I will also say something about canonical heights for a dynamical system, and a local-global formula for these.

This material is from Chapters 1-2 of the book by Bombieri and Gubler on heights [BG]; also Silverman's book *The Arithmetic of Dynamical Systems*, 3.1 - 3.5 and 5.9 [Si]. There is also a chapter on heights in Silverman's *Advanced Topics in the Arithmetic of Elliptic Curves* [Si2].

1 Heights on Projective Space

In general a "height function" is some real-valued function that defines the "arithmetic complexity" of a point on a variety over a number field. One hopes that such a height function will have two fundamental properties. First, only a finite number of points should have height less than any given bound. Second, the height should behave in a predictable way under automorphisms of the variety. For example, the standard logarithmic height on an elliptic curve approximately quadruples when the point is doubled, i.e.

$$h(2P) = 4h(P) + O(1).$$

This notion of height is used in the proof of the Mordell-Weil theorem; it is also useful in arithmetic dynamics.

(As an aside, it is often useful to have effective bounds for the big-O expressions that arise here. It seems these bounds can be attained but not without quite a bit of effort. If this is the sort of thing that interests you then keep it in mind throughout my talk and we can talk about where the difficulty lies.)

Definition 1.1. *We say a height function (a real-valued function on a set of points) satisfies the* finiteness property *if, for any real number B, the set of points with height less than B is finite.*

As a warm-up, define a height on $\mathbb{Q}$ by

$$H(p/q) = \max(|p|, |q|).$$

The finiteness property is clear.

This height extends in the obvious way to $\mathbb{P}^1_{\mathbb{Q}}$, where it can be phrased more uniformly as follows. Any point of $\mathbb{P}^1_{\mathbb{Q}}$ can be written uniquely as $(p : q)$ with p and q relatively prime integers. Then we take

$$H(p : q) = \max(|p|, |q|).$$

This notion generalizes to $\mathbb{P}^n_{\mathbb{Q}}$. Any point of $\mathbb{P}^{n+1}_{\mathbb{Q}}$ can be expressed uniquely as $(x_0 : x_1 : \cdots : x_n)$, where the x_i's are integers with no common factor. With this choice of coordinates, we define

$$H(x_0 : \cdots : x_n) = \max_i(|x_i|).$$

With this definition the finiteness property is again clear.

In generalizing to an arbitrary number field K, whose ring of integers may not be a PID, we cannot always guarantee that the coordinates be "integers with no common factor," i.e. algebraic integers in K generating the full ring of integers $\mathcal{O}_K$. We could get around this by scaling the height so the coordinates do not have to be normalized in the first place: specifically we scale by the norm of the fractional ideal generated by $x_0, \ldots, x_n$.

Over $\mathbb{Q}$, the finiteness property holds because only finitely many integers have absolute value less than a given bound. This remains true for imaginary quadratic fields, but it is not true for a general number field $K \subset \mathbb{C}$. We know from elementary number theory that K has more than one archimedean absolute value, and it is often more natural to treat them all on an equal footing. What turns out to be the correct definition is the following. For any point of projective space, choose homogenous coordinates $(x_0 : \cdots : x_n)$, and define

$$H(x_0 : \cdots : x_n) = \left(\frac{1}{|\mathrm{Nm}(x_0, \cdots, x_n)|} \prod_v \max_i |x_i|_v^{f_v} \right)^{\frac{1}{[K:\mathbb{Q}]}},$$

where v ranges over the archimedean valuations of K, and $f_v = 1$ if v is a real place, $f_v = 2$ if complex.

The height so defined is independent of the choice of homogenous coordinates, and invariant under extension of the field. We call it the *standard height function on projective space.*

Lemma 1.2. *The standard height defined above has the finiteness property.*

Proof. (Sketch.)

Let $\mathcal{I}$ be a set of ideals of $\mathcal{O}_K$, containing one ideal from each ideal class. Then $\mathcal{I}$ is finite, and any point of $\mathbb{P}^n_K$ can be written uniquely as $(x_0 : \ldots : x_n)$ so that the n coordinates generate an ideal in $\mathcal{I}$. Thus, it is enough to show for each fixed ideal I that only finitely many points with $(x_0, \ldots, x_n) = I$ have height less than any given finite bound; and for this it is enough to show that there are only finitely many choices of $x_0, \ldots, x_n$, up to multiplication by a unit, satisfying

$$\prod_v \max_i |x_i|_v^{f_v} < B,$$

for B an arbitrary real number.

Next, I claim that we can, by scaling by a unit of $\mathcal{O}_K$, guarantee that for all archimedean valuations v of K, we have

$$|x_n|_v > \epsilon^{f_v},$$

where $\epsilon > 0$ is a constant depending only on the choice of K. In the language of the proof of Theorem 38 of Marcus's *Number Fields*, the multiplicative monoid of $\mathcal{O}_K$ maps to "logarithmic space" by

$$x \mapsto (\log |x|_v)_v,$$

the unit group of $\mathcal{O}_K$ forms a lattice of maximal rank in the norm-one hyperplane of this logarithmic space. Now we can translate x_n by a unit to bring it into a suitably chosen fundamental domain for the action of this lattice, and the claim follows.

But now the bound

$$\prod_v \max_i |x_i|_v^{f_v} < B,$$

implies for each i and v that

$$|x_i|_v < \left(B\epsilon^{f_v - r}\right)^{\frac{1}{f_v}},$$

where $r = [K : \mathbb{Q}]$; and it is known that only finitely many x_i can satisfy such a bound. □

Given the appearance of archimedean valuations in our definition, it should not be surprising that the standard height can be defined as a product of local heights. The formula below is usually given as a definition in the literature. The proof is routine, and we omit it.

Lemma 1.3. *The standard height of a point $(x_0 : \ldots : x_n)$ in $\mathbb{P}^n_K$ is given as the product over all places of K,*

$$\left(\prod_v \max_i |x_i|_v^{e_v f_v}\right)^{\frac{1}{[K:\mathbb{Q}]}},$$

where e_v and f_v represent the ramification and inertia degrees, of v over the appropriate place of $\mathbb{Q}$.

It is often cleaner to use the logarithmic height, which we now define.

Definition 1.4. *The* logarithmic height *on projective space is given by*

$$h(x_0 : \cdots : x_n) = \log H(x_0 : \cdots : x_n).$$

Now we outline a proof of the second fundamental fact about heights: a degree-d map approximately multiplies the logarithmic height by d.

Theorem 1.5. *Let $f : \mathbb{P}^n_K \to \mathbb{P}^m_K$ be a map (i.e. a morphism of varieties). Then there is an constant C, depending only of f, such that for any (K-valued) point $p \in \mathbb{P}^n_K$, we have*

$$dh(p) - C < h(f(p)) < dh(p) + C.$$

(In fact this constant is effectively computable.)

Proof. (Sketch.)

First, note that f is given, in homogenous coordinates, by homogenous polynomials $f_0, \ldots, f_m$ of degree d, and we can take these polynomials to have coefficients in $\mathcal{O}_K$.

As in the proof of Lemma 1.2, we may assume p has integral coordinates, and the ideal generated by its coordinates belongs to a finite set $\mathcal{I}$ of ideals. Thus in particular the logarithmic height $h(p)$ differs by at most a constant from

$$\frac{1}{[K : \mathbb{Q}]} \left(\sum_v f_v \max_i \log(|x_i|_v) \right).$$

By an easy argument involving the triangle inequality, for any polynomial f_j of degree d, there is a constant C_0, such that

$$|f_j(x_0, \ldots, x_n)|_v < C_0 \max_i |x_i|^d.$$

The upper bound

$$h(f(p)) < dh(p) + C$$

now follows by staring at the definition of $h(f(p))$.

The lower bound is more difficult, as there is no reason a priori why the value of a polynomial should not be very *small* compared to the coefficients. The trick is to show (roughly) that the coordinates x_i can be recovered from the coordinates $f_j(x)$, and thus the values $f_j(x)$ cannot all be too small at once.

Since the polynomials f_j define a map from $\mathbb{P}^n$ to $\mathbb{P}^m$, they cannot simultaneously vanish unless all the x_i's are zero. Thus, by Hilbert's Nullstellensatz, the ideal generated by the f_j's contains $x_0^N, \ldots, x_n^N$, for some positive integer N. Thus we can bound the norms of x_i^N from above in terms of the height $h(f(p))$; and this gives the reverse inequality.

(On effectivity: the upper bound is easy. The lower bound requires an effective Nullstellensatz. See [MW]. On effectivity questions in general, see for example 2.2.12-13 and 2.5 of [BG].) □

We now have the following easy consequence for arithmetic dynamics, the topic of our seminar.

Corollary 1.6. *Any map $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ of degree 2 or greater has at most finitely many preperiodic points.*

Proof. By Theorem 1.5, we can bound the height of any preperiodic point of f. By Lemma 1.2, there are only finitely many points of bounded height. □

2 Heights on a Projective Variety

This material is presented in III.10 of [Si2] as well as sections 2.2-2.4 of [BG]. We follow the presentation in Silverman.

Definition 2.1. *For any set X, let $\mathbb{R}^X$ denote the $\mathbb{R}$-vector space of real valued functions on X, and $\mathcal{B}(X)$ the subspaces of bounded functions on X. By a* function up to $O(1)$ *on X we mean an element of the quotient $\mathbb{R}^X/\mathcal{B}(X)$.*

Given a projective variety X over a number field K and a very ample line bundle $\mathcal{L}$ on X, we define a height function up to $O(1)$ on X as follows. Let ϕ be the morphism $X \to \mathbb{P}^N_K$ defined in the usual way from the complete linear system of $\mathcal{L}$; note that ϕ is defined up to linear automorphism of the target. Now we set, for all $p \in X$,

$$h_{\mathcal{L}}(p) = h(\phi(x)),$$

where the height on the right-hand side is the standard height on projective space.

By Theorem 1.5, the function $h_{\mathcal{L}}$ is well-defined up to $O(1)$.

In the above construction, we started with a line bundle $\mathcal{L}$ and used the map to projective space given by the complete linear system on $\mathcal{L}$ to define a height. What if we had only used some sections on $\mathcal{L}$? The answer is given by the following theorem, which we do not prove.

Theorem 2.2. *Suppose X is a projective variety over a number field K, with two morphisms $\phi : X \to \mathbb{P}^n_K$ and $\psi : X \to \mathbb{P}^m_K$ such that the hyperplane sections of the two projective spaces pull back to the same line bundle $\mathcal{L}$ on X. Then we have for $p \in X$*

$$h(\psi(p)) - h(\phi(p)) = O(1).$$

This is Lemma III.10.4 of [Si2]. See also sections 2.2-2.3 of [BG].

Some remarks on the proof: consider for simplicity the special case where ϕ and ψ are closed embeddings, so $\mathcal{L}$ is very ample. Without loss of generality, suppose ϕ is in fact defined via the complete linear system on $\mathcal{L}$, and suppose the image of ψ is not contained in any hyperplane. Then there is a linear projection away from some linear subspace A, say $\pi : \mathbb{P}^n_K - A \to \mathbb{P}^m_K$, such that $\psi = \pi \circ \phi$. Now it is obvious that

$$h(\psi(p)) < h(\phi(p)) + O(1)$$

by Theorem 1.5. The difficulty is in the other direction.

Since π induces an isomorphism on the projectively embedded copies of X, we know the induced map is invertible, hence given affine-locally by polynomial functions. Then one argues that the behavior of the height can be bounded locally on open affines. The argument (here and in general) requires Nullstellensatz, as in the proof of Theorem 1.5.

Next, we need an easy technical lemma.

Lemma 2.3. *Let $\phi : \mathbb{P}^n_K \times \mathbb{P}^m_K \to \mathbb{P}^{nm+n+m}_K$ denote the Segre embedding. Then we have*

$$h(\phi(x, y)) = h(x) + h(y),$$

where on both sides h denotes the standard logarithmic height on the relevant projective space.

Proof. Follows immediately from the definitions. □

As an immediate consequence, we have the following.

Lemma 2.4. *Let X be a projective variety over a number field K, and let $\mathcal{L}$ and $\mathcal{L}'$ be two very ample line bundles on X. Then we have*

$$h_{\mathcal{L}+\mathcal{L}'} = h_{\mathcal{L}} + h_{\mathcal{L}'} + O(1).$$

Proof. Follows from Theorem 2.2 and Lemma 2.3. □

Now we can define heights (up to $O(1)$) with respect to a general line bundle.

Definition 2.5. *Let X be a projective variety over a number field K, and let $\mathcal{L}$ be a line bundle on X. Write $\mathcal{L}$ as the difference of two very ample line bundles $\mathcal{L}_1 - \mathcal{L}_2$, and define*

$$h_{\mathcal{L}} = h_{\mathcal{L}_1} - h_{\mathcal{L}_2},$$

up to $O(1)$. This function $h_{\mathcal{L}}$ is called the Weil height *on X with respect to $\mathcal{L}$.*

By Lemma 2.4, the height so defined is independent of the choice of $\mathcal{L}_1$ and $\mathcal{L}_2$. If $\mathcal{L}$ was ample to start with then the Weil height agrees with the height $h_{\mathcal{L}}$ defined earlier, justifying our use of the same symbol for both.

From here the following theorem (III.10.1 of [Si2]) is easy.

Theorem 2.6. *The Weil height satisfies the following properties.*

(a) On $\mathbb{P}^n_k$, the Weil height $h_{\mathcal{O}(1)}$ is the standard logarithmic height.

(b) For any line bundles $\mathcal{L}$ and $\mathcal{L}'$ on a projective variety X over K, we have

$$h_{\mathcal{L}+\mathcal{L}'} = h_{\mathcal{L}} + h_{\mathcal{L}'} + O(1).$$

(c) Let $\phi : X \to Y$ be a morphism of projective varieties over K, and let $\mathcal{L}$ be a line bundle on Y. Then we have

$$h_{\phi^*(\mathcal{L})}(x) = h_{\mathcal{L}}(\phi(x)) + O(1).$$

3 Canonical Heights and the Local Decomposition

We return to the study of dynamical systems. Following Silverman [Si], we restrict ourselves to the study of degree-$d \geq 2$ maps $\phi : \mathbb{P}^1_K \to \mathbb{P}^1_K$, but in fact everything should be true over $\mathbb{P}^n_K$.

We first introduce the *canonical height* for a dynamical system.

Theorem 3.1. *Let K be a number field. Given any map $\phi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of degree $d \geq 2$, there is a height function $h_\phi : \mathbb{P}^1_K \to \mathbb{R}$ such that:*

(a)

$$h_\phi - h = O(1),$$

and

(b) for any $x \in \mathbb{P}^1_K$, we have

$$h_\phi(\phi(x)) = d h_\phi(x),$$

with exact equality.

Proof. Take

$$h_\phi(x) = \lim_n d^{-n} h(\phi^n(x)).$$

Using Lemma 1.5 one shows that this limit exists, and that it differs from the standard logarithmic height by $O(1)$. Part (b) is trivial. □

Remark 3.2. *The Néron-Tate height on elliptic curves is defined as a canonical height for the doubling map (viewed as an automorphism of $\mathbb{P}^1$). It is a remarkable fact that this canonical height not only transforms nicely with respect to the doubling map, but also defines a quadratic form that respects the group law on the elliptic curve. See Theorem VIII.9.3 of [Si1]. I do not know whether this generalizes, say, to families of commuting maps ϕ_i from $\mathbb{P}^n$ to itself.*

The canonical height plays well with preperiodic points.

Corollary 3.3. *With notation as above, we have*

$$h_\phi(x) = 0$$

if and only if x is preperiodic for ϕ.

Proof. The forward direction uses the finiteness property of the standard height. The reverse direction is trivial. □

We saw in Lemma 1.3 above that the standard (logarithmic) height on projective space can be expressed as a sum of local heights. It would be nice to express the global canonical height as a sum of local terms, as well.

Let K be a nonarchimedean local field. There is (for general ϕ) no logarithmic height on $\mathbb{P}^1_K$ satisfying the conclusion of Theorem 1.5. To deal with this, we lift ϕ to a map Φ on $\mathbb{A}^2_K$, as in the proof of Theorem 1.5; we define the local canonical height on $\mathbb{A}^2_K$ by an averaging trick as in Theorem 3.1; and then we normalize and sum the local heights.

We define a norm on $\mathbb{A}^2_K$ by

$$|(x, y)|_v = \max(|x|_v, |y|_v).$$

By the usual arguments, for $x \in \mathbb{A}^2_K$, and any homogenous degree-d map $\Phi = (F, G)$ from $\mathbb{A}^2_K$ to itself, where F and G have no common factors, we have

$$\log |\Phi(x)|_v = d \log |x|_v + O(1).$$

Furthermore, in case v is archimedean and ϕ is unramified at v, we have equality:

$$\log |\Phi(x)|_v = d \log |x|_v .$$

Thus, we can define the *Green's function*

$$\mathcal{G}_\Phi(x) = \lim_n d^{-n} \log |\phi^n(x)|_v .$$

We obtain as usual the following theorem (Proposition 5.58 of [Si]).

Theorem 3.4. *The Green's function defined above satisfies the following properties.*

(a) For any x in $\mathbb{A}^2$ we have

$$\mathcal{G}_\Phi(\Phi(x)) = d\mathcal{G}_\Phi(x)$$

and

$$\mathcal{G}_\Phi(x) = \log |x|_v + O(1).$$

(b) If v is archimedean and the map Φ has good reduction at v, then we have exact equality in

$$\mathcal{G}_\Phi(x) = \log |x|_v .$$

(c) The Green function scales by

$$\mathcal{G}_\Phi(cx) = \mathcal{G}_\Phi(x) + \log |c|_v$$

$$\mathcal{G}_{c\Phi}(x) = \mathcal{G}_\Phi(x) + \frac{1}{d-1} \log |c|_v .$$

(d) The Green function $\mathcal{G}_\Phi$ is Hölder continuous.

Finally, we show that the Green's functions $\mathcal{G}_{\Phi,v}$, as v ranges over all places of K, decompose the global canonical height.

Theorem 3.5. *Suppose $\phi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a map of degree $d \geq 2$. Choose a lift Φ of ϕ to $\mathbb{A}^2_K$. Then we have*

$$h_\phi(x : y) = \sum_v e_v f_v \mathcal{G}_{\Phi,v}(x, y),$$

for all $x \in \mathbb{P}^{\nvDash}(K)$.

Proof. (Sketch.)

First, note that the sum on the right-hand side is finite. Indeed, for all but finitely many v, we see that v is archimedean, Φ is unramified at v, and x and y have absolute value 1 at v. Thus the sum is well-defined. Let $h_0(x, y)$ denote its value.

Next, by scaling properties of Green's functions, we show that $h_0(x, y) = h_0(cx, cy)$. Thus h_0 gives a function on $\mathbb{P}^1_K$.

Finally, we show that applying $h_0(\phi(x)) = dh_0(x)$, and $h_0(x) = h(x) + O(1)$.

Now by uniqueness of the canonical height, we have $h_0 = h_\phi$. □

References

[BG] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, Cambridge, 1990.

[MW] D. Masser and G. Wüstholz, Fields of large transcendence degree generated by values of elliptic functions, *Invent. Math.* 72 (1983), 407–464.

[Si] J. Silverman, *The Arithmetic of Dynamical Systems*, Springer, 2007.

[Si1] J. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1994.

[Si2] J. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer, 1994.