On Symmetric Polynomials

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1 Introduction

Take the polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$. By the Fundamental Theorem of Algebra, we know that $P(x) = 0$ has exactly n solutions over $\mathbb{C}$. Let these roots be the elements of the set $R = \{r_1, r_2, r_3, \dots, r_n\}$.

Vieta's Formulas give expressions for specific "combinations" of the roots of a polynomial in terms of the coefficients. For example,

$$\begin{aligned} r_1 + r_2 + \cdots + r_n &= -\frac{a_{n-1}}{a_n}, \\ r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n &= \frac{a_{n-2}}{a_n}, \\ &\vdots \\ r_1 r_2 \cdots r_n &= \frac{(-1)^n a_0}{a_n}. \end{aligned}$$

This can be proven by expanding the factored form of

$$P(x) = a_n (x - r_1)(x - r_2) \cdots (x - r_n)$$

and setting the coefficients equal to those of the original form.

We define the following class of expressions.

Definition 1. *Symmetrized Monomial.*

A symmetrized monomial *in n variables in which each term contains k variables is defined as*

$$\sum_{r_1, r_2, \dots, r_k \in R} r_1^{a_1} r_2^{a_2} \dots r_k^{a_k}$$

where $a_i \in \mathbb{N}$ for all $1 \leq i \leq k$. This sum is over all distinct monomials that can be obtained from $r_1^{a_1} r_2^{a_2} \cdots r_k^{a_k}$ by permuting the variables. We say that the degree of the symmetrized monomial is

$$d = \sum_{i=1}^{k} a_i.$$

Without loss of generality, we assume $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k$.

Example 2. *For $n = 4$, an example of a symmetrized monomial in which each term contains 3 variables is*

$$\sum_{r_1,r_2,r_3 \in R} r_1r_2r_3 = r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4.$$

We also have the following definition.

Definition 3. *Symmetric Polynomial.*

Let σ be a permutation of the index set $\{1, 2, \ldots, n\}$ i.e. $\sigma(i)$ would represent the ith element in the permutation. A symmetric polynomial $P(r_1, r_2, \ldots, r_n)$ *is a polynomial in n variables such that*

$$P(r_1, r_2, \ldots, r_n) = P(r_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(n)})$$

for all possible permutations σ.

We can see that any symmetric polynomial can be written as the linear combination of symmetrized monomials.

Example 4. *For $n = 4$, an example of a symmetric polynomial is*

$$2 \sum_{r_1,r_2,r_3 \in R} r_1r_2r_3 + 3 \sum_{r_1,r_2 \in R} r_1^2 r_2.$$

We have the following special class of symmetric polynomials.

Definition 5. *Elementary Symmetric Polynomials.*

The elementary symmetric polynomials *in n variables are the "combinations" from Vieta's Formulas, specifically*

$$\begin{aligned} s_1 &= r_1 + r_2 + \ldots r_n, \\ s_2 &= r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n, \\ &\vdots \\ s_n &= r_1r_2\cdots r_n. \end{aligned}$$

Example 6. *For $n = 4$, the elementary symmetric polynomials are*

$$\begin{aligned} s_1 &= r_1 + r_2 + r_3 + r_4, \\ s_2 &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4, \\ s_3 &= r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 \text{ and} \\ s_4 &= r_1r_2r_3r_4. \end{aligned}$$

The question arises as to whether all symmetric polynomials can be rewritten in terms of "simpler" symmetric polynomials, and more specifically, whether all symmetric polynomials can be rewritten in terms of elementary symmetric polynomials. We prove the following theorem.

Theorem 7. *Every symmetric polynomial can be expressed in terms of elementary symmetric polynomials.*

Example 8. *If $n = 3$, an example of a symmetric polynomial is*

$$3\sum_{r_1,r_2,r_3\in R} r_1^2 r_2 r_3 + \sum_{r_1,r_2\in R} r_1^2 r_2 = 3r_1^2 r_2 r_3 + 3r_1 r_2^2 r_3 + 3r_1 r_2 r_3^2 + r_1^2 r_2 + r_1 r_2^2 + r_2^2 r_3 + r_2 r_3^2 + r_3^2 r_1 + r_3 r_1^2.$$

This symmetric polynomial can be rewritten as

$$\begin{aligned} 3\sum_{r_1,r_2,r_3\in R} r_1^2 r_2 r_3 + \sum_{r_1,r_2\in R} r_1^2 r_2 &= 3\,(r_1 r_2 r_3)\left(\sum_{r_1\in R} r_1\right) \\ &\quad + \left(\sum_{r_1,r_2\in R} r_1 r_2\right)\left(\sum_{r_1\in R} r_1\right) - 3(r_1 r_2 r_3) \\ &= 3s_3 s_1 + s_2 s_1 - 3s_3. \end{aligned}$$

2 The Proof

We begin by proving the following lemma.

Lemma 9. *Every symmetrized monomial with degree d can be expressed in terms of elementary symmetric polynomials.*

Proof. We proceed by induction on d.

The base case is $d = 1$. The only symmetrized monomial with degree 1 is $\sum_{i=1}^{n} r_i$, which is an elementary symmetric polynomial, so the base case holds.

We assume that every symmetrized monomial of degree $1, 2, \ldots, d-1$ can be expressed in terms of elementary symmetric polynomials.

We wish to prove that every symmetrized monomial of degree d can be written in terms of symmetrized monomials with smaller degree and symmetrized monomials with more than k variables.

Let the symmetrized monomial

$$M = \sum_{r_1,r_2,\ldots,r_k} r_1^{a_1} r_2^{a_2} \cdots r_k^{a_k}, a_1 \geq a_2 \geq \cdots \geq a_k > 0.$$

Then we have

$$M = \sum_{r_1,r_2,\ldots,r_k\in R} r_1 r_2 \cdots r_k \cdot \sum_{r_1,r_2,\ldots,r_k\in R} r_1^{a_1-1} r_2^{a_2-1} \cdots r_k^{a_k-1} - \sum c_i M_i$$

for some symmetrized monomials M_i each with degree d and some constants c_i. We see that the first term on the right hand side of the equation is an elementary symmetric polynomial, and the second term is a symmetrized monomial of lower degree. Therefore, by our induction hypothesis, these two terms can already be written in terms of elementary symmetric polynomials.

Each of the symmetrized monomials M_i can be split into one of two categories, those which contain all n variables in each term and those which contain less than n variables in each term. In the first case, we can factor the elementary symmetric polynomial $r_1 r_2 \cdots r_n$ out a_k times to get a new symmetrized monomial with smaller degree, which has already been taken care of by our induction hypothesis.

We now look at the second possibility. From our rewritten form of M, we know that all of the symmetrized monomials formed by the product of our two sums must contain at least k of the n variables in each term. If they contain exactly k of the n variables in each term, then they must be part of S, so they will not fall into this category. Therefore, all of the symmetrized monomials in this category will have m variables in each term for some $k < m < n$.

Since the number of variables in each term of the symmetrized monomials in this category is always increasing, at some point, we are bound to get a symmetrized monomial with n variables in each term. From here, we continue as in the previous case, so this symmetrized monomial can also be expressed in terms of elementary symmetric polynomials.

□

Theorem 10. *Every symmetric polynomial can be expressed in terms of elementary symmetric polynomials.*

Proof. By definition, symmetric polynomials are linear combinations of symmetrized monomials, so from the above lemma, every symmetric polynomial can be written in terms of elementary symmetric polynomials.

□

3 Applications

Example 11. *Let $P(x) = x^5 + 2x^4 - x^3 + x^2 - 3x + 1$. If the roots of $P(x) = 0$ are r_1, r_2, r_3, r_4, and r_5, find*

$$r_1^2r_2^2r_3^2 + r_1^2r_2^2r_4^2 + r_1^2r_2^2r_5^2 + r_1^2r_3^2r_4^2 + r_1^2r_3^2r_5^2 \\ + r_1^2r_4^2r_5^2 + r_2^2r_3^2r_4^2 + r_2^2r_3^2r_5^2 + r_2^2r_4^2r_5^2 + r_3^2r_4^2r_5^2.$$

Solution 12. *We wish to compute $\sum\limits_{r_1,r_2,r_3 \in R} r_1^2r_2^2r_3^2$. Using Vieta's Formulas, we have*

$$
\begin{aligned}
\sum_{r_1,r_2,r_3\in R} r_1^2r_2^2r_3^2 &= \left(\sum_{r_1,r_2,r_3\in R} r_1r_2r_3\right)\left(\sum_{r_1,r_2,r_3\in R} r_1r_2r_3\right) \\
&\quad - 2\sum_{r_1,r_2,r_3,r_4\in R} r_1^2r_2^2r_3r_4 - 6\sum_{r_1,r_2,r_3,r_4,r_5\in R} r_1^2r_2r_3r_4r_5 \\
&= (s_3)^2 - 2\left(\sum_{r_1,r_2,r_3,r_4\in R} r_1r_2r_3r_4\right)\left(\sum_{r_1,r_2\in R} r_1r_2\right) \\
&\quad + 2\cdot 4\sum_{r_1,r_2,r_3,r_4,r_5\in R} r_1^2r_2r_3r_4r_5 - 6\sum_{r_1,r_2,r_3,r_4,r_5\in R} r_1^2r_2r_3r_4r_5 \\
&= s_3^2 - 2s_4s_2 + 8s_1s_5 - 6s_1s_5 \\
&= s_3^2 - 2s_4s_2 + 2s_1s_5 \\
&= \boxed{-1}.
\end{aligned}
$$