Elliptic curves, Hilbert modular forms, and the Hodge conjecture

Don Blasius

September 8, 2004

1 Introduction

1.1 Recall the following special case of a foundational result of Shimura ([S2, Theorems 7.15 and 7.16]) :

Theorem Let f be a holomorphic newform of weight 2 with rational Fourier coefficients $\{a_n(f)|n \geq 1\}$. There exists an elliptic curve E defined over $\mathbf{Q}$ such that, for all but finitely many of the primes p at which E has good reduction E_p, the formula

$$a_p(f) = 1 - N_p(E) + p$$

holds. Here $N_p(E)$ denotes the number of points of E_p over the field with p elements.

1.2. The first result of this type is due to Eichler ([E]) who treated the case where $f = f_{11}$ is the unique weight 2 newform for $\Gamma_0(11)$ and E is the compactified modular curve for this group. Later, in several works, Shimura showed that the Hasse-Weil zeta functions of special models (often called *canonical models*) of modular and quaternionic curves are, at almost all finite places v, products of the v-Euler factors attached to a basis of the Hecke eigenforms of the given level. These results give at once computations of the zeta functions of the Jacobians of these curves since $H^1_l(C) = H^1_l(Jac(C))$ for a smooth projective curve. However, it was only in late 1960's that the correspondence between individual forms and geometry came to be emphasized. In particular, in his proof of the above Theorem, Shimura identified $L(f, s)$ as the zeta function of a one dimensional factor E_f of the Jacobian variety. He also treated the case where the $a_n(f)$ are not rational; then E_f

is replaced by a higher dimensional factor (or, alternatively, quotient)A_f of the Jacobian ([S3] and [S4]).

By Tate's conjecture ([F]), the above correspondence $f \to E_f$ determines E_f up to an isogeny defined over $\mathbf{Q}$. Further, by works of Igusa, Langlands, Deligne, Carayol, and others, a completed result is known: the conductor N_E of E coincides with the conductor N_f of f, the above formula holds for all p such that $(N_f, p) = 1$, and corresponding but more complicated statements (the *l*ocal Langlands correspondence) hold at the primes p which divide N_f.

1.3. Our goal here is to give a *conditional* generalisation of Shimura's result to totally real fields F, i.e. to Hilbert modular forms. Thus, we replace f by a cuspidal automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of the adele group $GL_2(A_F)$. The weight 2 condition generalises to the requirements that (i) π_∞ belong to the lowest discrete series as a representation of $GL_2(F \otimes \mathbf{R})$, and (ii) π have central character ω_π equal to the inverse of the usual idelic norm. In the classical language of holomorphic forms on (disjoint unions of) products of upper-half planes, condition (i) asserts that on each product the form be of diagonal weight (2,...,2). For each finite place v of F at which π is unramified, we have the Hecke eigenvalue $a_v(\pi)$. We assume that the $a_v(\pi)$ belong to $\mathbf{Q}$ for all such v, and that the Hecke operators are so normalized that, for almost all v, the polynomial $T^2 - a_v(\pi)T + N_v$ has its zeros at numbers of size $N_v^{1/2}$, where N_v denotes the number of elements in the residue field of F at v. Here is the now standard conjectural generalisation, first published by Oda ([O]), of Shimura's result.

1.4. **Existence Conjecture.** For π as above, there exists an elliptic curve E defined over F such that for all but finitely many of the finite places v of F at which E has good reduction E_v,

$$a_v(\pi) = 1 - N_v(E) + N_v$$

holds. Here $N_v(E)$ is the number of points of E_v over the residue field at v. It should be noted that if the Existence Conjecture is proven, then the appropriate statements of the Langlands correspondence hold at all places. (This follows from ([T1], [T2]) and the Cebotarev theorem.)

1.5. Central to our argument is an unproved hypothesis of Deligne: the *Absolute Hodge Conjecture* ([D4]). This conjecture can be stated in several ways. For us, its categorical formulation is most directly useful. To recall it, let $M_{\mathbf{C}}$ denote the tensor category of motives for absolute Hodge cycles defined over $\mathbf{C}$ (cf. [D4]). By Hodge theory, the usual topological cohomology functor on varieties, which attaches to each projective smooth complex variety X its total cohomology ring $H_B^*(X, \mathbf{Q})$, takes values in the tensor category of polarisable rational Hodge structures. It extends to the category $M_{\mathbf{C}}$ and we denote this extended functor by ω_B. The rational Hodge structure attached to a motive M, defined over a subfield L of $\mathbf{C}$, is $M_B = \omega_B(M \otimes \mathbf{C})$, where $M \otimes \mathbf{C}$ is the base change of M to $\mathbf{C}$.

In this language the Absolute Hodge Conjecture asserts simply:

> *The functor ω_B is fully faithful.*

In fact we shall use precisely the assertion:

> *If M and N are motives for absolute Hodge cycles defined over $\mathbf{C}$, and M_B is isomorphic to N_B as Hodge structure, then M is isomorphic to N.*

Of course, the Absolute Hodge Conjecture (AHC) is trivially a consequence of the usual Hodge Conjecture. The AHC was proved for all abelian varieties over subfields of $\mathbf{C}$ (in particular, for all products of curves over $\mathbf{C}$) by Deligne, and as such it has been of great utility in the theory of Shimura varieties. We need to use it in this paper for a product of a Picard modular surface (e.g. an arithmetic quotient of the unit ball) and an abelian variety. In such a case the conjecture is unknown.

1.6. The main result here is the following

Theorem 1 *Suppose that the Absolute Hodge Conjecture is true. Then the Existence Conjecture is true.*

1.7. For background it is essential to recall the known cases of the Existence Conjecture.

1.7.1. It is an easy consequence of work of Hida ([H] and Faltings ([F]) in the cases covered by the following hypotheses **QC**:

QC1. [F:Q] is odd, or
QC2. π has a finite place at which π_v belongs to the discrete series.

In Section 2 below this case is deduced from Hida's work.

1.7.2. The conjecture is also known for all forms π of CM type. Here π is defined to be of CM type if there exists a non-trivial idele class character ϵ (necessarily of order 2) of F such that the representation $\pi \otimes \epsilon$ is isomorphic to π. See 2.2 below for some comments.

1.7.3. However, to our knowledge, the existence conjecture is not known for any non-CM type everywhere unramified representations π in the case where F is a real quadratic extension of $\mathbf{Q}$. Such representations exist and should be in bijection, for a given F, with the F-isogeny classes of non-CM elliptic curves over F which have good reduction at all finite places. One may regard these forms as the test case for any construction.

1.8. In recent years, many automorphic correspondences, more sophisticated than the Jacquet-Langlands correspondence, have been proven. It is natural to ask whether, at least in principle, there should exist some other such functorialities to a group of Hermitian symmetric type from which the sought E's can be directly constructed. However, some new ideas will be needed. Indeed, we have the following folklore result:

Theorem 2 *Every non-CM abelian variety which occurs as a quotient of the Picard variety of any smooth compactification of a Shimura variety is isogenous to a base change of a factor of the Jacobian of a quaternionic Shimura curve.*

The proof of this easy negative result is sketched, if not proven,below in 2.3. In view of this fact, it seemed reasonable to investigate whether any known principles (e.g. the Hodge Conjecture) could provide the additional abelian varieties needed for the Existence Conjecture.

1.9. Here is a brief outline of the proof of Theorem 1. There are 4 main steps:

1.9.1. We use a sequence of functorialities to find an orthogonal rank 3 motive M in the second cohomology of a Picard modular surface ([LR]) which has the same L-function as the symmetric square of a base change of π.

1.9.2. From the weight 2 Hodge structure M_B of M, we construct a rank 2 rational Hodge structure R of type (1,0) (0,1) whose symmetric square is isomorphic to M_B. We let $A = A_{\mathbf{C}}$ be an elliptic curve over $\mathbf{C}$ whose $H^1_B(A)$ is isomorphic as Hodge structure to R.

1.9.3. Using the Absolute Hodge Conjecture, we descend A to a curve, also denoted A, defined over a number field L which contains F.

1.9.4. Using the existence ([T1, BR2]) of a two dimensional l-adic representation attached to π, we find a D of dimension 1 inside the Weil restriction of scalars of A from L to F; by construction, D has the correct l-adic representations.

1.10. Since the result is in any case conditional, and for simplicity, we have usually treated only the case of rational Hecke eigenvalues where the sought variety really is an elliptic curve. However, the proof extends, with some changes, to the general case of forms with π_∞ belonging to the lowest discrete series. At the suggestion of the referee, we have indicated significant changes needed for the case of general Hecke field T_π, which is defined as the number field generated by the $a_v(\pi)$. In this case it is more convenient to state the conjecture cohomologically:

For π of weight $(2, ..., 2)$, having Hecke field T_π, there exists an abelian variety A_π, defined over F, such that $End(A_\pi) = T_\pi$, and such that for all but finitely many of the finite places v of F at which A_π has good reduction,

$$L_v(H^1_l(A_\pi), T_\pi, s) = L_v(\pi, s).$$

Here $L_v(H^1_l(A_\pi), T_\pi, s)$ is the L-function denoted $\zeta(s; A_\pi/F, T_\pi)$ in ([S2, Section 7.6]), and is denoted by $L_v(H^1_l(B_\pi), \sigma_l, s)$ below in 2.1, for the case $\sigma_l = id$.

1.11.Finally, we note that in several lectures on this topic the proof was given with alternative first steps using quaternionic surfaces. The proof given here is a little simpler, but the earlier construction is still useful. In particular, it enables the unconditional proof of the Ramanujan conjecture at all places for holomorphic Hilbert modular forms which are discrete series at infinity ([B]).

1.12. **Notations**. Throughout the paper, F denotes a totally real field, K_0 is a fixed quadratic imaginary extension of $\mathbf{Q}$ and $K = FK_0$. The letter L will denote a number field whose definition depends on context. The symbol π denotes a fixed cuspidal automorphic representation, as above, of $GL(2, A_F)$, i.e. which is (i) of lowest discrete series type at infinity, and (ii) has central character equal to the inverse of the norm.

1.13. **Acknowledgements**. This work was largely done during a visit to SFB 478 in Münster, Germany; this hospitality was greatly appreciated.

2 Known Cases of the Existence Conjecture and Negative Background

2.1. Hida ([H, Theorem 4.12]) generalised most of Shimura's result (1.1), to the curves defined by quaternion algebras over totally real fields. Thus, to use Hida's work, one first invokes the Jacquet-Langlands correspondence ([JL]) to find an automorphic representation π_Q with the same L-function as π but on the adele group associated to the muliplicative group of a suitable quaternion algebra Q over F. Here suitable means that the canonical models of the associated arithmetic quotients are curves defined over F. The hypotheses **QC** above are necessary and sufficient for such a pair (Q, π_Q) to exist.

Let $T = T_\pi$ be the Hecke field of π. It is a number field which is either totally real or a totally imaginary quadratic extension of a totally real field. By Hida ([H, Prop. 4.8])there exists an abelian variety B_π and a T_π-subalgebra T of $End(B_\pi)$, which is isomorphic to a direct sum of number fields, such that $H^1_l(B_\pi)$ is a free rank 2 $T \otimes \mathbf{Q}_l$-module. Further, for almost all finite places v of F, all l prime to v, and all morphisms $\sigma_l : T \to \overline{\mathbf{Q}_l}$,

$$L_v(H^1_l(B_\pi), \sigma_l, s) = L_v(\pi, \sigma_l, s)^d$$

where $d = [T : T_\pi]$, the L-factor on the left hand side is that of the σ_l-eigensubspace of $H^1_l(B_\pi) \otimes \overline{\mathbf{Q}_l}$, and that on the right hand side is the L-factor with coefficients in $\sigma_l(T_\pi)$ defined by applying σ_l to the coefficients of $L_v(\pi, s)$. (The equality makes sense if one recalls the usual convention that $\overline{\mathbf{Q}}$ is identified once and for all with subfields of both $\mathbf{C}$ and $\overline{\mathbf{Q}_l}$ via fixed embeddings.)

By ([BR1]), each σ_l-eigensubspace of $H^1_l(B_\pi) \otimes \overline{\mathbf{Q}_l}$, is absolutely irreducible. Hence, the commutant of the image of Galois in $End_{\mathbf{Q}_l}(H^1_l(B_\pi) \otimes \overline{\mathbf{Q}_l})$ is isomorphic to the matrix algebra $M_d(T_\pi \otimes \overline{\mathbf{Q}_l})$. By the Tate conjecture ([F]),this means that $End(B_\pi)$ is a simple algebra with center T_π and T is a maximal commutative semisimple subalgebra. By Albert's classification, $End(B_\pi)$ is isomorphic either to (i) $M_e(D)$ where D is a quaternion algebra with center T_π, or (ii) $M_d(T_\pi)$). This follows because we know in general that the rank over T of the toplogical cohomology $H^1_B(B_\pi)$ is a multiple of k, if $[D : T_\pi] = k^2$. To exclude the former case if $T_\pi = \mathbf{Q}$ is easy: D is split at all finite places and so, since there is only 1 infinite place, it must be split everywhere. However, if $T_\pi \neq \mathbf{Q}$, we need to use a standard argument which relies upon the fact that F has a real place. Let B_0 be a simple factor of B_π. Then $End(B_0) = D$. Note that complex conjugation defines a continous involution of the set of complex points $B_0(\mathbf{C})$ of B_0. Denote the associated involution on the rank 1 D module $H^1_B(B_0)$ by Fr_∞. Evidently, Fr_∞ has order 2, has eigenvalues 1 and -1 with equal multiplicity (since it interchanges the holomorphic and antiholomorphic parts of the Hodge splitting) and commutes with T_π. To conclude, note that the commutant of D in $End(B_0)$ is isomorphic to the opposite quaternion algebra D^{op} to D, and that Fr_∞ is a non-scalar element of this algebra. Since Fr_∞ is non-scalar, the algebra $\mathbf{Q}[X]/(X^2 - 1)$ embeds in D^{op}. But this algebra is not a field, and hence this case cannot occur.

Thus, $End(B_\pi) = M_d(T_\pi)$. If A_π denotes any simple factor, we have $End(A_\pi) = T_\pi$ and finally

$$L_v(H^1_l(A_\pi), \sigma_l, s) = L_v(\pi, \sigma_l, s)$$

for all σ_l, almost all v, and all l which are prime to v. QED.

Theorem 3 *The Existence Conjecture holds for all π which satisfy the hypotheses* **QC**.

Remark. It would be interesting to show that we can take $d = 1$ without invoking the Tate Conjecture.

2.2. The CM case. This follows from the work of Casselman ([S1]), the Tate conjecture for abelian varieties of CM type, and the fact that the holomorphic cusp forms of CM type (of weight 2) are exactly those associated by theta series (or automorphic induction) to algebraic Hecke characters of totally imaginary quadratic extensions of F and having a CM type for their infinity type. In more detail, if π is of CM type, then there exists a totally imaginary quadratic extension J of F and a Hecke character ρ of J such that $L(\pi, s) = L(\rho, s)$, where the equality is one of formal Euler products over places of F. Since the Hecke eigenvalues of π are rational, the field generated by the values of ρ on the finite ideles of J is a quadratic imaginary extension T of $\mathbf{Q}$. By Casselman's theorem ([S1], Theorem 6), there exists an elliptic curve E defined over J, having complex multiplication by $\mathcal{O}_T$ over J, such that $L(\rho, s)L(\overline{\rho}, s)$ is the zeta function of $H^1(E)$. Let RE

denote the restriction scalars (in the sense of Weil) from J to F of E. Let for each finite place λ of K, ρ_λ be the λ-adic representation of $Gal(\overline{\mathbf{Q}}/\mathbf{Q}$ attached to ρ by Weil. Then the induced representation $Ind_F^J(\rho_\lambda)$ has a $\mathbf{Q}_l$-rational character and hence, since F has a real place, can be defined over $\mathbf{Q}_l$. Note the L-function of the compatible system of all $Ind_F^J(\rho_\lambda)$ is $L(\pi,s)$. Since $H_l^1(E)$ is a free rank 2 $K\otimes\mathbf{Q}_l$-module, and the Galois action commutes with this structure, the Galois action on $H_l^1(E)$ is the direct sum of two copies of the $\mathbf{Q}_l$ model of $Ind_F^J(\rho_\lambda)$. Hence the commutant of the image of Galois is $M_2(\mathbf{Q}_l)$. Since this holds for all l, and using the Tate conjecture, we conclude that $End(RE)$ is $M_2(\mathbf{Q})$ and RE is isogenous to a square $E_0\times E_0$. Evidently, we have $L(\pi,s)=L(H^1(E_0),s)$.

2.3. We clarify the meaning of the statement in (1.8).

2.3.1. Let S be any Shimura variety in the sense of Deligne's axioms ([D1]). Assume first that S is smooth, since the cohomology of a lower level injects into that of S. Let S^* be a smooth compactification of S. For any variety X, the cohomology $H_B^*(X,\mathbf{Q})$ has a canonical mixed Hodge structure, due to Deligne ([D3]). By this theory, the image of the pure weight n Hodge structure $H_B^n(S^*,\mathbf{Q})$ is exactly the first step $W_nH_B^n(S,\mathbf{Q})$ of the weight filtration of the mixed Hodge structure on $H_B^n(S,\mathbf{Q})$. Since $H_B^0(S,\mathbf{Q})$ is pure of weight 0, and since the weights of the pure graded components of $H_B^n(S,\mathbf{Q})$ are at least n, the only source for a Hodge structure of type (1,0) (0,1) among the pure Hodge structures defined as quotients of the weight filtration $W_\bullet$ of $H_B^*(S,\mathbf{Q})$ is $W_1H_B^1(S,\mathbf{Q})$. Further, if such a Hodge structure occurs, then it is identified with $H_B^1(A)$ for a suitable factor of the Picard variety of any S^*. Suppose now that S is not smooth. Choosing a suitably large level (=small open compact), we find an S_1 such that the cohomology (as mixed Hodge structure) of S injects into that of S_1. So no new Hodge structures of the type (1,0) (0,1) occur.

2.3.2. The condition that S have $H_B^1(S,\mathbf{Q})\neq 0$ is very restrictive. Indeed, unless $dim(S)=1$, we have

$$H_B^1(S,\mathbf{Q})=IH_B^1(S,\mathbf{Q})=H_2^1(S,\mathbf{Q})$$

where IH^* denotes the usual intersection cohomology and H_2^1 denotes the L^2 cohomology. Now using the Künneth formula, we can assume that S is defined by an algebraic group G over $\mathbf{Q}$ whose semisimple part G_{ss} is almost simple. In a standard way (using the Matsushima formula, the Künneth formula for continuous cohomology, the Vogan-Zuckermann classification, and the strong approximation theorem), we see that $G_{ss}(\mathbf{R})$ can have at most one non-compact factor G_1 which must itself be of real rank 1. By the classification of groups of Hermitian symmetric type, the only possibilities for G_1 are that it be isogenous to $SU(n,1)$ for $n \geq 1$. Suppose $n > 1$. Then by ([MR]) $H^1_B(S, \mathbf{Q})$ is of CM type and so is the associated Picard variety. On the other hand, if $n = 1$, then S is a curve, and it is not hard to see, from the definition of reflex fields and the computation of the zeta function of these curves, that every factor of the Jacobian of S is isogenous to the base change of a factor of the Jacobian of a quaternionic Shimura curve. This completes our sketch of the proof.

3 Some functorialities.

3.1. We assume throughout the paper that $[F : \mathbf{Q}] > 1$. Further, from now until the last Section of the paper we insist that

(i) K is unramified quadratic over F,
(ii) π is unramified at every finite place of F,
(iii) the central character ω_π of π is $| * |_F^{-1}$, the inverse of the idelic norm.

These conditions will impose no restriction on our final result. Indeed, the first condition may be achieved, starting from any quadratic extension K of F, by a cyclic totally real base change from F to an extension $F\prime$, suitably ramified at the places of F where K is ramified, so that $K\prime = KF\prime$ is unramified over $F\prime$. For (ii), if π_v belongs to the discrete series at any finite place v, then the condtion **QC2** is satisfied, and there is nothing to prove. On the other hand, if π_v belongs to the principal series at each finite place, then there is always an abelian totally real base change which kills all ramification, as may easily seen by using the fact that the Galois representations attached to π by Taylor([T1, T2]) satisfy the local Langlands correspondence. Finally, since ω_π differs from $| * |_F^{-1}$ by a totally even character ψ of finite order, a base change to the field F_ψ associated to ψ by classfield theory establishes (iii). Of course, for the main case of this paper, it is part of our initial assumption. Note in any case that it ensures, since π is non-CM, that T_π is totally real.
3.2. Jacquet and Gelbart ([GJ]) have proven a correspondence Sym^2 from non-CM cuspidal automorphic representations of $GL(2, A_F)$ to cuspidal automorphic representations of $GL(3, A_F)$. The underlying local correspon-

dence is elementary to describe, at least at the finite places, since π is unramified. For such a place v, recall that the Hecke polynomial at v of π is

$$H_v(\pi)(T) = T^2 - a_v(\pi)T + N_v,$$

which we factor as $H_v(\pi)(T) = (T - \alpha_v(\pi))(T - \beta_v(\pi))$. Similarly, each cuspidal automorphic representation Π of $GL(3, A_F)$ has Hecke polynomials

$$H_v(\Pi)(T) = (T - r_v)(T - s_v)(T - t_v)$$

for v which are unramified for Π. Define

$$H_v^2(\pi)(T) = (T - \alpha^2)(T - N_v)(T - \beta_v^2).$$

Then by ([GJ]) there exists a unique cuspidal automorphic representation $Sym^2(\pi)$ of $GL(3, A_F)$ such that

$$H_v(Sym^2(\pi)) = H_v^2(\pi)$$

for all finite v.

An analogous result holds at the infinite places. Here the groups are $GL(2, \mathbf{R})$ and $GL(3, \mathbf{R})$ whose irreducible admissible representations are classified by conjugacy classes of semisimple homomorphisms $\sigma_v : W_{\mathbf{R}} \to GL(k, \mathbf{C})$, $(k = 2, 3)$, where $W_{\mathbf{R}}$ is the Weil group of $\mathbf{R}$. Of course, given $\sigma_v : W_{\mathbf{R}} \to GL(2, \mathbf{C})$, there is a naturally defined class $Sym^2(\sigma_v) : W_{\mathbf{R}} \to GL(3, \mathbf{C})$. (One definition is $Sym^2 = (\sigma_v) \otimes (\sigma_v)/det(\sigma_v)$.) Then the correspondence at infinite v is $\sigma_v(Sym^2(\pi)) = Sym^2(\sigma_v)$.

3.3. There is a base change correspondence ([AC, Thm. 4.2])

$$BC_F^K \{\text{cusp forms on } \mathrm{GL}_3(\mathbf{A}_\mathrm{F})\} \longmapsto \{\text{cusp forms on } \mathrm{GL}_3(\mathbf{A}_\mathrm{K})\}$$

which takes cusp forms to cusp forms. For π_3 a cuspidal automorphic representation of $GL(3, A_K)$, $BC_F^K(\pi_3)$ is characterized by the equality, for all but finitely many v,

$$L_w(BC_F^K(\pi_3), s) = L_v(\pi_3, s) L_v(\pi_3 \otimes \epsilon_{K/F}, s),$$

where $\epsilon_{K/F}$ is the idele class character of A_F^* associated by class-field theory to K/F. An analogous result holds at all places. At the infinite places, the groups are $GL(3, \mathbf{R})$ and $GL(3, \mathbf{C})$ whose irreducible admissible representations are classified by semisimple homomorphisms $\sigma : W_{\mathbf{R}} \to GL(3, \mathbf{C})$ and $\sigma : W_{\mathbf{C}} = \mathbf{C}^* \to GL(3, \mathbf{C})$, respectively. Then

$$\sigma((BC_F^K(\pi_3))_w) = \sigma(\pi_3)_v|W_{\mathbf{C}}$$

if w lies over v.

3.4. Let V be a vector space of dimension 3 over K and let $H : V \times V \to K$ be a non-degenerate Hermitian form relative to K/F. Let $G = U(H)$ denote the unitary group of H as an algebraic group over F. Assume that H is chosen so that G is quasi-split. Let $G^* = GU^*(H)$ be the group of rational similitudes of H.Then the base change of G to K (as algebraic group) is isomorphic to $GL(3, K)$ as algebraic group over K. In [R2] Rogawski established base change correspondences, also denoted BC_F^K, between automorphic forms on $G(A_F)$ and on $GL(3, A_K)$. Let η be the algebraic automorphism of G defined by $\eta(g) =^t \overline{g}^{-1}$ for all g in $GL(3, K)$ and extend η to $GL(3, A_K)$ in the natural way. Then a cusp form Π_3 on $GL(3, A_K)$ is in the image of base change BC_F^K from a global L-packet of automorphic representations of G iff $\Pi_3 \circ \eta$ is isomorphic to Π_3. Let $\Pi_3 = BC_F^K(Sym^2(\Pi)) \otimes |det|$. Then evidently $\Pi_3 \circ \eta \cong \Pi_3$ since this is so almost everywhere locally (since the form is a base change, the conjugation can be ignored; since the map $g \to^t g^{-1}$ takes a local unitary Π_w to its contragredient, it suffices to note that the local components of $BC_F^K(Sym^2(\pi)) \otimes |det|$ are self-dual.) By [R2] there exists an L-packet $\Pi(G)$ of $G(A_F)$ such that $BC_F^K(\Pi(G)) = BC_F^K(Sym^2(\Pi)) \otimes |det|$. Further, at each infinite place v, the members of the local L-packet $\Pi(G)_v$ belong to the discrete series; they are exactly the 3 discrete series representations $\pi(G)_v$ such that

$$dim(H^2(Lie(G), k_\infty, \pi(G)_v)) = 1.$$

Denoting any of these $\Pi(G)_v$ by $\Pi(G)_\infty$, put

$$\Pi(G)_\infty = \{\pi^+, \pi^-, \pi^0\},$$

where the members are respectively holomorphic, antiholomorphic, and neither, for the usual choice of complex structure on the symmetric space (=unit ball) attached to G. Since $BC_F^K(Sym^2(\Pi))$ is cuspidal, $\Pi(G)$ is stable and its structure is easy to describe. If $\pi(G) \in \Pi(G)$, then $\pi(G) = \pi_\infty(G) \otimes \pi_f(G)$. The $\pi_f(G)$ is independent of the choice of $\pi(G) \in \Pi(G)$, and the $\pi_\infty(G)$ is any one of 3^g representations of $G(F \otimes \mathbf{R})$ which arise as external tensor products of the elements of the local L-packets $\Pi(G)_v$.

3.5. Now let G_1 be the inner form of G which is

(i) isomorphic to G at all finite places,
(ii) isomorphic to $U(2,1)$, e.g. quasi-split, at the archimedean place v_1 of F defined by the given embedding of F into $\mathbf{R}$,
(iii) isomorphic to $U(3)$, e.g. compact, at the other archimedean places of F.

Then G_1 is an anisotropic group, also defined by a Hermitian form H_1; let G_1^* be the associated group of rational similitudes. In [R2], Rogawski proved a Jacquet-Langlands type of correspondence between L-packets on G and G_1. Since $\Pi(G)$ is stable cuspidal and $\Pi(G)_v$ is discrete series at each infinite place, there is a unique L-packet, all of whose members are automorphic, $\Pi(G_1)$ on G_1 such that

(i) $\Pi(G_1)_f = \Pi(G)_f$,
(ii) $\Pi(G_1)_{\infty_1} = \Pi(G)_\infty$,
(iii) $\Pi(G_1)_v$ consists solely of the trivial representation for all archimedean v other than ∞_1.

Further, (i) the central character of each member of $\Pi(G_1)$ is trivial and (ii) the multiplicity of any $\pi(G_1)$ in the discrete automorphic spectrum of G_1 is one. (These results are not stated explicitly in Chapter 14 of [R2] but they follow easily from Thm. 14.6.1 (comparison of traces) and Thm. 13.3.3, and in any case are well-known.)

3.6. Each automorphic representation $\pi(G_1)$ in $\Pi(G_1)$ extends uniquely to an automorphic representation $\pi(G_1^*)$ of G_1^* with trivial central character. Thus, we obtain on G_1^* a set $\Pi(G_1^*)$ of 3 automorphic representations with isomorphic finite parts $\pi(G_1^*)_f$ and whose infinite parts are identified, via projection of $G_1^*(\mathbf{R})$ onto the factor corresponding to F_{∞_1} as the members of $\Pi(G)_\infty = \{\pi^+, \pi^-, \pi^0\}$. For more details see [BR1, R1].

4 Picard modular surfaces.

4.1. The group G_1^* defines a compact Shimura variety Sh whose field of definition is easily seen to be K. Let U be an open compact subgroup of $(G_1^*)_f$. Then Sh is the projective limit over such U of projective varieties Sh_U, each of which is defined over K and consists of a finite disjoint union of projective algebraic surfaces. (It is customary to refer to any of the Sh_K as a Picard modular surface. See [D1, LR] for background.)

Let $U(1)$ be an open compact which is a product of hyperspecial maximal compact subgroups at each finite place, and let U be a normal subgroup of $U(1)$, sufficiently small so that Sh_U is non-singular. For a $(G_1^*)_f$ module V, let $V^{U(1)}$ denote the subspace of $U(1)$ invariants. It is a module for the Hecke algebra $\mathbf{H}_{U(1)}$ of compactly supported bi-$U(1)$ invariant functions on $(G_1^*)_f$. Note that $V^{U(1)}$ is a module for $\mathbf{H}_{U(1)}$.

The degree 2 cohomology $H_B^2(Sh, \overline{\mathbf{Q}})$ of Sh decomposes as a direct sum of isotypic $\pi(G_1^*)_f$ modules. Now let $V = H_B^2(Sh, \overline{\mathbf{Q}})(\pi(G_1^*)_f)$ denote the $\pi(G_1^*)_f$-isotypic component. Then, as usual, $V^{U(1)}$ is identified, using the Matsushima formula, with a 3 dimensional subspace of $H_B^2(Sh_U, \overline{\mathbf{Q}})$; it is an isotypic component for a representation $(\pi(G_1^*)_f)^{U(1)}$ of $\mathbf{H}_{U(1)}$.

4.2. **Q**-structure. Note that V has a natural $\overline{\mathbf{Q}}$ structure coming from $H_B^2(Sh, \overline{\mathbf{Q}})$.

Lemma. The subspace $V^{U(1)}$ of V is defined over $\mathbf{Q}$.

Proof. Let τ be an automorphism of $\mathbf{C}$ and let $V^{U(1)\tau}$ be the conjugate of $V^{U(1)}$ inside $H_B^2(Sh, \overline{\mathbf{Q}}) = H_B^2(Sh, \mathbf{Q}) \otimes \overline{\mathbf{Q}}$. Then $V^{U(1)\tau}$ is the $((\pi(G_1^*)_f)^\tau)^{U(1)}$ isotypic subspace of $H_B^2(Sh, \overline{\mathbf{Q}})$, so it is enough to show that

$$((\pi(G_1^*)_f)^\tau) = (\pi(G_1^*)_f.$$

Since these representations are unramified, and since, by the discussion of [R2, 12.2], unramified local L-packets for G_1 consist of single elements, it is enough to check, for each finite place v, the equality of Langlands classes

$$\sigma_v((\pi(G_1^*)_f)^\tau) = \sigma_v((\pi(G_1^*)_f).$$

However, it is not hard to check that

$$\sigma_v((\pi(G_1^*)_f)^\tau) = \sigma_v((\pi(G_1^*)_v)^\tau.$$

By the discussion of [R1, 4.1-2], these classes are in Galois equivariant bijection with associated L-factors. In our case, as is easily seen, the factor is given by

$$L_w(BC_F^K(Sym^2(\pi)), s),$$

where w is any extension of v. Since π has rational Hecke eigenvalues $L(BC_F^K(Sym^2(\Pi)), s)$ is an Euler product over reciprocals of Dirichlet polynomials with rational coefficients. Hence

$$(\sigma_v((\pi(G_1^*)_v)^\tau = (\sigma_v((\pi(G_1^*)_v$$

and the result follows.

4.2.1. The case $T_\pi \neq \mathbf{Q}$. Here, instead of the above Lemma, one shows that the smallest subspace W_B of $H_B^2(Sh, \overline{\mathbf{Q}})$ whose scalar extension to *overline*$\mathbf{Q}$ contains $V^{U(1)}$ is a 3-dimensional T_π vector space, where T_π is identified with the quotient of $\mathbf{H}_{U(1)}$ acting on W_B. The argument requires nothing new.

4.3 **Hodge structure**. Let M_B denote the $\mathbf{Q}$ vector space such that

$$M_B \otimes \overline{\mathbf{Q}} = V^{U(1)}$$

By the stability of the L-packet, each tensor product $\sigma^* \otimes \pi_f$ with $\sigma^* \in \Pi(G)_{\infty 1}$ is automorphic. The bigraded Matsushima formula ([BW])shows that the Hodge decomposition of M_B has the form

$$M_B \otimes \mathbf{C} = M^{(2,0)} \bigoplus M^{(1,1)} \bigoplus M^{(0,2)}$$

where the factors, each of dimension 1, are the contributions of σ^+, σ^0, and σ^- to the cohomology, respectively.

4.4 **Motive**. For each rational prime l, $M_l = M_B \otimes \mathbf{Q}_l$ is identified with a summand of the l-adic etale cohomology $H_B^2(Sh \times \overline{\mathbf{Q}}, \mathbf{Q}_l)$. Since the action of the Hecke algebra on $H_B^2(Sh, \mathbf{Q})$ is semisimple, there exists an element e which acts as an idempotent on $H_B^2(Sh, \mathbf{Q})$ and whose image is M_B. Interpreting, as usual, the action of Hecke operators via algebraic correspondences (c.f. [DM, BR2]), we see that the pair

$$M = (H_B^2(Sh, \mathbf{Q}), e)$$

defines a Grothendieck motive with associated ∞-tuple of realizations

$$M_r = (M_B, M_{DR}; M_l, l\, prime).$$

Here M_B is regarded as a rational Hodge structure, M_{DR}, a graded K-vector space, is the De Rham cohomology of M, and each M_l is a $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$-module. Each module is the image of the action of e in the corresponding cohomology theory of Sh. Since the classes of algebraic cycles are absolute Hodge cycles, every Grothendieck motive is also a motive for absolute Hodge cycles. Henceforth, by abuse of language, we regard M_r as such motive, and all motives will be motives for absolute Hodge cycles.

4.5 Polarization. There is ([DM] a well-known Tate twist operation which for any motive M and any $n \in \mathbf{Z}$ defines a new motive $M(n)$; we recall the properties of this operation only as needed. Further, any motive is polarisable. In our case, this means that there is a non-degenerate symmetric morphism of Hodge structures

$$\Psi_M : M_B(1) \otimes M_B(1) \to \mathbf{Q}$$

whose associated bilinear form has signature $(1, 2)$. Here $\mathbf{Q}$ has the "trivial" Hodge structure of type $(0, 0)$, and the Hodge decomposition of $M_B(1)$ coincides with that of M_B, but with each pair (p, q) of the bigrading replaced by $(p - 1, q - 1)$. The form induces, for each l, a $Gal(\overline{\mathbf{Q}}/K)$ -equivariant map

$$\Psi_l : M_l(1) \otimes M_l(1) \to \mathbf{Q}_l$$

where $\mathbf{Q}_l$ is the trivial Galois module. A similar remark applies to M_{DR}, but we will have no need of it. Likewise, an explicit realisation of the polarisation is given by the restriction to M of the cup-product on $H^2(Sh)$, but we don't need this fact.

4.5.2. If $T_\pi \neq \mathbf{Q}$, then the above construction provides us with a motive M of rank 3 over T_π such that T_π acts on each component of the Hodge decomposition by regular representation. The polarization Ψ_M has the form $\Psi_M = Tr_{K/F}(\Psi_0)$ with a T_π-linear symmetric bilinear form Ψ_0 taking values in $T_\pi \otimes$. The $T_\pi \otimes \mathbf{R}$-valued form $\Psi_M \otimes \mathbf{R}$ decomposes as a sum, indexed by the embeddings of T_π into $\mathbf{R}$ of forms of signature $(1, 2)$. In particular, the orthogonal group of this form is quasi-split at each infinite place.

4.6. **Construction of an elliptic curve over C**. For a Hodge structure H, let $Sym^2(H)$ denote its symmetric square.

Proposition. There exists, up to isomorphism, a unique two dimensional rational Hodge structure H_B having Hodge types $(1, 0)$ and $(0, 1)$ such that $Sym^2(H_B)$ is isomorphic to M_B.

4.7. **Proof.**

4.7.1 We follow [D2], Sections 3 and 4. Let $C^+ = C^+(M_B(1))$ be the even Clifford algebra of the quadratic module $M_B(1)$. A Hodge structure on a rational vector space X is given by a morphism of real algebraic groups $h : \mathbf{S} \to Aut(X \otimes \mathbf{R})$ where $\mathbf{S} = R_{\mathbf{C}/\mathbf{R}} G_m$ is the Weil restriction of scalars from $\mathbf{C}$ to $\mathbf{R}$ of the multiplicative group. Let $h = h_{M_B(1)}$ denote the morphism so defined for M. There is a canonical morphism with kernel G_m of the algebraic group $(C^+)^*$ into the group $SO(M_B(1), \Psi)$. Of course, $im(h)$ lies in $SO(M_B(1), \Psi)(\mathbf{R})$. The morphism h lifts uniquely to a morphism $h^+ : \mathbf{S} \to (C^+)^* \otimes \mathbf{R}$ such that the associated Hodge structure on C^+ has types $(1, 0)$ and $(0, 1)$ only. Note that in our case C^+ is a quaternion algebra with center $\mathbf{Q}$.

4.7.2. We must show that C^+ is a split algebra. If so, then denoting by W an irreducible 2 dimensional module for C^+, we know by [D2], 3.4 that $End(W)/center$ is isomorphic to $\Lambda^2(M_B(1))$, which is itself isomorphic to $M_B(1)$. (Note that the center of $End(W)$ is a rational sub-Hodge structure of type $(0, 0)$.) Hence $Sym^2(W) = (W \otimes W)/\Lambda^2(W)$ is isomorphic to M_B.)

4.7.3. To show that C^+ is split, we use the surface Sh and R. Taylor's l-adic representations ρ_l^T of $Gal(\overline{\mathbf{Q}}/F)$.

4.7.4. **Lemma**. The representations ρ_l^T and $Sym^2(\rho_l^T)$ are irreducible and remain irreducible when restricted to any finite subgroup of $Gal(\overline{\mathbf{Q}}/F)$.
Proof. Evidently, if the result holds for $Sym^2(\rho_l^T)$, then it holds for ρ_l^T. The former case will follow from [BR1], Theorem 2.2.1(b), once we exclude the case (iib) of that Theorem, i.e. that $Sym^2(\rho_l^T)$ is potentially abelian. To see this, observe that, since $Sym^2(\rho_l^T)$ occurs as the Galois action on M_l for a motive M whose M_B has 3 Hodge types (2,0), (1,1), and (0,2), the Hodge-Tate theory shows that the semisimple part of the Zariski-closure of the image of Galois, over any finite extension L of K, contains a non-trivial torus over $\mathbf{C}_l$. Considering now ρ_l^T, this means that if ρ_l^T is potentially abelian, the connected component of the Zariski closure over $\mathbf{C}_l$ is a non-central torus S of $GL(2)$. Hence the image of ρ_l^T must lie in the normalizer N of S inside $GL(2)$. Such an N is either abelian or has an abelian subgroup of index 2, and so the image of Galois is either abelian or is non-abelian but has an abelian subgroup of index 2. The first case contradicts ([BR1, Prop.2.3.1], [T2]), whereas the second means that π is of CM type. QED

For each l, let $\eta_l = Sym^2(\rho_l^T|_K)(1))$. Each η_l acts on $\mathbf{Q}_l^3$ which is isomorphic to the module $M_l(1)$ as Galois module. Let G_l denote the Zariski closure of the image of η_l in$End(\mathbf{Q}_l^3)$.

4.7.5. **Lemma**. G_l is a quasi-split special orthogonal group.

Proof. The image of $Sym^2 \otimes (det^{-1}) : GL(2) \to GL(3)$ is a quasi-split special orthogonal group $SO(3)_{qs}$. Hence we need only check that the image of η_l, automatically contained in this group, has Zariski closure equal to it. But $(\eta_l)|_L$ is irreducible for all finite extensions L of K. Hence G_l is a connected irreducible algebraic subgroup of $SO(3)_{qs}$. Fortunately, the only such subgroup is $SO(3)_{qs}$ itself.

4.7.6. **Completion of proof that C^+ is split**. Now we know that the Zariski closure of the Galois action on each $M_l(1)$ is the quasi-split orthogonal group. On the other hand, this irreducible action preserves the form Ψ_l. Since an irreducible orthogonal representation can preserve at most one quadratic form (up to homothety), this shows that the special orthogonal group of Ψ_l is $SO(3)_{qs}$ for all l. Hence for all primes l, the algebra $C^+ \otimes \mathbf{Q}_l$ is split. Hence it must be split at infinity as well, and so is a matrix algebra. QED.

4.7.7. If $T_\pi \neq \mathbf{Q}$, we must construct a rational Hodge structure on the underlying rational vector space of a 2 dimensional T_π vector space H_B so that (i) only Hodge types $(1,0)$ and $(0,1)$ occur, and (ii) T_π acts via endomorphisms of the Hodge structure. The argument of 4.7.1 provides such a T_π-linear Hodge structure h^+ on C^+, which is now quaternion algebra over T_π. Similarly, the argument through 4.7.5, using the T_π -linearSym^2, extends without difficulty to show that C^+ is split at all finite places. Finally, since $SO(\Psi_M)$ is quasi-split at each infinite place, so is $(C^+)^*$. So $(C^+)^*$ is everywhere locally,and hence globally, split.

4.8. The Hodge structure H_B defines a unique isogeny class of elliptic curves over $\mathbf{C}$ such that for any member A of this class, $H_B^1(A)$ is isomorphic as Hodge structure to H_B. We now choose and fix one such A.

4.8.1. If $T_\pi \neq \mathbf{Q}$, H_B defines an isogeny class of abelian varieties over $\mathbf{C}$, such that for any member $A = A_\pi$ of this class, $H_B^1(A)$ is isomorphic as Hodge structure to H_B. Indeed, the T_π action renders H_B automatically polarizable, which is all that needed to be checked.

5 Descent of A to $\overline{\mathbf{Q}}$.

5.1. Let τ be an automorphism of $\mathbf{C}$ over $\overline{\mathbf{Q}}$. Let $\tau(A)$ be the conjugate of A by τ.

Theorem 4 *Suppose the Absolute Hodge Conjecture holds. Then $\tau(A)$ is isogenous to A.*

5.2. **Proposition**. Let E_1 and E_2 be elliptic curves over $\mathbf{C}$ and suppose that the Hodge structures $Sym^2(H^1_B(E_1))$ and $Sym^2(H^1_B(E_2))$ are isomorphic. Then E_1 is isogenous to E_2.

5.3. **Proof**. E_1 is isogenous to E_2 iff $H^1_B(E_1)$ is isomorphic to $H^1_B(E_2)$ as rational Hodge structure. As in 4.7.1, let h_1 and h_2 be the Hodge structure morphisms defined for $H^1_B(E_1)$ and $H^1_B(E_2)$; these actions are each separately equivalent over $\mathbf{R}$ to the tautological action of $\mathbf{C}^* = \mathbf{S}(R)$ on $\mathbf{C}$, for a suitable choice of isomorphism of $\mathbf{C}$ with $\mathbf{R}^2$. Let $V_j = H^1_B(E_j)$ ($j \in \{1,2\}$),and let

$$h = (h_1, h_2) : \mathbf{S} \to GL(V_1, \mathbf{R}) \times GL(V_2, \mathbf{R})$$

be the product morphism. Let H be the smallest algebraic subgroup of $GL(V_1, \mathbf{Q}) \times GL(V_2, \mathbf{Q})$ which contains $im(h)$ over $\mathbf{R}$. Then H is a reductive algebraic group; it is the Mumford-Tate group of $E_1 \times E_2$ and the isomorphism classes of its algebraic representations over $\mathbf{Q}$ are in natural bijection with the isomorphism classes of rational Hodge structures contained in all tensor powers of all sums of V_1, V_2, and their duals. The projection of H to a factor is the Mumford-Tate group of the corresponding curve.

Put $W = (V_1 \otimes V_2) \otimes (V_1 \otimes V_2)$. Then of course $W \cong (V_1 \otimes V_1) \otimes (V_2 \otimes V_2)$. Since the action of H on $V_j \otimes V_j$ factors through the projection onto $GL(V_j)$, we see that the action of H on $V_j \otimes V_j$ decomposes as a sum of the 1-dimensional representation $det \circ pr_j$ and the 3 dimensional representation $Sym^2(V_j) \circ pr_j$. Further, the representations $det(pr_1)$ and $det(pr_2)$ are isomorphic since the corresponding Hodge structures are 1-dimensional, of type (1,1), and there is up to isomorphism only 1 such Hodge structure. Since,by assumption, $Sym^2(V_1)$ is isomorphic to $Sym^2(V_2)$, and $Sym^2(V_1)^* \cong Sym^2(V_1) \otimes (det(pr_1)^{-2}$, we see that $Sym^2(V_1) \otimes Sym^2(V_2)$ decomposes as the sum of a 1-dimensional $(det(pr_1))^2$, and an 8-dimensional representation. Hence W contains at least two copies of the representation $(det(pr_1))^2$. Now, V_1 and V_2 are non-isomorphic as Hodge structures iff they are non-isomorphic as H-modules iff $V_1 \otimes V_2$ contains no 1-dimensional summand isomorphic to $det(pr_1)$. This follows since $(V_1)^* \cong V_1 \otimes (det(pr_1)^{-1})$, and the morphisms of Hodge structure are exactly the vectors in $V_1^* \otimes V_2$ on which H acts via the trivial representation. Suppose now $V_1 \otimes V_2 \cong A \oplus B$ with A and B two-dimensional sub-Hodge structures. Evidently, one of the factors is purely of type $(1,1)$, and so the associated H representation is isomorphic to $det(pr_1) \oplus det(pr_1)$. Hence V_1 is isomorphic to V_2. (Of course, in this case we know more: both have complex multiplication, since if α and β are two non-homothetic elements of $Hom(E_1, E_2)$, $(\hat{\beta}) \circ \alpha$ is a non-scalar endomorphism of E_1.) Thus, we must only exclude the possibility that $V_1 \otimes V_2$ is an irreducible Hodge structure, i.e. corresponds to an irreducible representation of H. But in this case $V_1 \otimes V_2 \otimes V_1 \otimes V_2$ contains exactly one copy of $(det(pr_1))^2$, by Schur's Lemma. Since we have seen above that the multiplicity of $det(pr_1)$ is at least 2, this case cannot occur. QED.

5.4. **Remark**: This Proposition is easily adapted to the general case, provided one works always in the category of Hodge structures which are T_π-modules, working T_π-linearly.

5.5. **Proof of the Theorem**. By construction, the Hodge structure $Sym^2(H^1(A))$ is isomorphic to that of $M_B(\pi_f)$. Hence, by the Absolute Hodge Conjecture, there is an isomorphism in the category of motives over $\mathbf{C}$:

$$\phi : Sym^2(H^1(A)) \to M_B(\pi_f) \otimes \mathbf{C}.$$

Let τ be an element of $Aut(\mathbf{C}/\overline{\mathbf{Q}})$. Conjugating by τ, we get an isomorphism $\phi^\tau : (Sym^2(H^1(A)))^\tau \to M_B(\pi_f)^\tau$. Since $M_B(\pi_f)$ is defined over $\overline{\mathbf{Q}}$, $M_B(\pi_f)^\tau = M_B(\pi_f)$. Further, $(Sym^2(H^1(A)))^\tau \cong Sym^2(H^1(A^\tau))$. Hence $Sym^2(H^1(A^\tau)) \cong Sym^2(H^1(A))$. In particular, the Hodge structures $Sym^2(H^1_B(A^\tau))$ and $Sym^2(H^1_B(A))$ are isomorphic. By the Lemma, A^τ is isogenous to A.

5.6. **Corollary**. Suppose the Absolute Hodge Conjecture holds. Then A admits a model over a finite extension L of $\mathbf{Q}$.

Proof (of the Corollary). The complex isogeny class of A contains only countably many complex isomorphism classes of elliptic curves. For an elliptic curve B let $j(B)$ be its j-invariant. Then B_1 is ismorphic over $\mathbf{C}$ to B_2 iff $j(B_1) = j(B_2)$. Furthermore, $j(\tau(B)) = \tau(j(B))$ for all B. Hence, considering all automorphisms τ of $\mathbf{C}$ over $\overline{\mathbf{Q}}$, the set $\{\tau(j(A))\}$ is countable.

Let on the other hand z be any non-algebraic complex number. Then it is well-known that the set $\{\tau(z)\}$ is uncountable. Hence, $j(A)$ must be in $\overline{\mathbf{Q}}$, i.e. $\{\tau(j(A))\}$ is finite. Let $L = Q(j(A))$. Since any elliptic curve B admits a Weierstrass model over $Q(j(B))$, we are done.

5.7. **Remark**: In the general case $T_\pi \neq \mathbf{Q}$, we can, changing A within its isogeny class as needed, give A a principal polarisation P_A . Then the set of all j-invariants is replaced by the quasi-projective variety M_d, defined over $\mathbf{Q}$, which parametrises all isomorphism classes $\mathcal{B}$ of pairs (B, P_B) where B has dimension d, and P_B is a principal polarisation of B. Then $\mathcal{A} = (\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ defines a point $\nu(\mathcal{A})$ of M_d. The variety M_d is defined over $\mathbf{Q}$. Further, if τ is an automorphism of $\mathbf{C}$ over $\overline{\mathbf{Q}}$, $\tau(\mathcal{A}) = (\tau(\mathcal{A}), \tau(\mathcal{P}_{\mathcal{A}}))$ is also a principally polarised abelian variety and $\tau(\nu(\mathcal{A})) = \nu(\tau(\mathcal{A}))$. Again, the set of all isomorphism classes of pairs $\mathcal{B} = (\mathcal{B}, \mathcal{P}_{\mathcal{B}})$ where B is isogenous to A is countable. Since $\tau(A)$ *is* isogenous to A, this means that the set of all $\tau(\nu(\mathcal{A}))$ is countable. Hence $\nu(\mathcal{A})$ has algebraic coordinates. (This is standard. Choosing a suitable hyperplane H, defined over $\mathbf{Q}$ which does not contain $\nu((\mathcal{A}))$, the countable set $\tau(\nu(\mathcal{A}))$ is contained in an affine variety defined over $\mathbf{Q}$. But if V is an affine variety defined over $\mathbf{Q}$, and v is a point of V with complex coordinates such that $\tau(v)$ is countable, then the coordinates of v are all algebraic.) Now let L_0 be the number field generated by the coordinates of $\nu(\mathcal{A})$; it is called the field of moduli of $\mathcal{A}$. But it is known that every polarized smooth projective variety admits a model over a finite algebraic extension of its field of moduli. So (A, P_A), and hence A alone, is definable over a finite extension L of L_0. This completes our sketch of the general case.

6 Comparison of $H^1_l(A)$ and $\rho^T_l(\pi)$

6.1. Fix a prime l. The two dimensional $\mathbf{Q}_l$-vector space $H^1_l(A)$ is a $Gal(\overline{\mathbf{Q}}/L)$-module. Recall that $V^T_l(\pi)$ denotes the 2-dimensional $Gal(\overline{\mathbf{Q}}/F)$-module attached to π ([T1, BR2]); the Galois action has been denoted $\rho^T_l(\pi_f)$.

6.2. **Proposition**. Suppose the Absolute Hodge Conjecture holds. Then there exists a finite extension L_1 of L, containing K, such that $V^T_l(\pi)|_{L_1}$ is isomorphic as $Gal(\overline{\mathbf{Q}}/L_1)$-module to $H^1_l(A)|_{L_1}$.

6.3. **Proof**. The motive $Sym^2(H^1(A))$ is defined over L, and by construction, there is an isomorphism ι_B of Hodge structures between M_B and $Sym^2(H^1(A))_B$. Regarding ι_B as an element of $M_B^* \otimes Sym^2(H^1(A))_B$, it is a rational class of type $(0,0)$. By Deligne's theorem, $Gal(\overline{\mathbf{Q}}/L)$ acts continuously, via a finite quotient group, on the $\mathbf{Q}$-vector subspace of all rational classes of type $(0,0)$. Let L_1 be a finite extension of L, containing K, such that $Gal(\overline{\mathbf{Q}}/L)$ acts trivially on ι. Now, for a rational prime l, ι defines also an isomorphism

$$\iota_l : (M \otimes_K L_1)_l \to (Sym^2(H^1(A)) \otimes_L L_1)_l$$

which is $Gal(\overline{\mathbf{Q}}/L_1)$-equivariant.

6.4. Since the restriction to K of

$$Sym^2(\rho^T(\pi)_l) : Gal(\overline{\mathbf{Q}}/F) \to Aut(V_l^T)$$

is isomorphic to M_l, we now know that, over L_1, $Sym^2(\rho^T(\pi)_l)|L_1$ is isomorphic as a Galois module to $Sym^2(H^1(A))_l$. Since the dual of $Sym^2(\rho^T(\pi)_l)|L_1$ is isomorphic to $(Sym^2(\rho^T(\pi)_l)|L_1)(-2)$, this means that

$$(Sym^2(\rho^T(\pi)_l)|L_1) \otimes (Sym^2(H^1(A))_l)$$

contains χ_l^{-2}, where χ_l is the l-adic cyclotomic character. Let $\mathbf{Q}_l(-1)$ be $\mathbf{Q}_l$ with Galois action given by χ_l.

6.5. We now proceed as in the conclusion of the proof of the preceding proposition. Let $V_1 = H_l^1(A)$ and let $V_2 = V_l^T|L_1$. Then

$$V_1 \otimes V_1 \otimes V_2 \otimes V_2$$

is isomorphic to

$$(Sym^2(V_1) \oplus \mathbf{Q}_l(-1)) \otimes (Sym^2(V_2) \oplus \mathbf{Q}_l(-1))$$

which, from the above, contains the square of the inverse of the l-adic cyclotomic character χ_l^{-2} with multiplicity two. Hence, putting $W = V_1 \otimes V_2$, we see that $W \otimes W$ contains χ^{-2} with multiplicity two. If W is absolutely (i.e. $\overline{\mathbf{Q}}_l$) irreducible as a Galois module, then $W \otimes W$ contains χ_l^{-2} with multiplicity one. So W is reducible. If W decomposes as $X \oplus Y$, with X and Y irreducible and two dimensional, then consider the exterior square

$$\Lambda = \Lambda^2(X \oplus Y).$$

This decomposes as

$$det(X) \oplus X \otimes Y \oplus det(Y).$$

On the other hand

$$\Lambda = \Lambda^2(V_1 \otimes V_2)$$

which decomposes as

$$\Lambda = (Sym^2(V_1) \otimes det(V_2)) \oplus (det(V_1) \otimes Sym^2(V_2)).$$

The first calculation shows that Λ has at least two 1-dimensional summands, and, since $Sym^2(V_1) \cong Sym^2(V_2)$ is irreducible, the second shows that Λ has no 1-dimensional summands, this case is impossible. Hence $W = U \oplus Z$ with a 1-dimensional summand Z.

6.6. To finish the argument, denote the Galois action on Z by ψ. Then $V_1^* \otimes (V_2 \otimes \psi^{-1})(-1)$ contains the trivial representation. Since V_1 is irreducible, this means that V_1 is isomorphic to $(V_2 \otimes \psi^{-1})(-1)$. Since $det(V_1)$ and $det(V_2)$ both have the Galois action given by χ_l^{-1}, we conclude by taking determinants, that $(\psi\chi_l)^2 = 1$. Set $\mu = \psi\chi_l$. Enlarge L_1 by a finite extension L_2 such that $\mu|L_2$ is trivial. Then over L_2, $V_2 \cong V_1$. Relabeling L_2 as L_1 , we are done.

7 Completion of the Construction

7.1. **Preliminary** We now remove the conditions, in force since Section 3, that π be unramified over F, and $FK_0 = K$ be unramified over F, and we reinterpret the preceding constructions as commencing from the base change of π to the solvable totally real extension F_1 of F, over which $BC_F^{F_1}(\pi)$ is unramified, and such that F_1K_0 is unramified. Further, we henceforth let L (not L_1) denote any number field which is a field of definition of A and which satisfies the conclusion of the previous proposition.

7.2. Finally, let $B = R_{L/F}(A)$. Then the $Gal(\overline{\mathbf{Q}}/F)$-module $H_l^1(B)$ is isomorphic to the induced module $Ind_{L/F}(H_l^1(A))$. However, $H_l^1(A)$ is isomorphic to the restriction $Res_{L/F}(V_l^T)$ of the 2-dimensional $Gal(\overline{\mathbf{Q}}/F)$-module $V_l^T = V_l^T(\pi)$. Hence $H_l^1(B)$ is isomorphic to $V_l^T \otimes \Pi_{L/F}$ where $\Pi_{L/F}$ is the permutation representation defined by the action of $Gal(\overline{\mathbf{Q}}/F)$ on the set of F-linear embeddings of L into $\overline{\mathbf{Q}}$.

7.3. Since the trivial representation occurs in $\Pi_{L/F}$, we see that V_l^T occurs in $H_l^1(B)$. Let τ be any non-trivial irreducible constituent of $\Pi_{L/F} \otimes \overline{\mathbf{Q}}_l$. Then we claim that ρ_l^T is not a constituent of $\rho_l^T \otimes \tau$. To see this, just note that the multiplicity in question is the dimension of $((V_l^T)^* \otimes V_l^T \otimes \tau)^{Gal}$ where $Gal = Gal(\overline{\mathbf{Q}}/F)$. Since $(V_l^T)^* \otimes V_l^T = 1 \oplus Ad(V_l^T)$, and $1 \otimes \tau$ is irreducible, it is enough to check that $Ad(V_l^T) \otimes \tau$ contains no Galois invariants. But if J denotes the kernel of τ, then the restriction of $Ad(V_l^T) \otimes \tau$ to J is isomorphic to the direct sum of 3 copies of (the restriction to J of) $Ad(V_l^T)$. Since we have seen that $Ad(V_l^T) = Sym^2(V_l^T) \otimes \omega_\pi^{-1}$, and $Sym^2(V_l^T)$ remains irreducible on restriction to J, there are no invariants. Hence the multiplicity of V_l^T in $H_l^1(B)$ is 1.

7.4. Let D be the smallest abelian subvariety of B such that $H_l^1(D)$ contains the unique submodule of $H_l^1(B)$ which is isomorphic to V_l^T. We will show that D is an elliptic curve. Evidently D is simple and so $End(D)$ is a division algebra. In fact, $End(D)$ is a field. To see this let Z be the center of $End(D)$ and let $dim_Z(End(D)) = n^2$. Then over $Z \otimes \overline{\mathbf{Q}}_l$, $H_l^1(D) \otimes \overline{\mathbf{Q}}_l$ is a free module over the the matrix algebra $M_n(Z \otimes \overline{\mathbf{Q}}_l)$. Hence each irreducible Galois submodule of $H_l^1(D) \otimes \overline{\mathbf{Q}}_l$ must occur at least n times. Since V_l^T occurs once, this means $n = 1$, i.e. $End(D) = Z$. Evidently, $H_l^1(D)$ is a free rank 2 $Z \otimes \mathbf{Q}_l$-module.

7.5. Now we must show that $Z = \mathbf{Q}$. To see this, note that $H_l^1(D)$ is a free $Z \otimes \mathbf{Q}_l$-module. Put $Z \otimes \mathbf{Q}_l = Z_1 \oplus ... \oplus Z_t$, with local fields Z_j. Let e_j denote the idempotent of $Z \otimes \mathbf{Q}_l$ which has image the factor Z_j. Choose the indexing so that the Z_1 module $e_1(H_l^1(D))$ contains the $\mathbf{Q}_l$-submodule W isomorphic to V_l^T . Since the Galois action on V_l^T is irreducible, there is an embedding of $V_l^T \otimes Z_1$ into $e_1(H_l^1(D))$. But if $[Z_1 : \mathbf{Q}_l] > 1$, the commutant of this image would be non-abelian. Since Z is a field, this means $Z_1 = \mathbf{Q}_l$. Since l is arbitrary, the Cebotarev theorem ([CF], Exercise 6.2) forces $Z = \mathbf{Q}$. Thus the commutant of the image of Galois in $End(H^1(D)) \otimes \overline{\mathbf{Q}_l}$ is $\overline{\mathbf{Q}_l}$. This means that $H_l^1(D)$ is isomorphic to V_l^T, and so D is in fact the sought elliptic curve. This completes the construction.

7.6. If $T_\pi \neq \mathbf{Q}$, the arguments of Sections 6 and 7 proceed essentially unchanged, albeit T_π-linearly, using the free rank 2 $T_\pi \otimes \mathbf{Q}_l$-adic representations V_l^T of Taylor.

References

[AC] Arthur, J., Clozel, L., *Simple algebras, base change, and the advanced theory of the trace formula.* Annals of Mathematics Studies,

120. Princeton University Press, Princeton, NJ, 1989.

[B] Blasius, D., *Ramanujan Conjecture for Hilbert modular forms.* In preparation.

[BR1] Blasius, D.,Rogawski, J., *Tate classes and arithmetic quotients of the two-ball.* The zeta functions of Picard modular surfaces, 421–444, Univ. Montréal, Montréal, QC, 1992.

[BR2] Blasius, D., Rogawski, J., *Motives for Hilbert modular forms.* Invent. Math. 114 (1993), no. 1, 55–87.

[BW] Borel, A., Wallach, N., *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Lie Groups*, Annals of Mathematics Studies 94, Princeton University Press, Princeton, N.J., 1980.

[CF] Cassels, J.W.S., Fröhlich, A., Algebraic Number Theory, Thompson Book Company, Washington, D.C., 1967.

[D1] Deligne, P., *Travaux de Shimura.* Séminaire Bourbaki 1970-71, Exposé 389, Lecture Notes in Mathematics, 244. Springer-Verlag, Berlin-New York, 1971.

[D2] Deligne, P., *La conjecture de Weil pour les surfaces K3.* Invent. Math. 15 (1972), 206–226.

[D3] Deligne, P., *Théorie de Hodge II* Publ. Math. IHES 40, (1972), 5-57.

[D4] Deligne, P.,*Hodge cycles on abelian varieties* Hodge cycles, motives, and Shimura varieties. Lecture Notes in Mathematics, 900. Springer-Verlag, Berlin-New York, 1982.

[DMOS] Deligne, P., Milne, J., Ogus, A., Shih, K., *Hodge cycles, motives, and Shimura varieties.* Lecture Notes in Mathematics, 900. Springer-Verlag, Berlin-New York, 1982.

[DM] Deligne, P., Milne, J.S. *Tannakian Categories.* Hodge cycles, motives, and Shimura varieties. Lecture Notes in Mathematics, 900. Springer-Verlag, Berlin-New York, 1982.

[E] Eichler, Martin *Quaternäre quadratische Formen und die Riemannsche Vermütung für die Kongruenzzetafunktion.* Arch. Math. 5, (1954), 355–366.

[F] Faltings, G. *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern.* Invent. Math. 73 (1983), no. 3, 349–366.

[GJ] Gelbart, S., Jacquet, H., *A relation between automorphic representations of* GL(2) *and* GL(3). Ann. Sci. École Norm.Sup. (4) 11 (1978), no. 4, 471–542.

[H] Hida, H., *On abelian varieties with complex multiplication as factors of the Jacobians of Shimura curves.* Amer. J. Math. 103 (1981), no. 4, 726–776.

[JL] Jacquet, H., Langlands, R. P., *Automorphic forms on* GL(2). Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[L] Langlands, R.P., *Base change for* GL(2). Annals of Mathematics Studies, 96. Princeton University Press, Princeton, N.J.,1980.

[LR] Langlands, R.P., Ramakrishnan, D., *The zeta functions of Picard modular surfaces.* Université de Montréal, Centre de Recherches Mathématiques, Montréal, QC, 1992.

[MR] Murty, R., Ramakrishnan, D., *The Albanese of unitary Shimura varieties*The zeta functions of Picard modular surfaces. Université de Montréal, Centre de Recherches Mathématiques, Montréal, QC, 1992.

[O] Oda, T.,*Periods of Hilbert modular surfaces.* Progress in Mathematics, 19. Birkhä user, Boston, Mass., 1982.

[R1] Rogawski, J., *Analytic expression for the number of points mod p.* The zeta functions of Picard modular surfaces, 65–109, Univ. Montréal, Montréal, QC, 1992.

[R2] Rogawski, J.,*Automorphic representations of unitary groups in three variables.* Annals of Mathematics Studies, 123. Princeton University Press, Princeton, NJ, 1990.

[S1] Shimura, G., *On the zeta-function of an abelian variety with complex multiplication.* Ann. of Math. (2) 94 (1971), 504–533.

[S2] Shimura, G., *Introduction to the arithmetic theory of automorphic functions.* Kanô Memorial Lectures, No. 1. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971.

[S3] Shimura, Goro On elliptic curves with complex multiplication as factors of the Jacobians of modular function fields. Nagoya Math. J. 43 (1971), 199–208.

[S4] Shimura, G., *On the factors of the jacobian variety of a modular function field.* J. Math. Soc. Japan 25 (1973), 523–544.

[T1] Taylor, R., *On Galois representations associated to Hilbert modular forms.* Invent. Math. 98 (1989), no. 2, 265–280.

[T2] Taylor, R., *On Galois representations associated to Hilbert modular forms. II.* Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), 185–191, Ser. Number Theory, I, Internat. Press, Cambridge, MA, 1995.