Dynamics of mean-field spin glasses at subexponential time scales

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Contents

Mean-field spin glasses: Static

- Configuration space $\Sigma_n = \{-1, +1\}^n$.
- (Random) Hamiltonian $H_n = \{H_n(x) : x \in \Sigma_n\}$.

  p spin SK model
  H_n = centered Gaussians with covariance structure

$$\begin{aligned} \mathbb{E} H_n(x) H_n(x') &= n R_n(x, x')^p \;, \quad \text{where } p \geq 2 \text{ and} \\ R_n(x, x') &= \tfrac{1}{n} \sum_{i=1}^n x_i x_i'. \end{aligned}$$

  SK Model = 2 spin SK model.

- Gibbs measure at inverse temperature β:

$$\mu_{\beta,n}(x) = \frac{\tau_n(x)}{Z_n(\beta)} \;, \quad Z_n(\beta) \quad \text{partition function} \;,$$

  $\tau_n(x) = \exp(\beta H_n(x))$ Gibbs weight.

Add Dynamics

Intuitively:

Define continuous time Markov chain X_n on Σ_n with

- X_n is reversible w.r.t Gibbs measure.
- Only local jumps are possible.

Explicit construction...

Random Walk in Random Environment

- Random environment: Gibbs weights $\tau = \{\tau_n(x) : x \in \Sigma_n\}$.
- Jumps: $J_n = \{J_n(k) : \ k \in \mathbb{N}\}$ simple random walk on Σ_n.
  Initial distribution: uniform on Σ_n.
  Transition probabilities: $p_n(x, y) = 1/n$ if $dist(x, y) = 1$.
- Clock process

$$\widetilde{S}_n(m) = \sum_{k=0}^{m-1} \tau_n(J_n(k))\, e_k \ ,$$

  where $\{e_k : k \in \mathbb{N}\} \overset{iid}{\sim} \exp(1)$.

- Process of interest X_n is time change of J_n:

$$X_n(t) = J_n\left(\widetilde{S}_n^{\leftarrow}(t)\right) \ , \quad t > 0,$$

  where $\widetilde{S}_n^{\leftarrow}$ is right-inverse.

Aim

We want to study aging.

Intuitively

- Prepare system at initial time t_0.
- Leave system to itself.
- Wait time t_w. Perform measurement.
- Is it dependent on t_0?
  - If yes, then the system ages.

Formally

Let $C_n(t,s)$ be a time-correlation function of X_n. $C_n(t,s)$ ages if for some diverging sequence t_n

$$\lim_{n\to\infty} C_n(t_n,(1+\theta)t_n) = h(\theta), \quad \forall\theta > 0 \ .$$

Determine limiting distribution of clock process, i.e.
find jump scales $(a_n)_{n\in\mathbb{N}}$ and time scales $(c_n)_{n\in\mathbb{N}}$ s.t.

$$S_n(t) = \frac{1}{c_n} \sum_{k=1}^{\lfloor a_n t \rfloor} \tau_n(J_n(k))\, e_k \quad \Rightarrow \quad \text{non-degenerate limit}$$

Exponential time scales

Ben Arous, Bovier, Černý, 2008

$p \geq 3$: There exists $\xi(p)$ s.t. for all $\alpha \in (0, \min(1, \xi(p)\beta^{-1}))$ and

$$a_n = \sqrt{n} \exp\left(\tfrac{1}{2}\alpha^2 n\right), \quad c_n = \exp\left(\alpha\beta^2 n\right),$$

it holds

$$S_n \Rightarrow V_\alpha \ ,$$

where V_α is stable subordinator with index α.

Convergence holds $\mathcal{P}$-almost surely wrt J_n, in $\mathbb{P}$-law wrt τ

Bovier, Gayrard, 2010

Same result, but convergence holds

- $p > 4$: $\mathbb{P}$-a.s.
- $p \geq 3$: in $\mathbb{P}$-probability.

Ben Arous, Gun, 2011

Let $\alpha_n = n^{-c}$, $c \in \left(0, \frac{1}{2}\right)$ and $K_p = \beta^2 p$. For

$$a_n = \sqrt{2\pi n}\alpha_n^{-1}\beta \exp\left(\tfrac{1}{2}\alpha_n^2 n\beta^2\right) \quad , \quad c_n = \exp\left(\alpha_n n\beta^2\right) \ .$$

it holds

$$S_n^{\alpha_n} \Rightarrow M \ ,$$

where M is extremal process with d.f. $F(x) = e^{-K_p/x}$.

Convergence holds $\mathcal{P}$-a.s., in $\mathbb{P}$-law for $p \geq 4$ and $p = 2$ or $p = 3$ if $c < \frac{1}{4}$.

Bovier, Gayrard, S., 2011

Same result, but convergence holds

- $\mathbb{P}$-a.s. for $p \geq 4$, $p = 3$ if $c > \frac{1}{4}$
- in $\mathbb{P}$-probability for $p = 2$ and $p = 3$ if $c \leq \frac{1}{4}$.

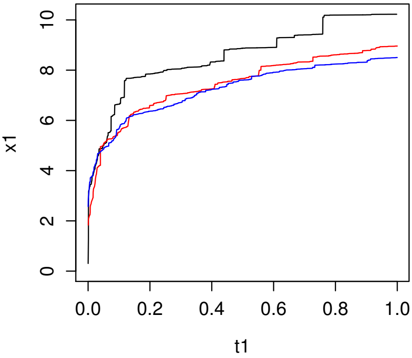
Subordinators and Extremal Processes

Let $\xi = \sum_{k \in \mathbb{N}} \delta_{s_k, x_k}$ Poisson point process with intensity measure $dt \times d\nu$, $\nu(u, \infty) \propto u^{-\alpha}$, $u > 0$, $\alpha \in (0, 1]$.

We can construct two different objects.

Stable subordinator

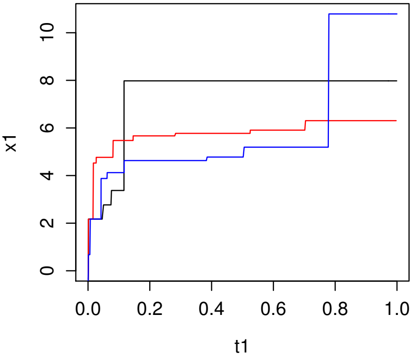
$$V_\alpha(t) = \int_0^t \int_0^\infty x \; d\xi(s, x).$$



Subordinators and Extremal Processes

Extremal process

$$\begin{aligned} M_\alpha(t) &= \sup\{x_k : \ s_k \le t\} \\ &= \sup\{\Delta V_\alpha(s) : s \le t\}. \end{aligned}$$



One-dimensional marginal

$$\mathbb{P}(M_\alpha(t) \le u) = \exp(-tu^{-\alpha}).$$

What happens to V_α if $\alpha \to 0$?

Extremal Processes

Kasahara, 86:

Every non-linear transformation of V_α converges to a degenerate limit as $\alpha \to 0$, i.e.

small α:

$$V_\alpha(t) \approx \max_{s \le t} \Delta V_\alpha(s).$$

Jumps are too big!

$$\mathbb{P}\left(V_\alpha(t) > u\right) \approx 1 - \exp(-tu^{-\alpha}).$$

Solution:

$$(V_\alpha(t))^\alpha \quad \approx \quad \max_{s \le t} (\Delta V_\alpha(s))^\alpha \approx \max_{s \le t} \Delta V_1(s) = M_1(t) \ ,$$

with one dimensional marginal

$$\mathbb{P}(M_1(t) \le u) = \exp(-tu^{-1}).$$

Sketch of proof

(i) Define for $Z_{n,k} = c_n^{-1} \sum_{i=n^2(k-1)+1}^{n^2 k} \tau_n(J_n(i))\, e_i$

$$\xi_n = \sum_{k\in\mathbb{N}} \delta_{\left\{\frac{k}{a_n},(Z_{n,k})^{\alpha_n}\right\}}.$$

Show $\mathbb{P}-$ a.s. / in $\mathbb{P}$ probability that $\xi_n \Longrightarrow \xi$,
where ξ PPP $(dt \times d\nu)$, $\nu(u,\infty) = K_p u^{-1}$.

(ii) Apply mappings

$$T_n: \quad m = \sum_{k\in\mathbb{N}} \delta_{t_k, j_k} \mapsto \left(\sum_{k\in\mathbb{N}} j_k^{1/\alpha_n}\right)^{\alpha_n},$$
$$T: \quad m \mapsto \sup\{j_k : k \in \mathbb{N}\}.$$

and show $T_n\xi_n \Longrightarrow T\xi$, $\mathbb{P}$-a.s./ in $\mathbb{P}$ probability.

Theorem 2 [Bovier, Gayrard, S., 11]
Define for $\varepsilon \in (0,1)$ and $\theta > 0$, the time correlation function by

$$\mathcal{C}_n^{\varepsilon}(\theta) \equiv \mathcal{P}\left(\left\{R_n\left(X_n(c_n), X_n((1+\theta)^{1/\alpha_n} c_n)\right) \geq 1-\varepsilon\right\}\right) .$$

Under the assumptions of Theorem 1,

$$\lim_{n\to\infty} \mathcal{C}_n^{\varepsilon}(\theta) = \frac{1}{1+\theta}, \quad \forall \varepsilon \in (0,1),\ \theta > 0.$$

- $\mathbb{P}$-a.s. for $p \geq 4$, $p = 3$ if $c > \frac{1}{4}$
- in $\mathbb{P}$-probability for $p = 2$, $p = 3$ if $c \leq \frac{1}{4}$.

Summary and Outlook

Summary

- Extended results from [Ben Arous, Gun, 11]
  - in law with respect to the environment,
  - to results that hold almost surely, respectively in probability.
- To this end we use similar methods as [Bovier, Gayrard, 10].

Outlook

- Infinite state space, eg. models on $\mathbb{Z}^d$.
- More complicated dynamics.

Thank you for your attention!

References

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