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Metastability in Stochastic Dynamics:
Random-Field Curie-Weiss-Potts Model

Metastability: A common phenomenon

The paradigm. Related to the dynamics of first order phase transitions

Change parameters quickly across the line of first order phase transition, the system reveals the existence of multiple time scales:

Short time scales.

- Existence of disjoint subsets M_i, viewed as metastable sets/states
- The system appears to be in a quasi-equilibrium within M_i

Larger time scales.

- Rapid transitions between metastable sets occur induced by random fluctuations

The goal. Understanding of quantitative aspects of dynamical phase transitions:

- expected time of a transition from a metastable to a stable state
- distribution of the exit time from a metastable state
- small eigenvalues and corresponding eigenvectors of the generator

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Stochastic spin models

We are interested in studying the stochastic dynamics of (disordered) spin systems, i.e. Markov process with

- State space $\mathcal{S}_\Lambda = \mathcal{S}^\Lambda$, where $\mathcal{S}$ finite set and e.g. $\Lambda \subset \mathbb{Z}^d$
- Hamiltonian $H_\Lambda \colon \mathcal{S}_\Lambda \to \mathbb{R}$
- Gibbs measure $\mu_{\Lambda,\beta}(\sigma) = Z_{\Lambda,\beta}^{-1} \exp\big(-\beta H_\Lambda(\sigma)\big)$
- Transition rates $p_{\Lambda,\beta}(\sigma,\eta)$ reversible with respect to $\mu_{\Lambda,\beta}$ and "local", i.e. essentially single site flips only.

Well understood situations

Low temperature limit. $\beta \to \infty$

- metastable states correspond to local minima of H_N
- exit from metastable states occur through minimal saddle points of H_N connecting one minimum to deeper ones, only a few path are relevant
- the mean exit time of a metastable state is proportional to $\exp\left(\beta(H_N(saddle) - H_N(min))\right)$
- normalized metastable exit times are $Exp(1)$ distributed

Mean-field models. $H_N(\sigma) = E(\varrho_N(\sigma))$ for some mesoscopic variable ϱ_N

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The random field Curie–Weiss–Potts Model and the dynamics

Random Hamiltonian.

$$H_N(\sigma) \;=\; -\frac{1}{N}\sum_{i,j=1}^{N}\delta(\sigma_i,\sigma_j) \;-\; \sum_{i=1}^{N}\sum_{r=1}^{q} h_r^i\,\delta(\sigma_i,r), \qquad \sigma\in\mathcal{S}_N\equiv\{1,\ldots,q\}^N$$

$\{h^i\}_{i\in\mathbb{N}}$ are i.i.d. random variables taking values in $\mathbb{R}^q$.

Gibbs measure. $\mu_N(\sigma) \;=\; Z_N^{-1}\,\exp\big(-\beta H_N(\sigma)\big)\,q^{-N}$

Equilibrium properties.

- J.M. Amaro de Matos, A.E. Patrick, V.A. Zagrebnov (JSP, 1992), C. Külske (JSP, 1997, 1998)
- G. Iacobelli, C. Külske (JSP, 2010)

Glauber dynamics. Discrete-time Markov chain $\{\sigma(t)\}_{t\in\mathbb{N}_0}$ on $\mathcal{S}_N$ reversible w.r.t. μ_N with Metropolis transition probabilities

$$p_N(\sigma,\eta) \;=\; \frac{1}{qN}\,\exp\big(-\beta\big[H_N(\eta)-H_N(\sigma)\big]_+\big)\,\mathbb{1}_{d_{\mathrm{H}}(\sigma,\eta)=1}$$

and $p_N(\sigma,\sigma)=\sum_\eta p_N(\sigma,\eta)$.

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Coarse graining and mesoscopic approximation

The entropic problem can be solved by passing on to

Mesoscopic variables.

$$\varrho^n : \mathcal{S}_N \to \Gamma^n \subset \mathbb{R}^{n \cdot q}, \qquad \varrho^n(\sigma) = \sum_{k=1}^{n} e^k \otimes \frac{1}{N} \sum_{i \in \Lambda_k} \delta_{\sigma_i}$$

- $\{\mathcal{H}_k\}_{k=1}^n$ is a partition of support of the distribution of the random field, $\mathrm{diam}\mathcal{H}_k < \varepsilon(n)$
- $\Lambda_k = \{i \in \{1, \dots, N\} \mid h^i \in \mathcal{H}_k\}$ is a random partition of $\{1, \dots, N\}$

Induced measure. $\mathcal{Q}^n = \mu_N \circ (\varrho^n)^{-1}$ on the set Γ^n

⚠ In general, $\{\varrho^n(\sigma(t))\}_{t \in \mathbb{N}_0}$ is not Markovian

Strategy. Approximate the original dynamics by Markovian dynamics on Γ^n which are reversible w.r.t. $\mathcal{Q}^n$ with

$$r^n(x, y) = \frac{1}{\mathcal{Q}^n(x)} \sum_{\sigma \in (\varrho^n)^{-1}(x)} \mu_N(\sigma) \sum_{\eta \in (\varrho^n)^{-1}(y)} p_N(\sigma, \eta).$$

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Mesoscopic free energy landscape

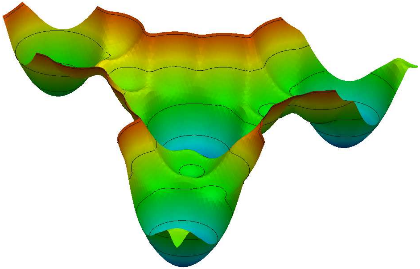
Sharp large deviation estimates

$$Z_N\, \mathcal{Q}^n(\boldsymbol{x}) \;=\; \frac{\exp\left(-N\beta\, F^n(\boldsymbol{x})\right)\,\left(1+o_N(1)\right)}{\prod_{k=1}^n (2\pi N)^{\frac{q-1}{2}} \sqrt{\left|\,\det\left[\pi_k\, \nabla^2 U_{|\Lambda_k|}\left(t^*(x^k/\pi_k)\right)\right]\right|}},$$

where $\pi_k = |\Lambda_k|/N$ and $F^n(\boldsymbol{x}) := E(\boldsymbol{x}) + \frac{1}{\beta}\sum_{k=1}^n \pi_k\, I_{|\Lambda_k|}(x^k)$

Critical points.

- Deterministic in the limit $N \to \infty$
- explicit expression for $F^n(\boldsymbol{x})$ at critical points



Main result

Let $\boldsymbol{m}$ be a local minimum of F^n and $\boldsymbol{M}$ the set of deeper local minima of F^n.

Theorem 1. *Suppose $\boldsymbol{z}$ be a unique critical point of index 1 separating $\boldsymbol{m}$ from $\boldsymbol{M}$ and denote by $A = (\varrho^n)^{-1}(\boldsymbol{m})$ and $B = (\varrho^n)^{-1}(\boldsymbol{M})$. Then, $\mathbb{P}_h$-a.s.,*

$$\mathbb{E}_\nu\left[\tau_B\right] = \frac{2\pi N}{\beta|\gamma_1|}\sqrt{\frac{\left|\det\left(I - 2\beta\,\nabla^2 U_N(2\beta z)\right)\right|}{\det\left(I - 2\beta\,\nabla^2 U_N(2\beta m)\right)}}\, e^{\beta N(F_N(z)-F_N(m))}\left(1 + o_N(1)\right)$$

where ν is a probability measure on A and

$$F_N(x) = \|x\|^2 - \frac{1}{\beta N}\sum_{i=1}^N \ln\left(\sum_{r=1}^q \frac{1}{q}\exp\left(2\beta z_r + \beta h_r^i\right)\right)$$

Previous and related work

- F. den Hollander and P. dai Pra (JSP, 1996) large deviations, logarithmic asymptotics
- P. Mathieu and P. Picco (JSP, 1998) Bernoulli distribution, up to polynomial errors
- A. Bovier, M. Eckhoff, V. Gayrard and M. Klein (PTRL, 2001) discrete distribution, up to a multiplicative constant
- A. Bianchi, A. Bovier and D. Ioffe (EJP, 2008) bounded continuous distribution, precise prefactor

Boundary value problems

Discrete generator. $\quad (L_N f)(\sigma) = \sum_{\eta \in \mathcal{S}_N} p_N(\sigma,\eta)\big(f(\eta) - f(\sigma)\big)$

Given $D \subset \mathcal{S}_N$ and functions $g, k \colon D^c \to \mathbb{R}$ and $u \colon D \to \mathbb{R}$

$$\begin{cases} (L_N f)(\sigma) - k(\sigma)\, f(\sigma) = -g(\sigma), & \sigma \in D^c \\ f(\sigma) = u(\sigma), & \sigma \in D, \end{cases}$$

Suppose $\min_{\eta \in D^c} k(\eta) \equiv \kappa > -1$ and $\mathbb{E}_\sigma\left[\tau_D\,(1+\kappa)^{-\tau_D}\right] < \infty$. Then

$$f(\sigma) = \mathbb{E}_\sigma\left[u(\sigma(\tau_D)) \prod_{s=0}^{\tau_D - 1} \frac{1}{1 + k(\sigma(s))} + \sum_{s=0}^{\tau_D - 1} g(\sigma(s)) \prod_{r=0}^{s} \frac{1}{1 + k(\sigma(r))}\right]$$

Mean hitting times. $w_D(\sigma) = \mathbb{E}_\sigma[\tau_D]$ solves

$$\begin{cases} (L_N w_D)(\sigma) = -1, & \sigma \in D^c \\ w_D(\sigma) = 0, & \sigma \in D \end{cases}$$

Equilibrium potential and capacities

Given $A, B \subset \mathcal{S}_N$ disjoint.

Equilibrium potential. $h_{A,B}(\sigma) = \mathbb{P}_\sigma\left[\tau_A < \tau_B\right]$ solves

$$\begin{cases} \left(L_N h_{A,B}\right)(\sigma) = 0, & \sigma \in (A \cup B)^c \\ h_{A,B}(\sigma) = \mathbb{1}_A(\sigma), & \sigma \in A \cup B \end{cases}$$

Equilibrium measure. $e_{A,B}(\sigma) = -\left(L_N h_{A,B}\right)(\sigma)$

Capacity.

$$\mathrm{cap}(A,B) = \sum_{\sigma \in B} \mu_N(\sigma)\, e_{A,B}(\sigma) = \frac{1}{2} \sum_{\sigma,\eta \in \mathcal{S}_N} \mu_N(\sigma)\, p_N(\sigma,\eta) \left(h_{A,B}(\sigma) - h_{A,B}(\eta)\right)^2$$

Dirichlet form. $\mathcal{E}(h,h) = \frac{1}{2} \sum_{\sigma,\eta \in \mathcal{S}_N} \mu_N(\sigma)\, p_N(\sigma,\eta) \left(h(\sigma) - h(\eta)\right)^2$

Connection between capacities and mean hitting times

Last exit biased distribution. $\nu_{A,B}$ measure on A

$$\nu_{A,B}(\sigma) = \frac{\mu_N(\sigma)\, e_{A,B}(\sigma)}{\mathrm{cap}(A,B)} = \frac{\mu_N(\sigma)\ \mathbb{P}_\sigma\left[\tau_B < \tau_A\right]}{\sum_{\eta \in A} \mu_N(\sigma)\ \mathbb{P}_\eta\left[\tau_B < \tau_A\right]}, \qquad \sigma \in A$$

Mean hitting time.

$$\mathbb{E}_{\nu_{A,B}}\left[\tau_B\right] = \frac{1}{\mathrm{cap}(A,B)} \sum_{\sigma \in \mathcal{S}_N} \mu_N(\sigma)\ h_{A,B}(\sigma)$$

The full beauty. To obtain sharp estimates for the mean hitting time, we need:

- precise control on capacities.
- some rough bounds on the equilibrium potential.

Averaged renewal equation. $A, B, X \subset \mathcal{S}_N$ mutually disjoint

$$\sum_{\sigma \in X} \nu_{X, A \cup B}(\sigma)\, h_{A,B}(\sigma) \le \min\left\{\frac{\mathrm{cap}(X,A)}{\mathrm{cap}\ X, B}, 1\right\}$$

Computation of capacities

Variational principles for capacities offers two convenient options for upper and lower bounds:

Dirichlet principle.

$$\mathrm{cap}(A,B) = \inf_{h\in\mathcal{H}_{A,B}} \frac{1}{2}\sum_{\sigma,\eta}\mu(\sigma)\,p(\sigma,\eta)\big(h(\sigma)-h(\eta)\big)^2$$

$\mathcal{H}_{A,B}$ is the space of functions with boundary constraints; minimizer harmonic function

Berman-Konsowa principle.

$$\mathrm{cap}(A,B) = \sup_{f\in\mathcal{U}_{A,B}} \mathbb{E}^f\left[\left(\sum_{(\sigma,\eta)\in\mathcal{X}}\frac{f(\sigma,\eta)}{\mu(\sigma)\,p(\sigma,\eta)}\right)^{-1}\right]$$

$\mathcal{U}_{A,B}$ is the space of unit flows; maximizer harmonic flow. $\mathbb{E}^f$ denotes the law of a directed Markov chain with transition probabilities proportional to the flow.

The program

The key step in the proof of the upper and lower bound on capacities is to

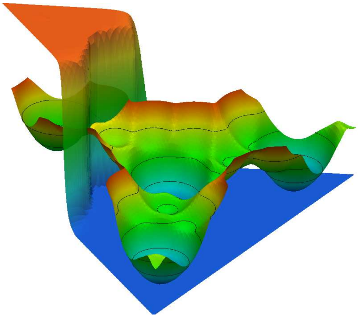
1. find a function which is almost harmonic in a small neighborhood of the relevant saddle point $\boldsymbol{z}$.

Two parameter family of test functions.

$$g(\boldsymbol{x}) \;=\; f(\langle \boldsymbol{v}, \boldsymbol{x} - \boldsymbol{z}, \rangle)$$

where $\boldsymbol{v} \in \Gamma^n$ and $|\gamma_1| \in \mathbb{R}_+$

$$f(s) \;=\; \sqrt{\frac{\beta N |\gamma_1|}{2\pi}} \int_{-\infty}^{s} \exp\left(-\tfrac{1}{2}\beta N |\gamma_1| u^2\right) \mathrm{d}u$$



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Two scale construction:

2. Construct a mesoscopic unit flow on variables x from the approximate harmonic function. This yields a good lower bound in the mesoscopic Dirichlet form.
3. Construct a subordinate microscopic unit flow for each mesoscopic path.
4. Use that the magnetic field is almost constant in any block Λ_k to show strong concentration properties along microscopic paths.

This yields a lower bound that differs from the upper bound only by a factor $1 + \mathcal{O}\big(\varepsilon(n)\big)$.

From average to pointwise estimates

Questions.

- Does the metastable time really depend on the last exit biased distribution ν?
- Under which conditions can we deduce pointwise estimates?

Heuristic.

The time spent in the starting well before reaching B is much larger then the mixing time of the dynamics conditioned to stay in the well:

$$\mathbb{E}_\sigma\left[\tau_B\right] \sim \mathbb{E}_\eta\left[\tau_B\right] \qquad \forall\, \sigma, \eta \in A.$$

After the system is mixed, the return times to A are i.i.d. random variables, and the number of returns to A is geometric. Provided that the mixing time is small enough respect to $\mathbb{E}_\nu[\tau_B]$, the metastable time is expected to be exponential distributed.

Main results

Let $\boldsymbol{m}$ and $\boldsymbol{M}$ be local minima in F^n and $A = (\varrho^n)^{-1}(\boldsymbol{m})$ and $B = (\varrho^n)^{-1}(\boldsymbol{M})$.

Theorem 2. *For n large enough,*

$$\mathbb{E}_\sigma\left[\tau_B\right] \;=\; \mathbb{E}_\eta\left[\tau_B\right]\left(1 + \mathcal{O}_N(1)\right)$$

for all $\sigma, \eta \in A$.

Theorem 3. *For n large enough and all $t > 0$*

$$\mathbb{P}_\sigma\left[\tau_B / \mathbb{E}_\sigma\left[\tau_B\right] > t\right] \;\to\; \mathrm{e}^{-t}, \qquad \textit{as} \quad N \to \infty$$

for all $\sigma, \eta \in A$.

Previous and related work

- D.A. Levin, M. Luczak, Y. Peres (PTRF, 2010) without random field, coupling construction
- A. Bianchi, A. Bovier and D. Ioffe (accepted Ann. Prob.) continuous distribution, coupling construction for Ising spins

Conclusions

What has been done so far.

- Sharp estimates on metastable exit times in a model without symmetry when entropy is relevant.
- Description of distribution of metastable exit times.
- Averaged version of renewal equations for harmonic functions.
- Construction of a coupling when the underlying single spin space is finite.

Future challenges.

- Control of the small eigenvalues of the generator!
- Hopfield model with infinitely many patterns.