Large deviations for Brownian intersection measures

Chiranjib Mukherjee

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Brownian paths do intersect

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- Dvoretzky, Erdös, Kakutani and Taylor showed S_t is non-empty with positive probability iff

$$\begin{cases} d = 2, \ p \in \mathbb{N} \\ d = 3, \ p = 2 \\ d \geq 4, \ p = 1. \end{cases}$$

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- A measure is naturally defined on S_t:

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- Goal: Make precise the above as $t \uparrow \infty$ (in particular, $d \geq 2$).

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$$s_d(\epsilon) = \begin{cases} \pi^{-p} \log^p(\frac{1}{\epsilon}) & \text{if } d = 2 \\ (2\pi\epsilon)^{-2} & \text{if } d = 3 \text{ and } p = 2 \\ \frac{2}{\omega_d(d-2)}\, \epsilon^{2-d} & \text{if } d \geq 3 \text{ and } p = 1. \end{cases}$$

Wiener Sausages

Intersection measure: scaling limit of Lebesgue measure on sausages

- Le Gall shows limit $\epsilon \downarrow 0$ gives the Brownian intersection measure:

$$\lim_{\epsilon \to 0} \ell_{\epsilon,t}(A) = \ell_t(A) \quad \text{in} \quad L^q \text{ for } q \in [1, \infty)$$

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- Want to study: Behavior of $\frac{1}{t}\ell_t^{(i)}$, as $t \uparrow \infty$.

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Our Goal: Similar statement for intersection measure ℓ_t, for *large* t?

Large deviations: diverging time

occupation measure and intersection measure

- $\ell_t^{(1)}, \dots, \ell_t^{(p)}$ occupation measures of p paths running until time t in a bounded domain B until first exit times $\tau_1, \dots, \tau_p$.

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- ℓ_t the intersection measure of p paths.

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- ℓ_t the intersection measure of p paths.
- Make sure no path exits B before time t: $\mathbb{P}_t(\cdot) = \mathbb{P}\big(\cdot \bigcap \{ t < \tau_1 \wedge \cdots \wedge \tau_p \} \big)$

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Intersection densities as product of occupation densities

$$\mathbb{P}_t\left(\frac{\ell_t}{t^p} \approx \mu; \frac{\ell_t^{(1)}}{t} \approx \mu_1, \ldots, \frac{\ell_t^{(p)}}{t} \approx \mu_p\right)$$

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else,

$$I = \infty \qquad \text{identically}$$

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Extension of classical theory

Theorem (König/M (2011))

The family of tuples $\left(\frac{\ell_t}{t^p}; \frac{\ell_t^{(1)}}{t}, \dots, \frac{\ell_t^{(p)}}{t}\right)$ *satisfies a LDP under* $\mathbb{P}_t$*, as* $t \uparrow \infty$*, with rate function*

$$I(\mu; \mu_1, \dots, \mu_p) = \frac{1}{2} \sum_{i=1}^{p} \|\nabla \psi_i\|_2^2$$

if μ *and* $\mu_1, \dots, \mu_p$ *have densities* ψ^{2p} *and* $\psi_1^2, \dots, \psi_p^2$ *respectively,* $\psi_i \in H_0^1(B)$, $\|\psi_i\|_2 = 1$ *and* $\psi^{2p} = \prod_{i=1}^{p} \psi_i^2$*, else* $I = \infty$*.*

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Corollary

The family of measures $\left(\frac{\ell_t}{t^p}\right)$ *satisfies a large deviation principle, under* $\mathbb{P}_t$*, as* $t \uparrow \infty$*, with rate function*

$$J(\mu) = \inf\left\{\frac{1}{2}\sum_{i=1}^{p} ||\nabla\psi_i||_2^2 : \psi_i \in H_0^1(B), ||\psi_i||_2 = 1, \prod_{i=1}^{p} \psi_i^2 = \frac{d\mu}{dx}\right\}$$

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$p = 1$: We recover classical Donsker-Varadhan theory for one path.

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[König and Mörters (2001)]:

$$\lim_{a\to\infty} a^{-\frac{1}{p}} \log \mathbb{P}\left[\ell(U) > a\right] = -\Theta(U)$$

for

$$\Theta(U) = \inf\left\{\frac{p}{2}\,||\nabla\psi||_2^2 \;:\; \psi \in H_0^1(B), ||1_U\psi||_{2p}^2 = 1\right\}.$$

Minimisers and path behavior
Euler-Lagrange equations

- Minimiser(s) to $\Theta(U)$ exist(s).
- Every minimising ψ solves

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Upshot: ψ^{2p} should be the large-a density of the intersection measure on U.

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- Upshot: Law of large numbers:

$$L \to \psi^{2p} \qquad \text{under } \mathbb{P}(\cdot|\ \ell(U) > a),\ a \uparrow \infty$$

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$$\lim_{a\uparrow\infty} \mathbb{P}[d(L,\mathcal{M}) > \epsilon \mid \ell(U) > a] = 0$$

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- Upshot: Law of large numbers:

$$L \to \psi^{2p} \qquad \text{under } \mathbb{P}(\cdot \mid \ell(U) > a),\ a \uparrow \infty$$

- Large deviations: What is the exponential decay rate?

Intersection measure until exit times

Large deviations: diverging mass

Theorem (König/M (2011))

The normalized probability measures $L = \frac{\ell}{\ell(U)}$ *satisfy a large deviation principle under* $\mathbb{P}(\cdot|\ell(U) > a)$, *as* $a \uparrow \infty$, *with rate function*

$$\Lambda(\mu) = \inf\left\{\frac{1}{2}\sum_{i=1}^{p} ||\nabla\psi_i||_2^2 : \psi_i \in H_0^1(B), \prod_{i=1}^{p} \psi_i^2 = \frac{d\mu}{dx}\right\} - \Theta(U).$$

Outlook

Questions we can chew on

- Study the variational formula for the rate function:

$$J(\mu) = \inf\left\{\frac{1}{2}\sum_{i=1}^{p}||\nabla\psi_i||_2^2 : \psi_i \in H_0^1(B), ||\psi_i||_2 = 1, \prod_{i=1}^{p}\psi_i^2 = \frac{d\mu}{dx}\right\}$$

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- Extend it to unbounded domains.: For $p = 2,\ B = \mathbb{R}^3$