Transition probabilities of Bethe ansatz solvable interacting particle systems

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Outline

1. Introduction to coordinate Bethe ansatz for ASEP
2. Borodin-Ferrari's PushASEP and its generalized model
3. Transition probability of the PushASEP by Bethe ansatz

Bethe ansatz technique in interacting particle systems

- Bethe ansatz : technique to obtain eigenvalues and eigenvectors of the Hamiltonian of 1D quantum spin-$\frac{1}{2}$ chains
- A generator of ASEP is a similarity transformation of the quantum spin chain Hamiltonian (XXZ model).
- ASEP with N particles on $\mathbb{Z}$ – *Bethe ansatz solvable*

ASEP - a model in KPZ universality class

Early works regarding RMT - TASEP

- Current fluctuation of TASEP with step initial condition – GUE TW distribution ; Johansson, Nagao and Sasamoto, Schütz and Rakos
- Limiting processes – $\mathcal{A}_2$-process (step initial), $\mathcal{A}_1$-process (flat initial) ; Prähofer and Spohn; Borodin, Ferrari, Prähofer and Sasamoto
- TASEP with particle dependent hopping rates ; Schütz and Rakos (via Bethe ansatz), Baik, Ben Arous and Peche (Last passage percolation, RMT)

Bethe ansatz in TASEP

- Quite successful !

Bethe ansatz

- Quantity of interest : $P_Y(X;t)$ where $X = (x_1, \cdots, x_N)$ with $x_1 < \cdots < x_N$.
- Time evolution

$$\begin{aligned}
\partial_t P_Y(X;t) &= -\hat{H} P_Y(X;t) \\
&= P_Y(x_1 - 1, x_2, \cdots, x_N; t) \\
&\quad + (1 - \delta_{x_2 - 1, x_1}) P_Y(x_1, x_2 - 1, x_3, \cdots, x_N; t) \\
&\qquad \vdots \\
&\quad + (1 - \delta_{x_N - 1, x_{N-1}}) P_Y(x_1, x_2, \cdots, x_N - 1; t) \\
&\quad - (N - \delta_{x_2 - 1, x_1} - \cdots - \delta_{x_N - 1, x_{N-1}}) P_Y(x_1, \cdots, x_N; t)
\end{aligned}$$

Idea

- Extend the physical region of definition of $P_Y(X;t)$ $\{(x_1,\cdots,x_N): x_i < x_{i+1}\}$ to $\mathbb{Z}^N$ by imposing a boundary condition

$$P_Y(\cdots,x_k,x_k,\cdots;t) = P_Y(\cdots,x_k,x_{k+1},\cdots;t).$$

Problem to solve

$$\begin{aligned}\partial_t P_Y(X;t) &= -\hat{H}P_Y(X;t)\\ &= \sum_i P_Y(x_1,\cdots,x_{i-1},x_i-1,x_{i+1},\cdots,x_N;t)\\ &\quad - NP_Y(x_1,\cdots,x_N;t)\end{aligned}$$

with

$$P_Y(\cdots,x_k,x_k,\cdots;t) = P_Y(\cdots,x_k,x_{k+1},\cdots;t).$$

and

$$P_Y(X;0) = \delta_Y(X).$$

Bethe ansatz solution

- A solution of DE is
$$\prod_j {\xi_j}^{x_j} e^{\varepsilon(\xi_j)t}$$
with
$$\varepsilon(\xi) = \frac{1}{\xi} - 1$$
and $\xi(\neq 0) \in \mathbb{C}$.
- Important observation : For $\sigma \in \mathbb{S}_N$,
$$\prod_j \xi_{\sigma(j)}^{x_j} e^{\varepsilon(\xi_j)t}$$
is another solution and
$$\sum_j \varepsilon(\xi_j)$$
is invariant under permutation.
- Any integral of a linear combination is also a solution.

Essence of Bethe ansatz

- Take a linear combination of $N!$ solutions and determine coefficients so that the linear combination satisfies BC.
- Bethe ansatz solution

$$\int \sum_{\sigma \in \mathbb{S}_N} A_\sigma \prod_j \xi_{\sigma(j)}^{x_j} \prod_j e^{\varepsilon(\xi_j)t} d^N \xi$$

with $A_\sigma = \prod S_{\beta\alpha}$ where $S_{\beta\alpha} = \frac{1-\xi_\beta}{1-\xi_\alpha}$ and the product is over all pairs (β, α) in σ such that $\alpha < \beta$ but $\sigma^{-1}(\alpha) > \sigma^{-1}(\beta)$

Theorem

(Schütz) Let $F_Y(X;t)$ be an $N \times N$ matrix with entries $F_{ij} = f_{i-j}(x_i - y_j; t)$ where

$$f_p(n;t) = \frac{1}{2\pi i} \oint_{|\xi|=1-0} e^{-(1-\xi^{-1})t}(1-\xi)^{-p}\xi^{n-1} d\xi$$

Then $P_Y(X;t) = \det F_Y(X;t)$.

Determinantal structure and special properties of $f_p(n;t)$ are essential for further works of

- Current distribution for step/flat initial conditions
- One-point distribution -TW(GUE, GOE)
- Limiting processes - $\mathcal{A}_2$, $\mathcal{A}_1$

In ASEP,

$$S_{\beta\alpha} = \frac{p + q\xi_\beta\xi_\alpha - \xi_\beta}{p + q\xi_\beta\xi_\alpha - \xi_\alpha}$$

non-determinantal structure

ASEP -Tracy and Widom's formula by Bethe ansatz

Theorem

$$P_Y(X;t) = \sum_{\sigma \in \mathbb{S}_N} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} A_\sigma \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\sum_i \varepsilon(\xi_i) t} d\xi_1 \cdots d\xi_N$$

where $\mathcal{C}_r$ *is a circle centered at zero with sufficiently small radius* r *so that all the poles of the integrand except at the origin lie outside* $\mathcal{C}_r$ *and* $A_\sigma = \prod S_{\beta\alpha}$ *where* $S_{\beta\alpha} = \frac{p + q\xi_\beta\xi_\alpha - \xi_\beta}{p + q\xi_\beta\xi_\alpha - \xi_\alpha}$ *and the product is over all pairs* (β, α) *in* σ *such that* $\alpha < \beta$ *but* $\sigma^{-1}(\alpha) > \sigma^{-1}(\beta)$ *and* $\varepsilon(\xi_i) = p\xi_i^{-1} + q\xi_i - 1$.

Based on the transition probability,

- Probability distribution of a single particle's position – time-integrated current (TW)
- Fredholm determinant representation of the distribution for step initial condition (TW)
- Asymptotic analysis – KPZ universality class (TW)

$$\frac{x_m(t) - c_1 t}{c_2 t^{1/3}} \to F_2$$

  in distribution as $m, t \to \infty$
- One-point probability distribution of Hopf-Cole solution to KPZ equation with narrow edge initial data (Amir-Corwin-Quastel, and Sasamoto-Spohn)

Borodin-Ferrari's model (2008) - PushASEP

Dynamics

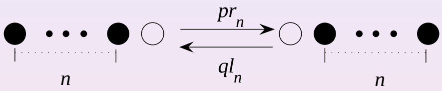
- Each particle has two Poisson clocks; one for left jump with rate L and the other for right jump with rate R.
- When the ith right clock rings, the ith particle jumps according to TASEP dynamics.
- When the ith left clock rings, the ith particle jumps to the nearest vacant site on its left – Pushing dynamics
- Interpolates between TASEP and Drop-push model which are in KPZ universality class.

Borodin-Ferrari's model - PushASEP

- Totally asymmetric dynamics.
- S-matrices are *separable*; determinantal transition probability by Bethe ansatz.
- For step and flat initial conditions; GUE TW and GOE TW, further, $\mathcal{A}_2$, $\mathcal{A}_1$ processes, respectively.

Question : Partially asymmetric dynamics ? Clock rates depending on the environment ?

Two-sided PushASEP



- $p + q = 1$
- Assume that pushing rates depend on the number of particles to be pushed.

Boundary Condition for $N = 2$

$$
\begin{aligned}
& pr_1 P_Y(x_1, x_2 - 1; t) - pr_2 P_Y(x_1 - 1, x_2 - 1; t) - p(r_1 - r_2) P_Y(x_1, x_2; t) \\
= \quad & -ql_1 P_Y(x_1 + 1, x_2; t) + ql_2 P_Y(x_1 + 1, x_2 + 1; t) + q(l_1 - l_2) P_Y(x_1, x_2; t)
\end{aligned}
$$

In the original B.C.

- Set both sides to be zero, $\lambda = l_2/l_1, \mu = r_2/r_1$
- Set $r_1 = l_1 = 1$, $\lambda + \mu = 1$

Then after letting $x_1 = x$, we have much simpler BC

$$P_Y(x, x; t) = \mu P_Y(x - 1, x; t) + \lambda P_Y(x, x + 1; t).$$

Alimohammadi et al.(1999) suggested this BC directly as a combined version of TASEP and Drop-push model.

Constraints for Bethe ansatz solvability

Alimohammadi et al. (1999)

$$
\begin{aligned}
r_n &= \frac{1}{1+\lambda/\mu+(\lambda/\mu)^2+\cdots+(\lambda/\mu)^{n-1}}, \qquad (\lambda+\mu=1)\\
l_n &= \frac{1}{1+\mu/\lambda+(\mu/\lambda)^2+\cdots+(\mu/\lambda)^{n-1}}, \qquad (r_1=l_1=1)
\end{aligned}
$$

This model generalizes Borodin-Ferrari model ($\mu \to 0$).
Question : Is this model in KPZ universality class ?

S-matrices for two-particle interaction

$$S^{\dagger}_{\alpha\beta} := -\frac{\xi_\alpha}{\xi_\beta} \cdot \frac{\mu + \lambda\xi_\alpha\xi_\beta - \xi_\beta}{\mu + \lambda\xi_\alpha\xi_\beta - \xi_\alpha} := \frac{\xi_\alpha}{\xi_\beta} \cdot S^{(\mu)}_{\beta\alpha}$$

Remarks

- *Non-determinantal*
- $q = 0$ but $\mu, \lambda \neq 0$; totally asymmetric (to the right only) but not determinantal.

Main result

Theorem

(L, 2011) The transition probability of the PushASEP is given in the form of

$$P_Y(X;t) = \sum_{\sigma \in \mathbb{S}_N} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} B_\sigma \prod_i \xi_{\sigma(i)}^{-x_i + y_{N-\sigma(i)+1} - 1} e^{\sum_i \varepsilon(\xi_i) t} d\xi_1 \cdots d\xi_N,$$

where $\mathcal{C}_r$ is a circle centered at zero with sufficiently small radius r so that all the poles of the integrand except at the origin lie outside $\mathcal{C}_r$ and

$$B_\sigma = \prod_{(\alpha,\beta)} S^\dagger_{\alpha\beta}.$$

The product is over all pairs (α, β) in a permutation $\sigma \in \mathbb{S}_N$ such that $\beta > \alpha$ with $\sigma^{-1}(\beta) > \sigma^{-1}(\alpha)$.

Further works

- Distribution of a single particle's position with special initial conditions; for example, step initial - in progress
  good signal : the same combinatorial identity
- Fredholm determinant representation
- Asymptotic analysis - TW distribution ? KPZ universality class ?