Christof Külske

Metastates in Markov chain driven mean-field models

Lattice spin models with a quenched random Hamiltonian, examples

Edwards-Anderson spinglass

$$H = -\sum_{\langle i,j\rangle} J_{i,j}\sigma_i\sigma_j$$

Spins: $\sigma_i \in \{1, -1\}$

Random couplings: $J_{i,j} \sim \mathcal{N}(0,1)$, i.i.d.

Random field Ising model:

$$H = -\sum_{\langle i,j\rangle} \sigma_i\sigma_j - \varepsilon \sum_i \eta_i\sigma_i$$

Random fields: $\eta_i = \pm 1$ with equal probability, i.i.d.

The **metastate** is a concept to capture the asymptotic volume-dependence of the Gibbs states

$$"\mu(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z}"$$

Quenched (fixed) randomness $\eta = (\eta_i)_{i \in \mathbb{Z}^d}$.
Probability distribution $\mathbb{P}(d\eta)$
Infinite volume spin configuration $\sigma = (\sigma_i)_{i \in \mathbb{Z}^d}$
Infinite volume Hamiltonian $H^\eta(\sigma)$ (given in terms of an interaction Φ^η)

Fixing a boundary condition $\bar{\sigma}$, define the *finite-volume Gibbs states*

$$\mu_\Lambda^{\bar{\sigma}}[\eta](d\sigma)$$

in the finite volume $\Lambda \subset \mathbb{Z}^d$
restricting the terms of the Hamiltonian to $\Lambda = \Lambda_n = [-n, n]^d$

Common for *translation-invariant systems*:

to have convergence of the finite-volume states

$$\mu^{\bar{\sigma}}_{\Lambda_n}[\eta = 0](d\sigma) \to \mu^{\bar{\sigma}}(d\sigma)$$

as n gets large

Common for *disordered systems*:

not to have convergence of the finite-volume states:

$$\mu^{\bar{\sigma}}_{\Lambda_n}[\eta](d\sigma)$$

might have many limit points when several Gibbs measures are available

Newman book, Bovier book

Külske: mean-field random field Ising

Bovier, Gayrard: Hopfield with many patterns

van Enter, Bovier, Niederhauser: Hopfield model with Gaussian fields (continuous symmetry)

van Enter, Netocny, Schaap: Ising ferromagnet on lattice with random boundary conditions

Arguin, Damron, Newman, Stein (2009): "Metastate-version" of uniqueness of groundstate for lattice-spinglass in 2 dimensions

Iacobelli, Külske 2010: Metastates in mean-field models with i.i.d. disorder

Cotar, Külske 2011, in preparation

measurably $\mu[\xi] = \int \nu w[\xi](d\nu)$ with $w[\xi]($ex$\mathcal{G}(\xi)) = 1$

Spin variables: $\sigma(i)$ taking values in a finite set E

Disorder variable: $\eta(i)$ taking values in a finite set E'

Sites: $i \in \{1, 2, \ldots, n\}$

$$\mathcal{P}(E) = \{\text{set of probability measures on } E\}$$

$$= \{(p(a))_{a\in E} : p(a) \geq 0, \sum_{a\in E} p(a) = 1\}$$

$$L_n = \text{empirical distribution} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\sigma(i)} \in \mathcal{P}(E)$$

$$F : \mathcal{P}(E) \to \mathbb{R},$$

twice continuously differentiable.

Local a priori measures $\alpha[b] \in \mathcal{P}(E)$

for any possible type of the disorder $b \in E'$.

Mean-field interaction F

A priori measures $\alpha = (\alpha[b])_{b \in E'}$

Disorder distribution $\pi \in \mathcal{P}(E')$

Definition* 1.** *The* ***disorder-dependent finite-volume Gibbs measures *are*

$$\mu_{F,n}[\eta(1), \ldots, \eta(n)](\sigma(1) = \omega(1), \ldots, \sigma(n) = \omega(n))$$
$$= \frac{1}{Z_{F,n}[\eta(1), \ldots, \eta(n)]} \exp\left(-nF\left(L_n^\omega\right)\right) \prod_{i=1}^{n} \alpha[\eta_i](\omega_i)$$

Frozen disorder: $\eta(i) \sim \pi$ i.i.d. over sites i

Definition 2. *Assume that, for every bounded continuous* $G : \mathcal{P}(E^\infty) \times (E')^\infty \to \mathbb{R}$ *the limit*

$$\lim_{n\uparrow\infty} \int \mathbb{P}(d\eta) G(\mu_n[\eta], \eta) = \int J(d\mu, d\eta) G(\mu, \eta)$$

exists. Then the conditional distribution $\kappa[\eta](d\mu) := J(d\mu|\eta)$ *is called the* AW-metastate on the level of the states*.*

Volume of b-like sites, given η:

$$\Lambda_n(b) = \{i \in \{1, 2, \ldots, n\}; \eta(i) = b\}$$

Frequency of the b-like sites:

$$\widehat{\pi}_n(b) = \frac{|\Lambda_n(b)|}{n}$$

empirical spin-distribution on the b-like sites:

$$\widehat{L}_n(b) = \frac{1}{|\Lambda_n(b)|} \sum_{i \in \Lambda_n(b)} \delta_{\sigma(i)}$$

vector of empirical distributions:

$$\widehat{L}_n = (\widehat{L}_n(b))_{b \in E'}$$

total empirical spin-distribution

$$L_n = \sum_{b \in E'} \widehat{\pi}_n(b) \widehat{L}_n(b)$$

Definition 3. *Consider the free energy minimization problem*

$$\widehat{\nu} \mapsto \Phi[\pi](\widehat{\nu})$$

on $\mathcal{P}(E)^{E'}$, with the free energy functional

$$\Phi : \mathcal{P}(E') \times \mathcal{P}(E)^{|E'|} \to \mathbb{R}$$
$$\Phi[\widehat{\pi}](\widehat{\nu}) = F\left(\sum_{b \in E'} \widehat{\pi}(b)\widehat{\nu}(b)\right) + \sum_{b} \widehat{\pi}(b) S(\widehat{\nu}(b)|\alpha[b])$$

where $S(p_1|p_2) = \sum_{a \in E} p_1(a) \log \frac{p_1(a)}{p_2(a)}$ is the relative entropy.

Non-degeneracy condition 1:

$\widehat{\nu} \mapsto \Phi[\pi](\widehat{\nu})$ has a finite set of minimizers $M^* = M^*(F, \alpha, \pi)$ with positive curvature.

Let $\widehat{\nu}_j$ be a fixed element in M^*. Let us consider the linearization of the free energy functional at the fixed minimizers as a function of $\tilde{\pi}$ around π, which reads

$$\Phi[\tilde{\pi}](\widehat{\nu}_j) - \Phi[\pi](\widehat{\nu}_j) = -B_j[\tilde{\pi} - \pi] + o(\|\tilde{\pi} - \pi\|)$$

This defines an affine function on the tangent space of field type measures $T\mathcal{P}(E')$ (i.e. vectors which sum up to zero, isomorphic to $\mathbb{R}^{|E'|-1}$), for any j.

Non-degeneracy condition 2:

No different minimizers j, j' have the same $B_j = B_{j'}$

Definition 4. *Call B_j the **stability vector of** $\widehat{\nu}_j$ and call*

$$R_j := \{x \in T\mathcal{P}(E'), \langle x, B_j \rangle > \max_{k \neq j} \langle x, B_k \rangle\}$$

***stability region of** $\widehat{\nu}_j$.*

THEOREM 5. *(Iacobelli, Külske, JSP 2010) Assume that the model satisfies the non-degeneracy assumptions 1 and 2. Define the weights*

$$w_j := \mathbb{P}_\pi(G \in R_j)$$

where G *taking values in* $T\mathcal{P}(E')$ *is a centered Gaussian variable with covariance*

$$C_\pi(b, b') = \pi(b) 1_{b=b'} - \pi(b)\pi(b')$$

Then the Aizenman-Wehr metastate on the level of the states equals

$$\kappa[\eta](d\mu) = \sum_{j=1}^{k} w_j \delta_{\mu_j[\eta]}(d\mu)$$

where $\mu_j[\eta] := \prod_{i=1}^{\infty} \gamma[\eta(i)](\,\cdot\,|\pi\widehat{\nu}_j)$ *with*

$$\gamma[b](a|\nu) = \frac{e^{-dF_\nu(a)}\alpha[b](a)}{\sum_{\bar{a}\in E} e^{-dF_\nu(\bar{a})}\alpha[b](\bar{a})}$$

Let us take the Potts model with quadratic interaction

$$F(\nu) = -\frac{\beta}{2}(\nu(1)^2 + \cdots + \nu(q)^2)$$

Let us take $E \equiv E'$ and π to be the equidistribution and switch to the specific case $\alpha[b](a) = \frac{e^{B1_{b=a}}}{e^B+q-1}$ (random field with homogenous intensity). The kernels become

$$\gamma[b](a|\nu) = \frac{e^{\beta\nu(a)+B1_{a=b}}}{\sum_{\bar{a}\in E} e^{\beta\nu(\bar{a})+B1_{\bar{a}=b}}}$$

We will be looking at measures in $\nu_{j,u} \in \mathcal{P}(E)$ of the form $\nu_{j,u}(j) = \frac{1+u(q-1)}{q}$, $\nu_{j,u}(i) = \frac{1-u}{q}$ for $i \neq j$. The stability vector for $\nu_{1,u}$ is given by

$$\widehat{B}_{\nu_{1,u}} = \begin{pmatrix} \frac{q-1}{q} \log \frac{e^{\beta u+B}+q-1}{e^{\beta u}+e^B+q-2} \\ -\frac{1}{q} \log \frac{e^{\beta u+B}+q-1}{e^{\beta u}+e^B+q-2} \\ \cdots \\ -\frac{1}{q} \log \frac{e^{\beta u+B}+q-1}{e^{\beta u}+e^B+q-2} \end{pmatrix}$$

the other ones are related by symmetry.

mean-field equation for u:

$$u = \frac{e^{\beta u}}{e^{\beta u} + e^{B} + (q-2)} - \frac{1}{e^{\beta u + B} + (q-1)}$$

$u = 0$ is always a solution

for $B = 0$: mean-field equation for Potts without disorder

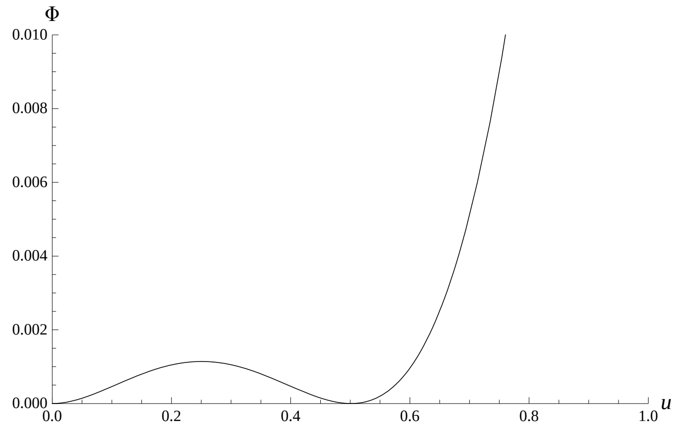
the non-trivial solution u is to be chosen iff $\Phi[\pi](u) < \Phi[\pi](u = 0)$

$B = 0$: first order transition at the critical inverse temperature $\beta = 4 \log 2$

B takes small enough positive values: line in the space of temperature and coupling strength B of an equal-depth minimum at $u = 0$ and a positive value of $u = u^*(\beta, q)$

Along this line the set of Gibbs measures is strictly bigger then the set of states which are seen under the metastate.

The Plot shows the graph of $u \mapsto \Phi[\pi](\hat{\Gamma}(\nu_{j,u}))$ for $B = 0.3, q = 3, \beta = 4 \log 2 + 0.03203$ at which there is the first order transition.



$$\kappa[\eta](d\mu) = \frac{1}{3}\sum_{j=1}^{3} \delta_{\mu_j[\eta]}$$

with

$$\mu_j[\eta] = \prod_{i=1}^{\infty} \gamma[\eta(i)](\,\cdot\,|\nu_{j,u=u^*(\beta,q)})$$

since $\widehat{B}_{\nu_{1,u=0}} = 0$ lies in the convex hull of the three others

Concentration of the total empirical spin vector follows from finite-volume Sanov:

$$\mu_{F,n}[\eta(1),\dots,\eta(n)](d(L_n,\pi M^*)\geq\varepsilon)$$
$$\leq \prod_{b\in E'}(n\widehat{\pi}_n(b)+1)^{2|E|}\exp\left(-n\inf_{\substack{\widehat{\nu}\in\widehat{M}_n:\\ d(\widehat{\pi}_n\widehat{\nu},\pi M^*)\geq\varepsilon}}\Phi[\widehat{\pi}_n](\widehat{\nu})+n\inf_{\widehat{\nu}'\in\widehat{M}_n}\Phi[\widehat{\pi}_n](\widehat{\nu}')\right)$$

$\widehat{\pi}_n$: empirical field-type distribution

This explains the importance of the spin-rate-function $\Phi[\eta](\widehat{\nu})$

for not too atypical $\widehat{\pi}_n$.

How to get weights w_j?

Fluctuations of type-empirical distribution on CLT-scale:

$$X_{[1,n]}[\eta] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\delta_{\eta_i} - \pi) \to G$$

Define n-dependent *good-sets* $\mathcal{H}_n^{\delta_n}$ of the realization of the randomness

$$\mathcal{H}_{i,n}^{\delta_n} := \left\{ \eta \in (E')^n : X_{[1,n]}[\eta] \in R_{i,\delta_n} \right\}$$

$$\mathcal{H}_n^{\delta_n} := \bigcup_{i=1}^{k} \mathcal{H}_{i,n}^{\delta_n}$$

where $R_{i,\delta_n} := \{x \in T\mathcal{P}(E') : \langle x, B_i \rangle - \max_{k \neq i} \langle x, B_k \rangle > \delta_n\}$, and

(a) $\delta_n \downarrow 0$, but

(b) $\sqrt{n}\, \delta_n \uparrow \infty$

(a) Get full proba of $\mathcal{H}_n^{\delta_n}$ in the limit of $n \uparrow \infty$.

(b) Have concentration of $\widehat{L}_n$ around a given minimizer $\widehat{\nu}_j$ on $\mathcal{H}_{j,n}^{\delta_n}$.

Suppose F is a local function, depending on m coordinates of spins and random fields.

Then:

$$\lim_{n\uparrow\infty}\int_{\mathcal{H}_{j,n}^{\delta_n}}\mathbb{P}_\pi(d\eta)F(\mu_n[\eta],\eta)=w_j\int_{(E')^m}\pi^{\otimes m}(d\eta)F\Big(\prod_{i=1}^m\gamma[\eta(i)](\,\cdot\,|\pi\widehat{\nu}_j),\eta\Big)$$

Productification with only local influence of randomness conditional on stability region R_j.

Disorder variable: $\eta(i)$ taking values in a finite set E'

Markov chain transition matrix $M = (M(i,j)_{i,j\in E'})$, ergodic

Invariant distribution $\pi \in \mathcal{P}(E')$

Fact. For an ergodic finite state Markov chain, the standardized occupation time measure of the form $\sqrt{n}(\widehat{\pi}_n - \pi)$ converges in distribution, as n tends to infinity, to a centered Gaussian distribution G with a covariance matrix Σ_M on the $|E'| - 1$ dimensional vector space $T\mathcal{P}(E')$.

Warning: Ergodicity of the Markov chain does not imply that Σ_M has the full rank $|E'| - 1$

Consider the case $q = |E'| = 3$ of a general doubly stochastic matrix in the form

$$M = \begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & -1+a+b+c+d \end{pmatrix}, \quad a, b, c, d \in (0,1).$$

$$\Sigma_M = \begin{pmatrix} \frac{2}{9} + \frac{2(1+b(2-6c)+2c-2d+a(-5+6d))}{27(-1+a+bc+d-ad)} & -\frac{1}{9} - \frac{b(5-6c)+5c-2(1+d)+a(-2+6d)}{27(-1+a+bc+d-ad)} & -\frac{1}{9} - \frac{4-8a-b-c-6bc-2d+6ad}{27(-1+a+bc+d-ad)} \\ -\frac{1}{9} - \frac{b(5-6c)+5c-2(1+d)+a(-2+6d)}{27(-1+a+b+c+d-ad)} & \frac{2}{9} + \frac{2(1+b(2-6c)+2c-5d+a(-2+6d))}{27(-1+a+b+c+d-ad)} & -\frac{1}{9} - \frac{4-2a-b-c-6bc-8d+6ad}{27(-1+a+bc+d-ad)} \\ -\frac{1}{9} - \frac{4-8a-b-c-6bc-2d+6ad}{27(-1+a+bc+d-ad)} & -\frac{1}{9} - \frac{4-2a-b-c-6bc-8d+6ad}{27(-1+a+bc+d-ad)} & \frac{2}{9} - \frac{2(-4+b+c+6bc+a(5-6d)+5d)}{27(-1+a+bc+d-ad)} \end{pmatrix}$$

***THEOREM* 6.** *With Formentin, Reichenbachs (2011).*
Assume full rank occupation time covariance Σ_M*.*

Suppose the non-degeneracy conditions 1) and 2) on the spin model. Then the metastate on the level of the spin measures exists and

$$\kappa[\eta](d\mu) = \sum_{j=1}^{k} w_j \delta_{\mu_j[\eta]}(d\mu) \text{ for } \mathbb{P}_\pi\text{-a.e. } \eta.$$

The weights are $w_j = \mathbb{P}_{\Sigma_M}(G \in R_j)$ *where* G *is a centered gaussian on* $T\mathcal{P}(E')$ *with covariance* Σ_M*.*

Degenerate (but ergodic) Markov chain which also has the equidistribution as its invariant measure, nonreversible

$$M = \begin{pmatrix} 0 & 1 & 0 \\ p & 0 & 1-p \\ 1-p & 0 & p \end{pmatrix}$$

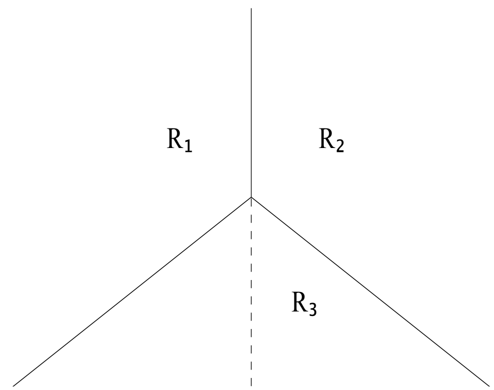


Figure 1: The Gaussian limiting distribution of $\sqrt{n}(\widehat{\pi}_n - \pi)$ concentrates on the dashed line that for upper half coincides with the boundary between the stability regions R_1 and R_2.

The metastate takes the following unusual form due to almost degeneracies:

THEOREM 7. *The Metastate in the 3-state random field Potts model defined above, driven by the degenerate MC above has the form*

$$\kappa[\eta] = \frac{1}{2}\delta_{\mu^3[\eta]}$$

$$+ \frac{1}{3}\delta_{\frac{1}{2}\mu^1[\eta]+\frac{1}{2}\mu^2[\eta]} + \frac{1}{9}\delta_{p(\beta,B)\mu^1[\eta]+(1-p(\beta,B))\mu^2[\eta]} + \frac{1}{18}\delta_{(1-p(\beta,B))\mu^1[\eta]+p(\beta,B)\mu^2[\eta]}$$

Here the function $p(\beta, B)$ is computable in terms of the mean-field parameter u and is strictly bigger than $1/2$ in the phase transition regime.

NO SYMMETRY BETWEEN STATE 1 AND STATE 2!

Since $N\widehat{\pi}_N(1) - N\widehat{\pi}_N(2) \in \{0, 1\}$

state 1 gets slightly bigger weight