The spectrum of dynamically defined operators

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Schrödinger operators

The discrete Laplacian acting on the square summable sequences $\ell^2(\mathbb{Z})$ is given by

$$\Delta\psi(n) = \psi(n+1) + \psi(n-1). \tag{1}$$

For a potential, i.e. a bounded sequence, $V : \mathbb{Z} \to \mathbb{R}$, we call $H = \Delta + V$ a Schrödinger operator.

For $V(n)$ i.i.d.r.v. with distribution supported in $[a, b]$, $H = \Delta + V$ is called the Anderson model. we have that the spectrum is given by

$$\sigma(H) = \mathrm{range}(V) + \sigma(\Delta) = [a-2, b+2]. \tag{2}$$

For $V(n) = 2\lambda\cos(2\pi(n\omega + x))$ with ω irrational and $\lambda \neq 0$, we have the Almost–Mathieu operator. Then the spectrum is always a Cantor set.

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Number theory of $\omega n \pmod 1$ and $\omega n^2 \pmod 1$

$\omega n \pmod 1$ and $\omega n^2 \pmod 1$ are both equidistributed in $[0,1]$.

Let $N \geq 2$ and define

$$\{\beta_1 < \beta_2 < \cdots < \beta_N\} = \{\omega n \pmod 1\}_{n=1}^N$$

and

$$\{\gamma_1 < \gamma_2 < \cdots < \gamma_N\} = \{\omega n^2 \pmod 1\}_{n=1}^N.$$

Then the set of lengths

$$\{l_j = \beta_{j+1} - \beta_j, \quad j = 1, \dots, N-1\}$$

consists of just three elements, whereas

$$\{\ell_j = \gamma_{j+1} - \gamma_j, \quad j = 1, \dots, N-1\}$$

obey Poisson statistics for *generic* ω.

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The skew-shift Schrödinger operator

Define the skew-shift $T : \mathbb{T}^2 \to \mathbb{T}^2$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$,

$$T(x, y) = (x + 2\omega, x + y) \pmod 1. \tag{3}$$

Then $\omega n^2 = T^n(\omega, 0)_2 \pmod 1$.

It thus makes sense instead of considering the potential $V(n) = f(n^2\omega \pmod 1)$, to consider potentials given by

$$V(n) = \lambda f(T^n(x, y)) \tag{4}$$

for $f : \mathbb{T}^2 \to \mathbb{R}$. These then form an ergodic family of potentials.

Conjecture: For sufficiently regular f, the spectrum of $\Delta + V$ consists of finitely many intervals and is Anderson localized. This means it behaves as in the random case.

Progress: Large coupling ($\lambda \gg 1$): Bourgain–Goldstein–Schlag, Bourgain, Bourgain–Jitomirskaya, K.
Small coupling ($0 < \lambda \ll 1$): Bourgain. **largely open**
Necessity of regularity: Avila–Bochi–Damanik, Boshernitzan–Damanik.

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$$V(n) = 2\lambda \cos(2\pi\omega n^2)$$

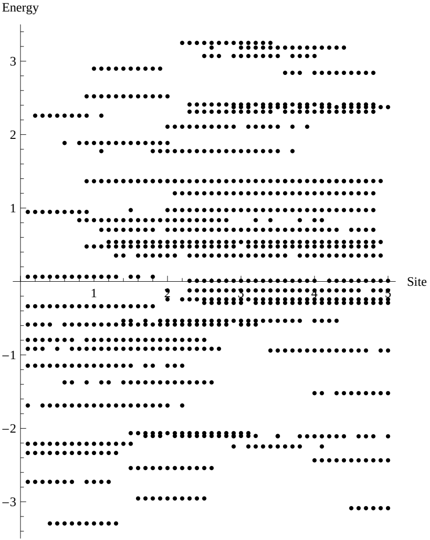
$$\lambda = 0.9$$
$$\omega = \sqrt{2}.$$

$$H : \ell^2([1,50]) \to \ell^2([1,50])$$

$$\begin{pmatrix} b(1) & 1 & & \\ 1 & b(2) & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & b(50) \end{pmatrix}$$

$$Hu_j = E_j u_j \text{ for } j = 1, \dots, 50$$

Black dot at (n, E_j) if $|u_j(n)| \geq 0.01$.



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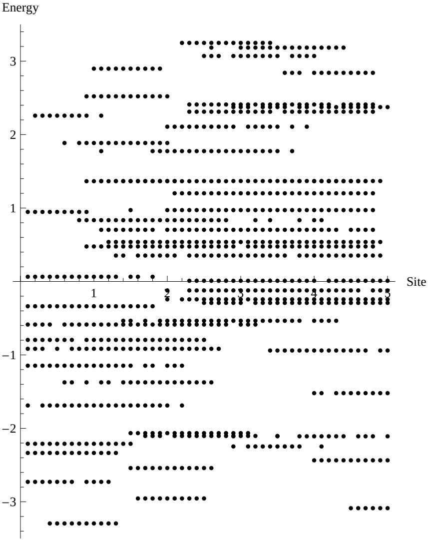

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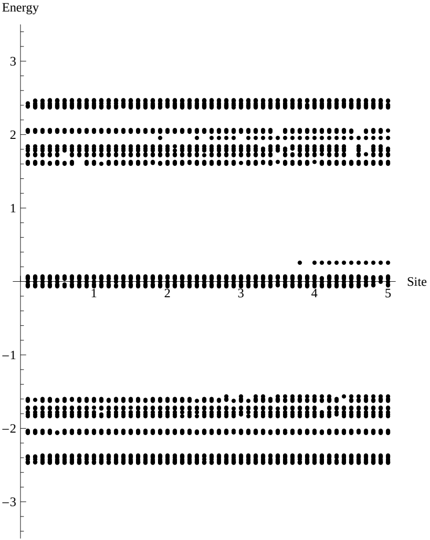

Verblunsky coefficients and orthogonal polynomials on the unit circle

It turns out that this problem can be solved explicitely for the unitary analog of Schrödinger operators: CMV matrices.

Given a sequence of Verblunsky coefficients

$$\alpha(n) \in \mathbb{D} = \{z \in \mathbb{C} : \quad |z| < 1\}. \tag{5}$$

Define a sequence of monic polynomials by the Szegő recursion

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha(n)}\Phi_n^*(z) \tag{6}$$

with $\Phi_n^*(z) = z^n\overline{\Phi_n(\overline{z}^{-1})}$.

Then there exists an unique probability measure μ supported on $\partial\mathbb{D}$ such that Φ_n are the polynomials obtained by orthogonalizing $1, z, \dots$ in $L^2(\partial\mathbb{D}, \mu)$.

One can also view μ as the spectral measure of the CMV matrix $\mathcal{C}$ corresponding to the Verblunsky coefficients $\alpha(n)$.

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Rotating the Verblunsky coefficients

Recall

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha(n)}\Phi_n^*(z). \tag{7}$$

Define $\tilde{\alpha}(n) = \mathrm{e}^{-2\pi\mathrm{i}\theta(n+1)}\alpha(n)$. Then one has for $\widetilde{\Phi}_n(z) = \mathrm{e}^{2\pi\mathrm{i}n\theta}\Phi_n(\mathrm{e}^{-2\pi\mathrm{i}\theta}z)$ that

$$\widetilde{\Phi}_{n+1}(z) = z\widetilde{\Phi}_n(z) - \overline{\tilde{\alpha}(n)}\widetilde{\Phi}_n^*(z). \tag{8}$$

Hence we see that the measure $\tilde{\mu}$ corresponding to $\tilde{\alpha}$ is just the measure μ rotated by $\mathrm{e}^{2\pi\mathrm{i}\theta}$.

In particular, that

$$\sigma(\mathcal{C}) = \mathrm{e}^{2\pi\mathrm{i}\theta}\sigma(\widetilde{\mathcal{C}}). \tag{9}$$

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Skew-shift Verblunsky coefficients

Given $\lambda \in \mathbb{D}$, define the function $f(x, y) = \lambda \mathrm{e}^{2\pi \mathrm{i} y}$ and the Verblunsky coefficients

$$\alpha_{x,y}(n) = f(T^n(x, y)) = \lambda \mathrm{e}^{2\pi \mathrm{i}(\omega n(n-1)+nx+y)}. \qquad (10)$$

From the previous results, we have

$$\sigma(\mathcal{C}_{\tilde{x},y}) = \mathrm{e}^{2\pi \mathrm{i}(x-\tilde{x})}\sigma(\mathcal{C}_{x,y}). \qquad (11)$$

From minimality of the skew-shift, we have

$$\sigma(\mathcal{C}_{\tilde{x},\tilde{y}}) = \sigma(\mathcal{C}_{x,y}). \qquad (12)$$

Since $\sigma(\mathcal{C}_{x,y}) \subseteq \partial\mathbb{D}$ is non-empty, we obtain

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For every x, y, we have

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Anderson localization

Using similar arguments, one also obtains

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Let $\lambda \in \mathbb{D}$ and $x \in \mathbb{R}$. For almost-every $y \in \mathbb{R}$, the CMV matrix $\mathcal{C}_{x,y}$ has pure point spectrum with exponentially decaying eigenfunctions.

This is what is called Anderson localization.

Proof: Define the Lyapunov exponent

$$L(z) = \lim_{N\to\infty} \frac{1}{N} \int_{\mathbb{T}^2} \log \|A_N^z(x,y)\| d(x,y) \qquad (14)$$

where $A_N^z(x,y) = A^z(T^n(x,y)) \cdots A^z(x,y)$ is the transfer matrix

$$A^z(x,y) = \frac{1}{\sqrt{1-|\lambda|^2}} \begin{pmatrix} z & -\overline{\lambda} \mathrm{e}^{-2\pi \mathrm{i} y} \\ -\lambda \mathrm{e}^{2\pi \mathrm{i} y} z & 1 \end{pmatrix}. \qquad (15)$$

One then shows $L(\mathrm{e}^{2\pi \mathrm{i} t}) = L(\mathrm{e}^{2\pi \mathrm{i} s})$ as before. Since $\alpha(n) \neq 0$, one thus must have $L(\mathrm{e}^{2\pi \mathrm{i} t}) > 0$. Standard results then imply the localization claim.

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Using similar arguments, one also obtains

Theorem

Let $\lambda \in \mathbb{D}$ and $x \in \mathbb{R}$. For almost-every $y \in \mathbb{R}$, the CMV matrix $\mathcal{C}_{x,y}$ has pure point spectrum with exponentially decaying eigenfunctions.

This is what is called Anderson localization.

Proof: Define the Lyapunov exponent

$$L(z) = \lim_{N\to\infty} \frac{1}{N} \int_{\mathbb{T}^2} \log \|A_N^z(x,y)\| d(x,y) \tag{14}$$

where $A_N^z(x,y) = A^z(T^n(x,y)) \cdots A^z(x,y)$ is the transfer matrix

$$A^z(x,y) = \frac{1}{\sqrt{1-|\lambda|^2}} \begin{pmatrix} z & -\overline{\lambda}\mathrm{e}^{-2\pi \mathrm{i} y} \\ -\lambda \mathrm{e}^{2\pi \mathrm{i} y} z & 1 \end{pmatrix}. \tag{15}$$

One then shows $L(\mathrm{e}^{2\pi \mathrm{i} t}) = L(\mathrm{e}^{2\pi \mathrm{i} s})$ as before. Since $\alpha(n) \neq 0$, one thus must have $L(\mathrm{e}^{2\pi \mathrm{i} t}) > 0$. Standard results then imply the localization claim.

Future work

It should be tedious but possible to extend these results to Verblunsky coefficients of the form

$$\alpha_{x,y}(n) = \lambda \mathrm{e}^{2\pi \mathrm{i}(\omega n(n-1)+nx+y)} + \varepsilon g(T^n(x,y)), \qquad (16)$$

where $g : \mathbb{T}^2 \to \mathbb{R}$ is real-analytic and $\varepsilon > 0$ is small enough.

More interestingly, one should be able to compute the eigenvalue statistics of this operator.

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