Free energy of a particle in high-dimensional Gaussian potentials with isotropic increments

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Gaussian fields with isotropic increments

- **Gaussian random field**: $X_N = \{X_N(u) : u \in \mathbb{R}^N\}$.
- $X_N(u)$ centred **Gaussian**, $u \in \mathbb{R}^N$.
- **Isotropic increments**:

$$\mathbb{E}\left[(X_N(u) - X_N(v))^2\right] = D\left(\frac{1}{N}\|u-v\|_2^2\right) =: D_N(\|u-v\|_2^2), \quad u,v \in \mathbb{R}^N.$$

- **NB!** $N \gg 1$.
- **Any (admissible)** $D : \mathbb{R}_+ \to \mathbb{R}_+$.

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Complete classification of the correlators

A.M. Yaglom (1957):

1. **Isotropic field:**

$$\mathbb{E}\left[X_N(u)X_N(v)\right] = B\left(\frac{1}{N}\|u-v\|_2^2\right), \quad u,v \in \Sigma_N,$$

$$B(r) = c_0 + \int_0^{+\infty} \exp\left(-t^2 r\right) \nu(\mathrm{d}t),$$

$c_0 \in \mathbb{R}_+$, $\nu \in \mathscr{M}_{\text{finite}}(\mathbb{R}_+)$.

$$D(r) = 2(B(0) - B(r)).$$

2. **Non-isotropic field with isotropic increments:**

$$D(r) = \int_0^{+\infty} \left[1 - \exp\left(-t^2 r\right)\right] \nu(\mathrm{d}t) + A \cdot r, \quad r \in \mathbb{R}_+,$$

$A \in \mathbb{R}_+$, $\nu \in \mathscr{M}((0;+\infty))$

$$\int_0^{+\infty} \frac{t^2 \nu(\mathrm{d}t)}{t^2+1} < \infty.$$

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A particle subjected to a rugged potential:

- **Particle state space**:

$$S_N := S^N, \quad S \subset \mathbb{R}, \quad \textbf{or} \quad S_N := \{u \in \mathbb{R}^N : \|u\|_2 \le L\sqrt{N}\}, \quad L > 0.$$

- **Partition function**:

$$Z_N(\beta) := \int_{S_N} \mu_N(\mathrm{d}u) \exp\left(\beta \sqrt{N} X_N(u)\right), \quad \beta \in \mathbb{R}_+.$$

- **Log-partition function**:

$$p_N(\beta) := \frac{1}{N} \log Z_N(\beta).$$

- **Q:**

$$\lim_{N \to +\infty} p_N(\beta) =: p(\beta) = \text{?}$$

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Parisi-type functional

- **Regularised derivative:**

$$D'^{,M}(r) := \begin{cases} D'(r), & r \in [1/M; +\infty), \\ M, & r \in [0; 1/M). \end{cases}$$

- **Parisi terminal value problem:**

$$\begin{cases} \partial_q f(y,q) + \frac{1}{2} D'^{,M}(2(r-q)) \left(\partial^2_{qq} f(y,q) + x(q) \left(\partial_y f(y,q) \right)^2 \right) = 0, \\ f(y,1) = h(y), \quad q \in (0,r), \quad y \in \mathbb{R}. \end{cases}$$

- **Spin glass order parameter:**
  $x \in \mathscr{X}(r) := \{ x : [0,r] \to [0,1] \mid$ càdlàg $\uparrow, x(0) = 0, x(r) = 1 \}$.
- **Boundary conditions (product state space)**

$$h_\lambda(y) := \log \int_S \mu(\mathrm{d}u) \exp\left(\beta u y + \lambda u^2 \right), \quad y \in \mathbb{R}, \quad \lambda \in \mathbb{R}.$$

- **Parisi-type functional:**

$$\mathscr{P}(\beta, r)[x] := \lim_{M \uparrow +\infty} \left(\inf_{\lambda \in \mathbb{R}} \left[f^{(M)}_{r,x,h_\lambda}(0,0) - \lambda r \right] - \frac{\beta^2}{2} \int_0^1 x(q) \mathrm{d}\theta^{(M)}_r(q) \right),$$

$$\theta^{(M)}(q) := -q D'^{,M}(q) - D(q), \quad q \in \mathbb{R}_+.$$

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Variational formula

Effective size of the state space:

$$d := \sup_N \left(\frac{1}{N} \sup_{u \in S_N} \|u\|_2^2 \right).$$

Theorem

$$p(\beta) := \sup_{r\in[0;d]} \inf_{x\in\mathscr{X}(r)} \mathscr{P}(\beta,r)[x], \quad \textit{almost surely and in } L^1.$$

Heuristics: "localisation"

- **Covariance structure:**

$$\mathbb{E}\left[X_N(u)X_N(v)\right] = \frac{1}{2}\left(D_N(\|u\|_2^2) + D_N(\|v\|_2^2) - D_N(\|u-v\|_2^2)\right), \quad u,v \in \mathbb{R}^N.$$

- **Overlap:**

$$\langle u,v\rangle_N := \frac{1}{N}\sum_{i=1}^N u_i v_i, \quad u,v \in \mathbb{R}^N.$$

- **Fix** $r \in [0;d]$**:**

$$\mathbb{E}\left[X_N(u)X_N(v)\right] = D(r) - \frac{1}{2}D(2(r - \langle u,v\rangle_N)),$$
$$\|u\|_2^2 = \|v\|_2^2 = rN.$$

- $\Rightarrow$ **Localisation.**

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Particle in a rotationally invariant box

- **Particle state space:**

$$S_N := \{u \in \mathbb{R}^N : \|u\|_2 \le L\sqrt{N}\}.$$

- **A priori measure:** $\mu_N \in \mathscr{M}_{\text{finite}}(S_N)$ **:**

$$\frac{\mathrm{d}\mu_N}{\mathrm{d}\lambda_N}(u) := \exp\left(\sum_{i=1}^N f(u_i)\right), \quad u = (u_i)_{i=1}^N \in \mathbb{R}^N, \quad f : \mathbb{R} \to \mathbb{R},$$

$f(u) := h_1 u - h_2 u^2$, $h_1 \in \mathbb{R}$, $h_2 \in \mathbb{R}_+$.

- **Fyodorov, Sommers (2007):**

$$\mathscr{C}\mathscr{S}(\beta, r)[x] := \frac{1}{2}\left[\log(r - q_{\max}) + \int_0^{q_{\max}} \frac{\mathrm{d}q}{\int_q^r x(s)\mathrm{d}s} + h_1^2 \int_0^r x(q)\mathrm{d}q - h_2 r\right]$$
$$+ \frac{\beta^2}{2}\left(D'(2(r - q_{\max})) + \int_0^{q_{\max}} D'(2(r-q))x(q)\mathrm{d}q\right),$$
$$x \in \mathscr{X}(r).$$

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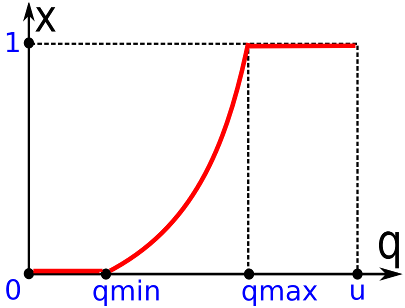
A test function



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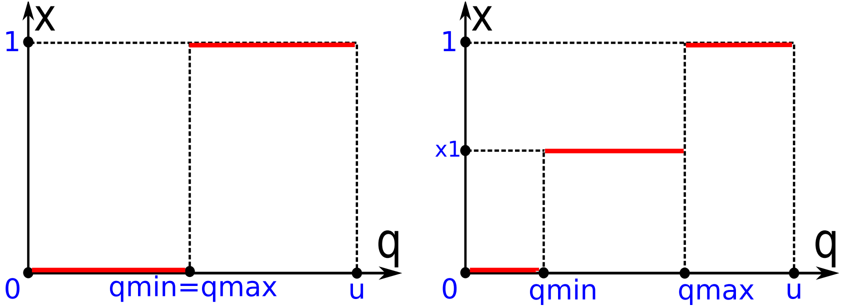
Short range: solution of the variational problem

Short range: $D(r) := B(0) - B(r)$

$$\boxed{3[D'''(r)]^2/2 - D''(r)D'''(r) =: S(r) > 0, \quad u \in \mathbb{R}_+,}$$

Derrida's random energy model (REM) behaviour:



$\beta \in [0; \beta_c) \Rightarrow$ **RS optimiser** $\qquad \beta \in (\beta_c; +\infty] \Rightarrow$ **1-RSB optimiser**

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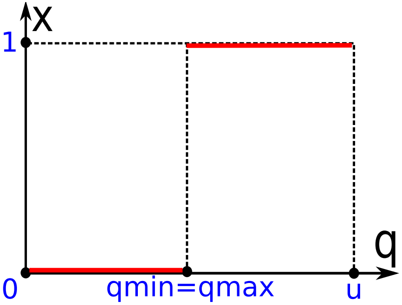
Long range: solution of the variational problem

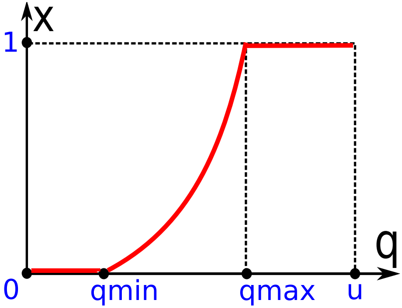
Long range: if **D** satisfies

$$\boxed{{\color{red}S}(r) < 0, \quad u \in \mathbb{R}_+,}$$

Full RSB:



$\beta \in [0; \beta_c) \Rightarrow$ **RS optimiser**



$\beta \in (\beta_c; +\infty] \Rightarrow$ **FRSB optimiser**

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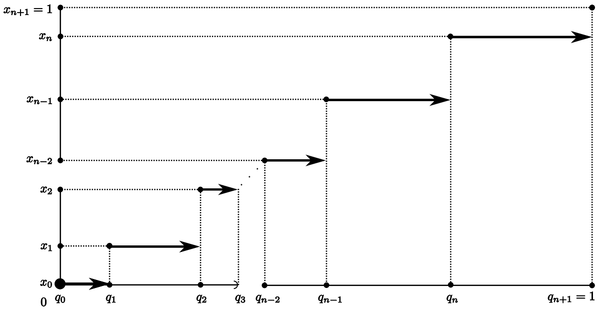
Critical range: logarithmic correlations

Long range: **D** satisfies

$$\boxed{{\color{red}S}(r) = 0, \quad u \in \mathbb{R}_+,}$$

- $D(r) = \log(c+r)$, $c > 0 \Rightarrow$ **REM-behaviour** (at the level of $\mathbb{E}$)
- $D(r) = \sum_{k=0}^{n+1} K_i \log(c_i + r)$ ($c_0 > c_1 > \ldots > c_{n+1}$ and $K_i > 0$) $\Rightarrow$ **generalised REM-behaviour** (n-RSB):



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Sketch of proof

Compare with a class of hierarchically correlated fields.

- **“Generalised random energy model”**: $r \in [0;d]$:

$$\mathrm{Cov}\left[a(\alpha^{(1)}), a(\alpha^{(2)})\right] = -D'(2(r - q(\alpha^{(1)}, \alpha^{(2)}))), \quad \alpha^{(1)}, \alpha^{(2)} \in \mathbb{N}^n,$$

  where

$$0 = q_0 < q_1 < \ldots < q_n < q_{n+1} = r,$$

  ultrametric overlap:

$$q(\alpha^{(1)}, \alpha^{(2)}) := q_{\max\left\{k \in [1;n] \cap \mathbb{N} : [\alpha^{(1)}]_k = [\alpha^{(2)}]_k\right\}}.$$

- **Comparison process:**

$$A(u, \alpha) := \sum_{i=1}^{N} u_i a_i(\alpha), \quad u \in S_N, \quad \alpha \in \mathbb{N}^n.$$

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Multiplicative probability cascades

- Let $\{g_i\}_{i=1}^{2^N}$ be i.i.d. r.v. of **Gumbel** extremal-type (say, Gaussian)

$$\sum_{i=1}^{2^N} \delta_{\exp(u_N(g_i)/x)} \xRightarrow[N\to+\infty]{} \xi(x), \quad x \in (0;1).$$

  where $u_N(y) = a_N y + b_N$ is the **linear extreme value normalisation**.
- **Ruelle (1987)**:

$$\xi(x) := \{\delta_{\xi(x)_i}\}_{i\in\mathbb{N}} := \mathrm{PPP}(\mathbb{R}_+ \ni t \mapsto xt^{-x-1}), \quad x \in (0;1)$$

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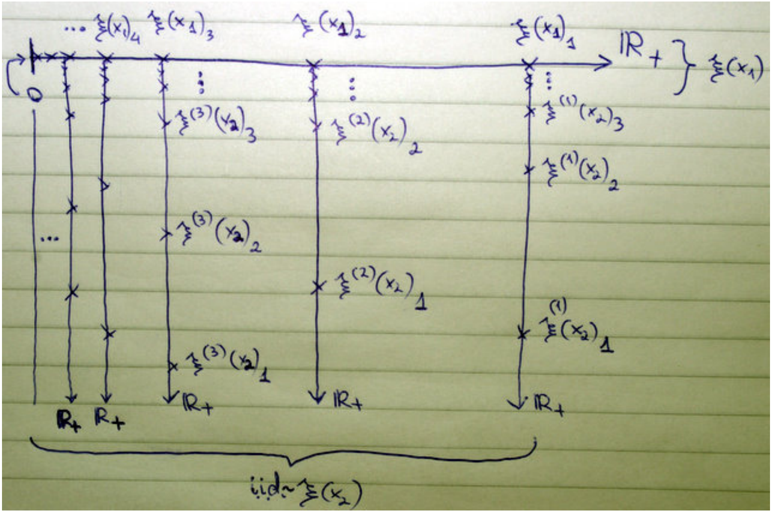
Cascades

Ruelle's probability cascades (RPC):

$$\left\{\xi(x_1)_{i_1}\cdot\xi^{(i_1)}(x_2)_{i_2}\cdot\xi^{(i_1,i_2)}(x_3)_{i_3}\cdots\xi^{(i_1,i_2,\dots,i_{n-1})}(x_n)_{i_n} : i=(i_1,\dots,i_n)\in\mathbb{N}^n\right\}$$
$$=: \mathrm{RPC}(x_1,\dots,x_n),$$

where $0 < x_1 < \ldots < x_n < 1$, $\xi^{(i_1,i_2,\dots,i_{k-1})}(x_k)$ i.i.d. $\xi(x_k)$.

RPC construction (sketch)



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Comparison

- **Interpolation**:

$$H_t(u,\alpha) := \sqrt{t}X(u) + \sqrt{1-t}A(u,\alpha), \quad t \in [0;1], \quad u \in S_N, \quad \alpha \in \mathbb{N}^n.$$

- **Extended free energy functional**:

$$\Phi_N(x)[H_t] := \frac{1}{N}\mathbb{E}\left[\log\left(\int_{S_N}\mu_N(\mathrm{d}u)\sum_{\alpha\in\mathbb{N}^n}\mathrm{RPC}(x)_\alpha \exp\left(\beta\sqrt{N}H_t(u,\alpha)\right)\right)\right].$$

- **Fundamental theorem of calculus:**

$$p_N(\beta) = \Phi_N(x)[H_1] = \Phi_N(x)[H_0] + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Phi_N(x)[H_t]\mathrm{d}t,$$

where

  - **nonlinear summand** $= \Phi_N(x)[H_0]$.
  - **linear summand** + (annoying) **remainder** $= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Phi_N(x)[H_t]$.

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