Ising models on power-law random graphs

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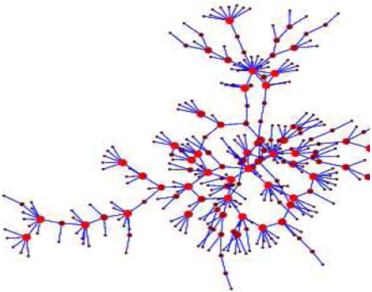
Prague School on Mathematical Statistical Physics
August 29–September 9, 2011

Where innovation starts

Introduction

There are many complex real-world networks, e.g.,

- Social networks (friendships, business relationships, *sexual contacts,* ...);
- Information networks (World Wide Web, citations, ...);
- Technological networks (Internet, airline routes, ...);
- Biological networks (protein interactions, neural networks,...).



Sexual network Colorado Springs, USA (Potterat, et al., '02)

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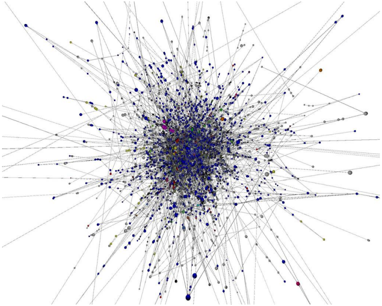

Small part of the Internet (http://www.fractalus.com/steve/stuff/ipmap/)

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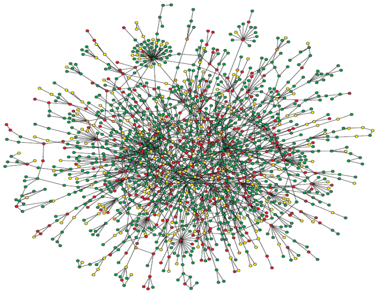
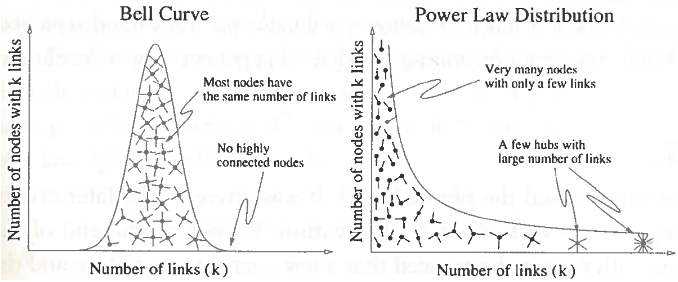


Yeast protein interaction network
(Jeong, et al., '01)

Properties of complex networks

Power-law behavior

Number of vertices with degree k is proportional to $k^{-\tau}$.



Barabási, Linked, '02

Small worlds

Distances in the network are small

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Ising model

Ising model: paradigm model in statistical physics for *cooperative behavior.*

When studied on complex networks it can model for example *opinion spreading* in society.

We will model complex networks with *power-law random graphs.*

What are effects of *structure* of complex networks on *behavior* of Ising model?

Definition of the Ising model

On a graph G_n, the *ferromagnetic Ising model* is given by the following Boltzmann distributions over $\sigma \in \{-1, +1\}^n$,

$$\mu(\sigma) = \frac{1}{Z_n(\beta, B)} \exp\left\{\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B \sum_{i\in[n]} \sigma_i\right\},$$

where

- $\beta \geq 0$ is the inverse temperature;
- B is the external magnetic field;
- $Z_n(\beta, B)$ is a normalization factor (the *partition function*), i.e.,

$$Z_n(\beta, B) = \sum_{\sigma\in\{-1,1\}^n} \exp\left\{\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B \sum_{i\in[n]} \sigma_i\right\}.$$

Power-law random graphs

In the *configuration model* a graph $G_n = (V_n = [n], E_n)$ is constructed as follows.

- Let D have a certain distribution (the *degree distribution*);
- *Assign* D_i half-edges to each vertex $i \in [n]$, where D_i are i.i.d. like D (Add one half-edge to last vertex when the total number of half-edges is odd);
- *Attach* first half-edge to another half-edge *uniformly at random;*
- Continue until all half-edges are connected.

Special attention to power-law degree sequences, i.e.,

$$\mathbb{P}[D \geq k] \leq ck^{-(\tau-1)}, \qquad \tau > 2.$$

Local structure configuration model for $\tau > 2$

Start from random vertex i which has degree D_i.

Look at neighbors of vertex i, probability such a neighbor has degree $k+1$ is approximately,

$$\frac{(k+1)\sum_{j\in[n]} \mathbb{1}_{\{D_j=k+1\}}}{\sum_{j\in[n]} D_j}$$

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Let K have distribution (the *forward degree* distribution),

$$\mathbb{P}[K=k] = \frac{(k+1)\mathbb{P}[D=k+1]}{\mathbb{E}[D]}.$$

Locally tree-like structure: a branching process with offspring D in first generation and K in further generations. Also, *uniformly sparse.*

Pressure in thermodynamic limit ($\mathbb{E}[K] < \infty$)

Theorem (Dembo, Montanari, '10)

For a locally tree-like *and* uniformly sparse *graph sequence* $\{G_n\}_{n\geq 1}$ *with* $\mathbb{E}[K] < \infty$, *the pressure per particle,*

$$\psi_n(\beta, B) = \frac{1}{n} \log Z_n(\beta, B),$$

converges, for $n \to \infty$, *to*

$$\varphi_h(\beta, B) \equiv \frac{\mathbb{E}[D]}{2} \log\cosh(\beta) - \frac{\mathbb{E}[D]}{2} \mathbb{E}[\log(1 + \tanh(\beta)\tanh(h_1)\tanh(h_2))] + \mathbb{E}\left[\log\left(e^B \prod_{i=1}^{D} \{1 + \tanh(\beta)\tanh(h_i)\} + e^{-B} \prod_{i=1}^{D} \{1 - \tanh(\beta)\tanh(h_i)\}\right)\right].$$

Pressure in thermodynamic limit ($\mathbb{E}[D] < \infty$)

Theorem (DGvdH, '10)

Let $\tau > 2$. Then, in the configuration model, the pressure per particle,

$$\psi_n(\beta, B) = \frac{1}{n} \log Z_n(\beta, B),$$

converges almost surely, for $n \to \infty$, to

$$\varphi_h(\beta, B) \equiv \frac{\mathbb{E}[D]}{2} \log\cosh(\beta) - \frac{\mathbb{E}[D]}{2} \mathbb{E}[\log(1 + \tanh(\beta)\tanh(h_1)\tanh(h_2))]$$
$$+ \mathbb{E}\left[\log\left(e^B \prod_{i=1}^{D} \{1 + \tanh(\beta)\tanh(h_i)\}\right.\right.$$
$$\left.\left.+ e^{-B} \prod_{i=1}^{D} \{1 - \tanh(\beta)\tanh(h_i)\}\right)\right].$$

Tree recursion

Proposition

Let K_t be i.i.d. like K and $B > 0$. Then, the recursion

$$h^{(t+1)} \stackrel{d}{=} B + \sum_{i=1}^{K_t} \operatorname{atanh}(\tanh(\beta)\tanh(h_i^{(t)})),$$

has a unique fixed point h^*_β.

Interpretation: the *effective field* of a vertex in a *tree* expressed in that of its neighbors.

Uniqueness shown by showing that effect of *boundary conditions* on generation t vanishes for $t \to \infty$.

Correlation inequalities

Lemma (Griffiths, ’67, Kelly, Sherman, ’68)

For a ferromagnet with positive external field, the magnetization *at a vertex will not decrease, when*

- *The* number of edges *increases;*
- *The external* magnetic field *increases;*
- *The* temperature *decreases.*

Lemma (Griffiths, Hurst, Sherman, ’70)

For a ferromagnet with positive external field, the magnetization *is* concave *in the external fields, i.e.,*

$$\frac{\partial^2}{\partial B_k \partial B_\ell} m_j(\underline{B}) \leq 0.$$

Outline of the proof

$$\lim_{n\to\infty} \psi_n(\beta, B)$$

$$= \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \left[\psi_n(0, B) + \int_0^{\varepsilon} \frac{\partial}{\partial\beta'} \psi_n(\beta', B) \mathrm{d}\beta' + \int_{\varepsilon}^{\beta} \frac{\partial}{\partial\beta'} \psi_n(\beta', B) \mathrm{d}\beta' \right]$$

$$= \varphi_h(0, B) + 0 + \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{\beta} \frac{\partial}{\partial\beta'} \varphi(\beta', B) \mathrm{d}\beta'$$

$$= \varphi_h(\beta, B).$$

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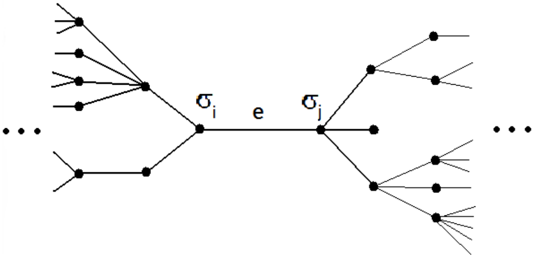
Internal energy

$$\frac{\partial}{\partial\beta}\psi_n(\beta, B) = \frac{1}{n}\sum_{(i,j)\in E_n}\left\langle\sigma_i\sigma_j\right\rangle_\mu = \frac{|E_n|}{n}\frac{\sum_{(i,j)\in E_n}\left\langle\sigma_i\sigma_j\right\rangle_\mu}{|E_n|}$$

$$\longrightarrow \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle\sigma_i\sigma_j\right\rangle_\mu\right]$$



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Internal energy

$$\frac{\partial}{\partial \beta}\psi_n(\beta, B) = \frac{1}{n}\sum_{(i,j)\in E_n} \left\langle \sigma_i \sigma_j \right\rangle_\mu = \frac{|E_n|}{n}\frac{\sum_{(i,j)\in E_n}\left\langle \sigma_i \sigma_j \right\rangle_\mu}{|E_n|}$$

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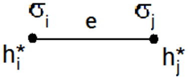

$$\frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle \sigma_i \sigma_j \right\rangle_\mu\right] \longrightarrow \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle \sigma_i \sigma_j \right\rangle_e\right]$$

Derivative of φ

$$\frac{\partial}{\partial\beta}\varphi_{h^*_\beta}(\beta, B) = \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle\sigma_i\sigma_j\right\rangle_e\right].$$

$$\varphi_h(\beta, B) = \frac{\mathbb{E}[D]}{2}\log\cosh(\beta) - \frac{\mathbb{E}[D]}{2}\mathbb{E}[\log(1+\tanh(\beta)\tanh(h_1)\tanh(h_2))]$$
$$+\mathbb{E}\left[\log\left(e^B\prod_{i=1}^{D}\{1+\tanh(\beta)\tanh(h_i)\} + e^{-B}\prod_{i=1}^{D}\{1-\tanh(\beta)\tanh(h_i)\}\right)\right]$$

- Show that we can ignore dependence of h^*_β on β;
  (*Interpolation* techniques. Split analysis into two parts, one for *small degrees* and one for *large degrees*)
- Compute the derivative with assuming β fixed in h^*_β.

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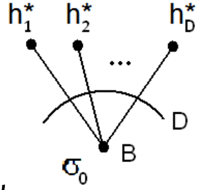
Thermodynamic quantities

Corollary

Let $\tau > 2$. Then, in the configuration model, a.s.:
The magnetization *is given by*

$$m(\beta, B) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \langle \sigma_i \rangle_\mu = \frac{\partial}{\partial B} \varphi_{h^*}(\beta, B) = \mathbb{E}\left[\langle \sigma_0 \rangle_{\nu_{D+1}} \right].$$



The susceptibility *is given by*

$$\chi(\beta, B) \equiv \lim_{n\to\infty} \frac{\partial M_n(\beta, B)}{\partial B} = \frac{\partial^2}{\partial B^2} \varphi_{h^*}(\beta, B).$$

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Define the *magnetization* on G_n as

$$m_n(\beta, B) = \frac{1}{n} \sum_{i=1}^{n} \langle \sigma_i \rangle_\mu.$$

Then, the *spontaneous magnetization*,

$$m(\beta, 0+) = \lim_{B \downarrow 0} m(\beta, B) \begin{cases} = 0, & \beta < \beta_c; \\ > 0, & \beta > \beta_c. \end{cases}$$

The *critical inverse temperature* β_c is given by

$$\mathbb{E}[K](\tanh \beta_c) = 1.$$

Note that, for $\tau \in (2, 3)$, we have $\mathbb{E}[K] = \infty$, so that $\beta_c = 0$.

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Critical exponents

Predictions by physicists (e.g. Leone, Vázquez, Vespignani, Zecchina, '02).

Critical behavior of *magnetization* m, and *susceptibility* χ.

	$m(\beta, 0^+), \beta \downarrow \beta_c$	$m(\beta_c, B), B \downarrow 0$	$\chi(\beta, 0^+), \beta \downarrow \beta_c$
$\tau > 5$	$\sim (\beta - \beta_c)^{1/2}$	$\sim B^{1/3}$	$\sim (\beta - \beta_c)^{-1}$
$\tau \in (3, 5)$	$\sim (\beta - \beta_c)^{1/(\tau-3)}$	$\sim B^{1/(\tau-2)}$	
$\tau \in (2, 3)$	$\sim (\beta - \beta_c)^{1/(3-\tau)}$	$\sim B^{1}$	$\sim (\beta - \beta_c)^{1}$

Distances in power-law random graphs

Let H_n be the graph distance between two *uniformly chosen connected* vertices in the configuration model. Then:

- For $\tau > 3$ and $\mathbb{E}[K] > 1$ (vdH, Hooghiemstra, Van Mieghem, '05),

$$H_n \sim \log n,$$

- For $\tau \in (2, 3)$ (vdH, Hooghiemstra, Znamenski, '07),

$$H_n \sim \log \log n;$$