Spectral zeta function & quantum statistical mechanics on Sierpinski carpets

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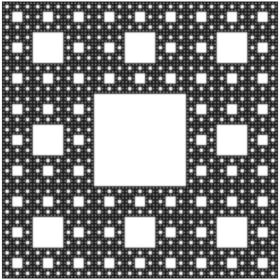
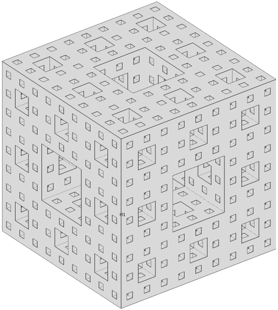
Generalized Sierpinski carpets



Sierpinski carpet

($m_F = 8$, $d_{\mathrm{H}} = \log 8 / \log 3$)



Menger sponge

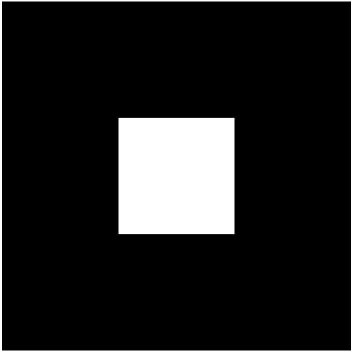
($m_F = 20$, $d_{\mathrm{H}} = \log 20 / \log 3$)

- Constructed via an IFS of affine contractions $\{f_i\}_{i=1}^{m_F}$.
- They are infinitely ramified fractals. (*Translation:* Analysis is hard.)
- Brownian motion and harmonic analysis on SCs have been studied extensively by Barlow & Bass (1989 onwards) and Kusuoka & Zhou (1992).
- Uniqueness of BM on SCs [Barlow, Bass, Kumagai & Teplyaev 2008].

Outline

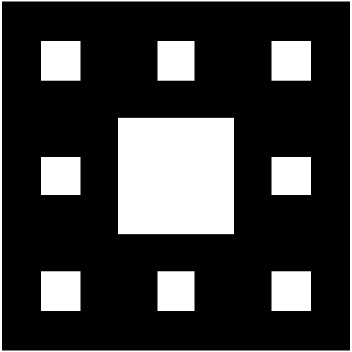
1. What we know about diffusion on Sierpinski carpets
   - Hausdorff dimension, walk dimension, and spectral dimension
   - Estimates of the heat kernel trace
2. Spectral zeta function on Sierpinski carpets
   - Simple poles give the carpet's "complex dimensions"
   - Meromorphic continuation to $\mathbb{C}$
3. Applications: Ideal quantum gas in Sierpinski carpets
   - Bose-Einstein condensation & its connection to Brownian motion

Dirichlet energy in the Barlow-Bass construction



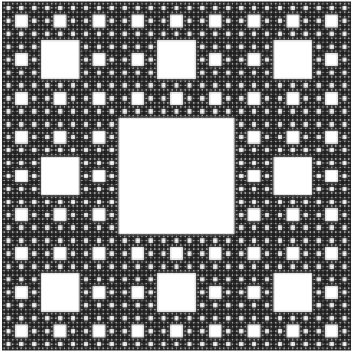
Reflecting BM W_t^1, Dirichlet energy $E_1(u) = \int_{F_1} |\nabla u(x)|^2 \mu_1(dx)$.

Dirichlet energy in the Barlow-Bass construction



Reflecting BM W_t^2, Dirichlet energy $E_2(u) = \int_{F_2} |\nabla u(x)|^2 \mu_2(dx)$.

Dirichlet energy in the Barlow-Bass construction



Time-scaled BM $X_t^n = W_{a_n t}^n$, DF $\mathcal{E}_n(u) = a_n \int_{F_n} |\nabla u(x)|^2 \mu_n(dx)$.

Dirichlet energy in the Barlow-Bass construction

$$\text{Time-scaled BM } X_t^n = W_{a_n t}^n, \text{ DF } \mathcal{E}_n(u) = a_n \int_{F_n} |\nabla u(x)|^2 \mu_n(dx).$$

- $\mu_n \rightharpoonup \mu = C(d_H\text{-dim Hausdorff measure})$ on carpet F.
- $a_n \asymp \left(\frac{m_F \rho_F}{l_F^2}\right)^n$, $\rho_F =$ resistance scale factor. No closed form expression known.
- BB showed that there exists subsequence $\{n_j\}$ such that, resp., the laws and the resolvents of X^{n_j} are tight. Any such limit process is a BM on the carpet F.
- If X is a limit process and T_t its semigroup, define the Dirichlet form on $L^2(F)$ by

$$\mathcal{E}_{BB}(u) = \sup_{t>0} \frac{1}{t} \langle u - T_t u, u \rangle \quad \text{with natural domain.}$$

Denote by Δ the corresponding Laplacian. Note $\mathcal{E}_{BB}$ is self-similar:

$$\mathcal{E}_{BB}(u) = \sum_{i=1}^{m_F} \rho_F \cdot \mathcal{E}_{BB}(u \circ f_i).$$

Heat kernel estimates on GSCs

Theorem (Barlow, Bass, $\cdots$)

$$p_t(x,y) \asymp C_1 t^{-d_{\mathrm{H}}/d_{\mathrm{W}}} \exp\left(-C_2 \left(\frac{|x-y|^{d_{\mathrm{W}}}}{t}\right)^{\frac{1}{d_{\mathrm{W}}-1}}\right).$$

Here $d_{\mathrm{H}} = \log m_F / \log l_F$ *(Hausdorff),* $d_{\mathrm{W}} = \log(\rho_F m_F)/\log l_F$ *(walk).*

$d_{\mathrm{S}} = 2\dfrac{d_{\mathrm{H}}}{d_{\mathrm{W}}} = 2\dfrac{\log m_F}{\log(m_F \rho_F)}$ is the spectral dimension of the carpet.

For any carpet, $d_{\mathrm{W}} > 2$ and $1 < d_{\mathrm{S}} < d_{\mathrm{H}}$, indicative of sub-Gaussian diffusion.

Theorem (Hambly '08, Kajino '08)

There exists a $(\log \rho_F)$*-periodic function* G*, bounded away from* 0 *and* ∞*, such that the heat kernel trace*

$$K(t) := \int_F p_t(x,x)\mu(dx) = t^{-d_{\mathrm{H}}/d_{\mathrm{W}}}\left[G(-\log t) + o(1)\right] \quad \text{as } t \downarrow 0.$$

A better estimate of heat kernel trace

Consider GSCs with Dirichlet b.c. on exterior boundary, and Neumann b.c. on the interior boundaries.

Theorem (Kajino '09, in prep '11)

For any GSC $F \subset \mathbb{R}^d$, there exist continuous, $(\log \rho_F)$-periodic functions $G_k : \mathbb{R} \to \mathbb{R}$ for $k = 0, 1, \cdots, d$ such that

$$K(t) = \sum_{k=0}^{d} t^{-d_k/d_{\mathrm{W}}} G_k(-\log t) + \mathcal{O}\left(\exp\left(-ct^{-\frac{1}{d_{\mathrm{W}}-1}}\right)\right) \quad \text{as} \quad t \downarrow 0,$$

where $d_k := d_{\mathrm{H}}\left(F \cap \{x_1 = \cdots = x_k = 0\}\right)$.

Remark. $G_0 > 0$ and $G_1 < 0$. Numerics suggest that G_0 is nonconstant.

Recall that the analogous result for manifolds M^d is

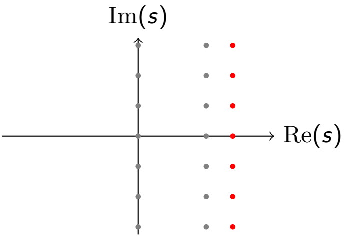
$$K(t) = \sum_{k=0}^{d} t^{-(d-k)/2} G_k + \mathcal{O}(\exp(-ct^{-1})) \quad \text{as} \quad t \downarrow 0.$$

Spectral zeta function $\zeta_\Delta(s,\gamma) := \mathrm{Tr}\frac{1}{(-\Delta+\gamma)^s} = \sum_{j=1}^\infty (\lambda_j+\gamma)^{-s}$

(Mellin integral rep)

$$\zeta_\Delta(s,\gamma) = \frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-\gamma t} K(t)\frac{dt}{t}\,, \quad \mathrm{Re}(s) > \frac{d_S}{2}.$$

(Poles of $\zeta_\Delta(s,0)$, for GSC $\subset \mathbb{R}^2$)



Using the Mittag-Leffler decomposition, we find that the simple poles of $\zeta_\Delta(s)$ are (at most)

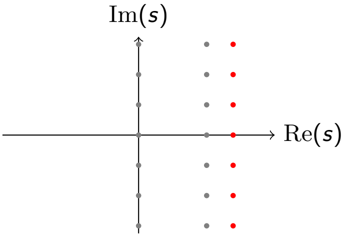
$$\bigcup_{k=0}^{d}\bigcup_{p\in\mathbb{Z}}\left\{\frac{d_k}{d_W}+\frac{2\pi p i}{\log\rho_F}\right\} := \bigcup_{k=0}^{d}\bigcup_{p\in\mathbb{Z}}\{d_{k,p}\},$$

with residues $\mathrm{Res}\left(\zeta_\Delta, d_{k,p}\right) = \dfrac{\hat{G}_{k,p}}{\Gamma\left(d_{k,p}\right)}$.

The poles of the spectral zeta fcn encode the dims of the relevant spectral volumes:

- On manifolds M^d: $\left\{\frac{d}{2}, \frac{(d-1)}{2}, \cdots, \frac{1}{2}, 0\right\}$.
- On fractals: $\left\{\frac{d_k}{d_W} + \frac{2\pi p i}{\log\rho_F}\right\}_{k,p}$. (Complex dims, à la Lapidus)

Meromorphic continuation of ζ_Δ



Theorem (Steinhurst & Teplyaev '10)

$\zeta_\Delta(\cdot, \gamma)$ *has a meromorphic continuation to all of* $\mathbb{C}$.

- The exponential tail in the HKT estimate is essential for the continuation.
- In particular, $\zeta_\Delta(s, 0)$ is analytic for $\mathrm{Re}(s) < 0$.
- *Application:* Casimir energy $\propto \zeta_\Delta(-1/2)$.

Application to quantum statistical mechanics on GSC

Consider a gas of N bosons confined to a domain F.

- The N-body wavefunction is symmetric under particle exchange, so the Hilbert space is $\mathcal{H}_N = \mathrm{Sym}\left(L^2(F)^{\otimes N}\right)$.
- $H_N = \sum_{j=1}^N K(x_j) + \sum_{i<j}^N V(x_i - x_j)$ is the Hamiltonian on $\mathcal{H}_N$.

In the grand canonical ensemble,

- $\mathcal{F} = \oplus_{N=0}^\infty \mathcal{H}_N$ is the Fock space; $d\Gamma(H) = \overline{\oplus_{N=0}^\infty H_N}$ the second quantization.
- The Gibbs state at inverse temp $\beta > 0$ and chemical potential μ is given by $\omega_{\beta,\mu}(\cdot)$, where for any self-adjoint operator A one has

$$\omega_{\beta,\mu}(A) = \Xi^{-1}\mathrm{Tr}_{\mathcal{F}}(e^{-\beta d\Gamma(H-\mu\mathbf{1})}A), \text{ with GC part. fcn. } \Xi = \mathrm{Tr}_{\mathcal{F}}(e^{-\beta d\Gamma(H-\mu\mathbf{1})}).$$

- For ideal Bose gas ($V \equiv 0$), $\log \Xi = -\mathrm{Tr}_{\mathcal{H}_1} \log(1 - e^{-\beta(H-\mu)})$.
- For an ideal massive Bose gas ($K = -\underline{\Delta}$) in a carpet of side length L,

$$\log \Xi_L(\beta,\mu) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{L^2}{\beta}\right)^t \Gamma(t)\zeta_R(t+1)\zeta_\Delta\left(t, -L^2\mu\right) dt.$$

Bose-Einstein condensation in GSC

Consider the unbounded carpet $F_\infty = \bigcup_{n=0}^\infty l_F^n F$. We exhaust F_∞ by taking an increasing family of carpets $\{\Lambda_n\}_n = \{l_F^n F\}_n$.

Proposition

As $n \to \infty$, the density of Bose gas in a GSC at (β, μ) is

$$\rho_{\Lambda_n}(\beta,\mu) = \frac{1}{(4\pi\beta)^{d_{\mathrm{S}}/2}\hat{G}_{0,0}} \sum_{m=1}^\infty e^{m\beta\mu} G_0\left(-\log\left(\frac{m\beta}{(l_F)^{2n}}\right)\right) m^{-d_{\mathrm{S}}/2} + o(1).$$

In particular, $\overline{\rho}(\beta) := \limsup_{n\to\infty} \rho_{\Lambda_n}(\beta, 0) < \infty$ iff $d_{\mathrm{S}} > 2$.

If the total density $\rho_{\mathrm{tot}} > \overline{\rho}(\beta)$, then the excess density must condense in the lowest eigenvector of the Hamiltonian $\to$ BEC.

Observe also that

$$\rho_L(\beta,\mu) = \frac{1}{CL^{d_{\mathrm{S}}}} \sum_{m=1}^\infty e^{m\beta\mu} K\left(\frac{m\beta}{L^2}\right).$$

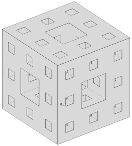
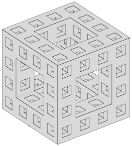
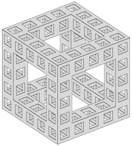
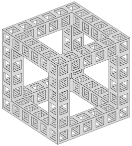
$\sum_{m=1}^\infty K(mt) < \infty \iff$ BM is transient.

Criterion for BEC in GSC

Theorem

For an ideal massive Bose gas in an unbounded GSC, the following are equivalent:

1. *Spectral dimension* $d_S > 2$.
2. *(The Brownian motion whose generator is) the Laplacian is transient.*
3. *BEC exists at positive temperature.*

	MS(3,1)	MS(4,2)	MS(5,3)	MS(6,4)
Rigorous bnds on d_S [Barlow & Bass '99]	$2.21 \sim 2.60$	$2.00 \sim 2.26$	$1.89 \sim 2.07$	$1.82 \sim 1.95$
Numerical d_S [C.]	2.51...	-	2.01...	-
BEC exists?	Yes	Yes	Yes (?)	No

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