Lecture 9

Extremal local extrema

In this lecture we focus on the structure of the set of values of the DGFF close (by order unity) to the absolute maximum. Specifically, we will state the main result on this proved jointly with O. Louidor and then discuss Liggett's theory of invariant measures for point processes evolving (or Dysonized) by independent Markov chains. This characterizes the subsequential limits of the resulting process; uniqueness will be deferred to the next lecture.

9.1. Extremal level sets

In the previous lectures we showed that the maximum of the DGFF in V_N is tight around the sequence m_N defined in (7.8). By Exercise 3.4, this applies to any sequence $\{D_N \colon N \geq 1\}$ of admissible discretizations of a continuum domain $D \in \mathfrak{D}$. Once the tightness of the maximum is in place, additional conclusions of interest can be derived concerning the structure of the *extremal level set*

$$\Gamma_N^D(t) := \big\{x \in D_N \colon h_x^{D_N} \geq m_N - t\big\}. \tag{9.1}$$

These are the content of:

thm-DZ1 **Theorem 9.1 [Ding and Zeitouni, 2012]** *There are $c, C \in (0,\infty)$ such that*

$$\lim_{t\to\infty} \liminf_{N\to\infty} P\big(\mathrm{e}^{ct} \leq |\Gamma_N^D(t)| \leq \mathrm{e}^{Ct}\big) = 1. \tag{9.2}$$

thm-DZ2 **Theorem 9.2 [Ding and Zeitouni, 2012]** *There is $c \in (0,\infty)$ such that*

$$P\Big(\exists x, y \in \Gamma_N^D(c \log\log r) \colon r < |x-y| < \frac{N}{r}\Big) = 0 \tag{9.3}$$

We will not prove these theorems as that amounts to further (albeit rather ingenious) manipulations with square boxes and Gibbs-Markov property for the DGFF therein. Instead we observe that these results paint the following picture: *The extremal level set $\Gamma_N^D(t)$ is the union of a finite number of islands of bounded size at distances* *of order* N. Naturally, this leads to the consideration of the extreme value empirical measure on $D \times \mathbb{R}$ defined by

$$\eta_N^D := \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h_x^{D_N} - m_N}. \tag{9.4}$$

Note that $\eta_N^D(D \times [-t, \infty)) = |\Gamma_N^D(t)|$ and so η_N^D indeed captures the size of the level set for all t at the same time. In addition, η_N^D also keeps track of how the points are distributed in the level set, very much like as the measures discussed in Lectures 2-5. Note, however, that he no normalization is required.

As it turns out, the structure of the problem at hand does not make η_N^D the most natural object to work with after all. This is because each high value of the DGFF will come along with a whole *cluster* of high values — the values of the field at points at finite distances. Moreover these "adjacent" values are heavily correlated and so it is not advantageous to let their relative positions wash out from the picture completely. Thus, denoting by

$$\Lambda_r(x) := \{ y \in \mathbb{Z}^2 \colon |x - y| < r \} \tag{9.5}$$

the r-neighborhood of x in (say) Euclidean norm, instead of η_N^D we will consider a *structured* version of this measure,

$$\eta_{N,r}^D := \sum_{x \in D_N} 1_{\{h_x^{D_N} = \max_{y \in \Lambda_r(x)} h_y^{D_N}\}} \delta_{x/N} \otimes \delta_{h_x^{D_N} - m_N} \otimes \delta_{\{h_x^{D_N} - h_{x+z}^{D_N} \colon z \in \mathbb{Z}^2\}}. \tag{9.6}$$

This measure picks one reference point in each "island" — namely the local maximum in r-neighborhood thereof — and records the scaled position and reduced value at this point along with the shape of the entire configuration in the neighborhood thereof. Picking the local maximum for the reference point makes a very natural choice although other choices would work as well.

Let $\mathrm{PPP}(\mu)$ denote the Poisson point process with (σ-finite) intensity measure μ. The main theorem to be proved in these lectures is now as follows:

thm-extremal-vals **Theorem 9.3 [B-Louidor 2013, 2014, 2016]** *There is a probability measure ν on $[0, \infty)^{\mathbb{Z}^2}$ and, for each $D \in \mathfrak{D}$, there is a random Borel measure Z^D on D with $Z^D(D) \in (0, \infty)$ a.s. such that the following holds for any admissible approximating sequence $\{D_N \colon N \geq 1\}$ and any sequence $\{r_N\}$ with $r_N \to 0$ and $N/r_N \to \infty$:*

$$\eta_{N,r_N}^D \xrightarrow{\text{law}} \mathrm{PPP}\big(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h} \mathrm{d}h \otimes \nu(\mathrm{d}\phi)\big), \tag{9.7}$$

where (as before) $\alpha := 2/\sqrt{g}$.

The convergence in law is with respect to the vague convergence on Radon measures on $D \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^d}$. Note that the expression we saw in the convergence of intermediate level sets appears here, albeit now as the random intensity measure of the Poisson process. (The PPP process with a random intensity is to be understood as follows: First sample the random intensity and then generate the Poisson process using this sample of the intensity measure.)

9.2. Distributional invariance

For the rest of this lecture we will focus only on the first two coordinates of the process above. First we ask the reader to verify:

Exercise 9.4 *Show that, for any* $f \in C_c(\overline{D} \times (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}^{\mathbb{Z}^2})$, *the sequence of random variables* $\{\langle \eta^D_{N,r_N}, f\rangle \colon N \geq 1\}$ *is tight.*

In addition, we pose the following standard fact from the theory of point processes:

ex:9.5 **Exercise 9.5** *Suppose* $\mathscr{X}$ *be a locally compact, separable Hausdorff space and let* η_N *be a sequence of random integer-valued Radon measures on* $\mathscr{X}$. *Assume* $\{\langle \eta_N, f\rangle \colon N \geq 1\}$ *is tight for each* $f \in C_c(\mathscr{X})$. *Prove that there is a sequence* $N_k \to \infty$ *such that* $\langle \eta_{N_k}, f\rangle$ *converges in law to a random variable that takes the form* $\langle \eta, f\rangle$ *for some random Radon measure* η *on* $\mathscr{X}$. *Prove that* η *is integer valued.*

Our method to prove Theorem 9.3 is very much like what we did in the lectures on intermediate level sets (although the chronological development started with extremal level sets): extract a subsequential limit of the processes of interest and then derive enough properties of these to identify their law uniquely. In this lecture we go only part of the way by proving:

prop-subseq **Theorem 9.6 [Poisson structure of subsequential limits]** *Any weak subsequential limit* η^D *of the processes* $\{\eta^D_{N,r_N} \colon N \geq 1\}$ *restricted to just the first two coordinates takes the form*

$$\eta^D \overset{\text{law}}{=} \text{PPP}\big(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h\big) \tag{9.8}$$

for some random Borel measure Z^D *on* D *with* $Z^D(D) \in (0, \infty)$.

As we will demonstrate, this arises from the fact that *every* subsequential limit measure η^D has the following distributional symmetry:

prop-Dyson **Proposition 9.7 [Invariance under Dysonization]** *For any* η^D *as above (projected on the first two coordinates) and any function* $f \in C_c(\overline{D} \times (\mathbb{R} \cup \{\infty\}))$, *we have*

$$E\big(\mathrm{e}^{-\langle \eta, f\rangle}\big) = E\big(\mathrm{e}^{-\langle \eta, f_t\rangle}\big), \quad t > 0, \tag{9.9}$$

where

$$f_t(x, h) := -\log E^0\big(\mathrm{e}^{-f(x, h + B_t - \frac{\alpha}{2}t)}\big) \tag{9.10}$$

with $\{B_t \colon t \geq 0\}$ *denoting the standard Brownian motion.*

Let us pause to explain why we refer to this as "invariance under Dysonization." The tightness mentioned earlier ensures that η^D is a point measure, i.e.,

$$\eta^D = \sum_{i \in \mathbb{N}} \delta_{x_i} \otimes \delta_{h_i}. \tag{9.11}$$

(The fact that η^D is infinite and yet finite on compact sets follows from Theorems 9.1–9.2.) Now consider a collection $\{B^{(i)}_t \colon t \geq 0\}_{i=1}^{\infty}$ of i.i.d. standard Brownian motions and set

$$\eta^D_t := \sum_{i \in \mathbb{N}} \delta_{x_i} \otimes \delta_{h_i + B^{(i)}_t - \frac{\alpha}{2}t}. \tag{9.12}$$

Of course, for $t > 0$ this may no longer be a good point measure as we cannot *a priori* guarantee that $\eta_t^D(C) < \infty$ for any compact set. Nonetheless, we can use the Monotone Convergence Theorem to perform the following computations:

$$\begin{aligned} E\bigl(\mathrm{e}^{-\langle \eta_t, f\rangle}\bigr) &= E\Bigl(\prod_{i\in\mathbb{N}} \mathrm{e}^{-f(x_i, h_i + B_t^{(i)} - \frac{\alpha}{2}t)}\Bigr) \\ &= E\Bigl(\prod_{i\in\mathbb{N}} \mathrm{e}^{-f_t(x_i,h_i)}\Bigr) = E\bigl(\mathrm{e}^{-\langle \eta, f_t\rangle}\bigr)\,, \end{aligned} \tag{9.13}$$

E:9.13

where in the middle equality we passed the expectation with respect to each Brownian motion inside the infinite product. Proposition 9.7 then tells us

$$E\bigl(\mathrm{e}^{-\langle \eta_t, f\rangle}\bigr) = E\bigl(\mathrm{e}^{-\langle \eta, f\rangle}\bigr), \quad t \geq 0, \tag{9.14}$$

and since this holds for all f as above,

$$\eta_t \overset{\text{law}}{=} \eta, \quad t \geq 0. \tag{9.15}$$

Thus, Dysonizing each point of η^D by the diffusion $t \mapsto B_t - \frac{\alpha}{2}t$ in the second coordinate preserves the law of η^D.

Proof of Proposition 9.7, main idea. Writing h for the DGFF on D_N, let h', h'' be independent copies of h. For any $s \in [0,1]$, we can then realize h as

$$h = \sqrt{1-s}h' + \sqrt{s}\,h''. \tag{9.16}$$

Choosing $s := t/(g\log N)$, we then get

$$h = \sqrt{1 - \frac{t}{g\log N}}\; h' + \frac{\sqrt{t}}{\sqrt{g\log N}}\, h'' \tag{9.17}$$

E:9.17

Now pick x at or near local maximum of h' of order $m_N + O(1)$. Expanding the first square-root into Taylor polynomial and using that $h'_x = m_N + O(1) = 2\sqrt{g}\,\log N + O(\log\log N)$ yields

$$\begin{aligned} h_x &= h'_x - \frac{1}{2}\frac{t}{g\log N}\, h'_x + \frac{\sqrt{t}}{\sqrt{g\log N}}\, h''_x + O\Bigl(\frac{1}{\log N}\Bigr) \\ &= h'_x - \frac{t}{\sqrt{g}} + \frac{\sqrt{t}}{\sqrt{g\log N}}\, h''_x + O\Bigl(\frac{\log\log N}{\log N}\Bigr) \end{aligned} \tag{9.18}$$

The covariance of the second field on the right of (9.17) satisfies

$$\mathrm{Cov}\Bigl(\frac{\sqrt{t}}{\sqrt{g\log N}}\, h''_x, \frac{\sqrt{t}}{\sqrt{g\log N}}\, h''_y\Bigr) = \begin{cases} t + o(1), & \text{if } |x-y| < r_N, \\ o(1), & \text{if } |x-y| > N/r_N. \end{cases} \tag{9.19}$$

Thus, the second field behaves as a *constant* normal random variable on the whole island of radius r_N around x, while the random variables on different islands can be regarded as more or less independent.

If x is a local maximum of h'_x of order $m_N + O(1)$, we thus have to show that the *gap* between h'_x and the second largest value in an r_N-neighborhood of x stays positive with probability tending to 1 as $N \to \infty$. Once the errors of all approximations above are smaller than the gap, x will also be the local maximum of the field h. One can easily checks that, in the limit, the effect of the second term on the right of (9.17) is that of independent Brownian motions. The drift term comes from the linear shift by $t/\sqrt{g}$ noting that $1/\sqrt{g} = \alpha/2$. □

9.3. Dysonization-invariant point processes

Our next task is to use the distributional invariance articulated in Proposition 9.7 to show that the law of η^D must take the form in Proposition 9.6. A first natural idea is to look at moments of η^D, i.e., measures μ_n on $\mathbb{R}^n$ defined via

$$\mu_n(A_1 \times \cdots \times A_n) := E\big[\eta^D(D \times A_1) \ldots \eta^D(D \times A_n)\big]. \tag{9.20}$$

By the smoothness of the kernel associated with $t \mapsto B_t - \frac{\alpha}{2}t$, these would have to have density with respect to the Lebesgue measure on $\mathbb{R}^n$ which would then satisfy a PDE whose solutions can be classified. Unfortunately, this reasoning is all useless because, as it turns out, all moments of η^D are infinite. We thus have to proceed along a different line of argument. Fortunately, a 1978 paper of T. Liggett does all what we need to do, so this is what we will discuss next.

Liggett 1978 paper's interest lies non-interacting, interacting particle systems on a general state space. Roughly speaking, these are collection of particles that evolve according to independent Markov chains. The setting we will consider is as follows: Let $\mathscr{X}$ be a locally compact, separable Hausdorff space and, writing $\mathcal{B}(\mathscr{X})$ for the class of Borel sets in $\mathscr{X}$, let $\mathsf{P}\colon \mathscr{X} \times \mathcal{B}(\mathscr{X}) \to [0,1]$ be a transition kernel of a Markov chain on $\mathscr{X}$. Letting $\mathbb{N}^\star := \mathbb{N} \cup \{0\} \cup \{\infty\}$, we are interested in point processes on $\mathscr{X}$ which we take to mean the random elements of

$$\mathcal{M} := \big\{\mathbb{N}^\star\text{-valued Radon measures on } (\mathscr{X}, \mathcal{B}(\mathscr{X}))\big\}. \tag{9.21}$$

Note that the "Radon" qualifier implies that the measure is inner and outer regular with a finite value on every compact subset of $\mathscr{X}$. Calling such measures "point processes" is meaningful in light of:

Exercise 9.8 *Every $\eta \in \mathcal{M}$ takes the form*

$$\eta = \sum_{i=1}^{N} \delta_{x_i} \tag{9.22}$$

E:9.22

where $N \in \mathbb{N}^\star$ and, if $N > 0$, then $\{x_i\colon i = 1, \ldots, N\}$ is a collection of points from $\mathscr{X}$ (repetitions are possible) so that $\eta(C) < \infty$ for every $C \subset \mathscr{X}$ compact.

Given a representation of $\eta \in \mathcal{M}$ of the form (9.22), we now consider a collection of independent Markov chains $\{X_n^{(i)}\colon n \geq 0\}_{i=1}^N$ such that

$$P\big(X_0^{(i)} = x_i\big) = 1, \quad i = 1, \ldots, N. \tag{9.23}$$

Then we define

$$\eta_n := \sum_{i=1}^{N} \delta_{X_n^{(i)}} \tag{9.24}$$

Notice we are not claiming that $\eta_n \in \mathcal{M}$ for $n \geq 1$; in fact, easy counterexamples can be produced for the contrary whenever $N = \infty$. Nonetheless, this will be guaranteed for every element of,

$$\mathscr{I} := \bigl\{\eta\colon \text{random element of } \mathcal{M} \text{ such that } \eta_1 \overset{\text{law}}{=} \eta\bigr\}. \tag{9.25}$$

which is the set of all *invariant laws* for the above evolution $\eta \mapsto \{\eta_n\colon n \geq 0\}$. Indeed,

$$\eta_1 \overset{\text{law}}{=} \eta \quad \Rightarrow \quad \eta_n \in \mathcal{M} \text{ a.s. } \forall n \geq 0 \tag{9.26}$$

Let P^n denote the n-th (convolution) power of the kernel P. The following observation from Liggett's paper is attributed to folklore. Hence the subtitle of:

thm-Liggett-folk **Theorem 9.9 [Liggett's folklore theorem]** *Suppose* P *obeys the following "strong dispersivity" assumption*

$$\forall C \subset \mathscr{X} \text{ compact:} \qquad \sup_{x \in \mathscr{X}} \mathsf{P}^n(x, C) \underset{n\to\infty}{\longrightarrow} 0\,. \tag{9.27}$$ E:9.27

Then

$$\mathscr{I} := \bigl\{\mathrm{PPP}(M)\colon M = \text{(random) Radon measure on } \mathscr{X} \text{ s.t. } M \overset{\text{law}}{=} M\mathsf{P}\bigr\}. \tag{9.28}$$

where $M\mathsf{P}(\cdot) := \int M(\mathrm{d}x)\mathsf{P}(x,\cdot)$.

Before we set out to prove the theorem, let us verify its easier part — namely, that all $\mathrm{PPP}(M)$ with $M \overset{\text{law}}{=} M\mathsf{P}$ lie in $\mathscr{I}$. This will immediately follow from:

lemma-9.9 **Lemma 9.10** *Let M be a Radon measure on $\mathscr{X}$. Then*

$$\eta \overset{\text{law}}{=} \mathrm{PPP}(M) \quad \Rightarrow \quad \eta_n \overset{\text{law}}{=} \mathrm{PPP}(M\mathsf{P}^n), \quad \forall n \geq 0. \tag{9.29}$$

Proof. We start by noting that η being a sample from $\mathrm{PPP}(M)$ for some Radon measure M is equivalent to saying that, for every $f \in C_c(\mathscr{X})$ with $f \geq 0$,

$$E\bigl(\mathrm{e}^{-\langle \eta, f\rangle}\bigr) = \exp\Bigl\{-\int M(\mathrm{d}x)(1 - \mathrm{e}^{-f(x)})\Bigr\}. \tag{9.30}$$ E:9.30

The argument (9.13) shows

$$E\bigl(\mathrm{e}^{-\langle \eta_n, f\rangle}\bigr) = E\bigl(\mathrm{e}^{-\langle \eta, f_n\rangle}\bigr)\,, \tag{9.31}$$ E:9.31

where $f_n(x) := -\log[(\mathsf{P}^n\mathrm{e}^{-f})(x)]$. This equivalently reads as $\mathrm{e}^{-f_n} = \mathsf{P}^n\mathrm{e}^{-f}$ and so from (9.30–9.31) and the fact that $\mathsf{P}^n 1 = 1$ we get

$$E\bigl(\mathrm{e}^{-\langle \eta, f_n\rangle}\bigr) = \exp\Bigl\{-\int M(\mathrm{d}x)\,\mathsf{P}^n(1 - \mathrm{e}^{-f})(x)\Bigr\}. \tag{9.32}$$

The integral can be written as the right-hand side of (9.30) with M replaced by $M\mathsf{P}^n$. Since the right-hand side of (9.30) characterizes $\mathrm{PPP}(M)$, we are done. $\square$

We are now ready to give:

Proof of Theorem 9.9. In light of (9.10) we only need to verify that every element of $\mathscr{I}$ takes the form $\mathrm{PPP}(M)$ for some M satisfying $M \overset{\text{law}}{=} M\mathsf{P}$. Let $\eta \in \mathscr{I}$ and pick $f \in C_c(\mathscr{X})$ with $f \geq 0$. Since e^{-f} equals 1 outside a compact set, the strong dispersivity condition (9.27) then implies

$$\sup_{x\in\mathscr{X}}\bigl|(\mathsf{P}^n\mathrm{e}^{-f})(x) - 1\bigr| \underset{n\to\infty}{\longrightarrow} 0. \tag{9.33}$$

Recalling the definition of $f_n(x)$, this implies the existence of $\epsilon_n \downarrow 0$ such that

$$(1-\epsilon_n)\mathsf{P}^n(1-\mathrm{e}^{-f})(x) \leq f_n(x) \leq (1+\epsilon_n)\mathsf{P}^n(1-\mathrm{e}^{-f})(x), \quad x \in \mathscr{X}. \tag{9.34}$$

It follows there is a random $\widetilde{\epsilon}_n \in [-\epsilon_n, \epsilon_n]$ such that

$$\langle \eta, f_n\rangle = (1+\widetilde{\epsilon}_n)\bigl\langle \eta, \mathsf{P}^n(1-\mathrm{e}^{-f})\bigr\rangle. \tag{9.35}$$

Denoting $\eta\mathsf{P}^n(\cdot) := \int \eta(\mathrm{d}x)\mathsf{P}^n(x,\cdot)$, from $\eta \overset{\text{law}}{=} \eta$ and (9.31) we get

$$E\bigl(\mathrm{e}^{-\langle \eta, f\rangle}\bigr) = E\bigl(\mathrm{e}^{-(1+\widetilde{\epsilon}_n)\langle \eta\mathsf{P}^n,(1-\mathrm{e}^{-f})\rangle}\bigr)\,. \tag{9.36}$$ E:9.36

Since every $g \in C_c(\mathscr{X})$ can be written as $g = 1 - \mathrm{e}^{-f}$ for some $f \in C_c(\mathscr{X})$, perturbation arguments show that $\{\langle \eta\mathsf{P}^n, g\rangle \colon n \geq 0\}$ is tight for all $g \in C_c(\mathscr{X})$. Using Exercise 9.5, we can extract a subsequence such that all of these random variables converge in law; the limit is then of the form $\langle M, g\rangle$ for some random Radon measure M on $\mathscr{X}$. As $\widetilde{\epsilon}_n \to 0$ in L^∞, we conclude

$$E\bigl(\mathrm{e}^{-\langle \eta, f\rangle}\bigr) = E\bigl(\mathrm{e}^{-\langle M,(1-\mathrm{e}^{-f})\rangle}\bigr), \quad f \in C_c(\mathscr{X}). \tag{9.37}$$ E:9.37

In light of (9.30) this proves that $\eta \overset{\text{law}}{=} \mathrm{PPP}(M)$. Replacing n by $n+1$ in (9.36) shows that (9.37) holds with M replaced by $M\mathsf{P}$. Since every function in $C_c(\mathscr{X})$ takes the form $(1-\mathrm{e}^{-f})$, from (9.37) we infer that $M\mathsf{P}$ is equidistributed to M. $\square$

9.4. Characterization of subsequential limits

For the problem of the (two-coordinate) extremal process of the DGFF, the relevant setting is

$$\mathscr{X} := \overline{D} \times (\mathbb{R} \cup \{\infty\}) \tag{9.38}$$

and, given any $t > 0$, the transition kernel given by

$$\mathsf{P}_t\bigl((x,h), A\bigr) := P\bigl((x, h + B_t - \tfrac{\alpha}{2}t) \in A\bigr). \tag{9.39}$$

We leave it to the reader to verify:

Exercise 9.11 *For each $t > 0$, the kernel P_t has the strong dispersivity property* (9.27).

Hence we get:

cor-9.12 **Corollary 9.12** *Every subsequential limit η^D of processes $\{\eta^D_{N,r_N}\colon N \geq 1\}$ (projected on the first two coordinates) takes the form*

$$\eta^D \overset{\text{law}}{=} \text{PPP}(M) \tag{9.40}$$

where $M = M(\mathrm{d}x\mathrm{d}h)$ is a Radon measure on $\overline{D} \times (\mathbb{R} \cup \{\infty\})$ such that

$$M\mathsf{P}_t \overset{\text{law}}{=} M, \quad t > 0. \tag{9.41}$$ E:9.40

Moving back to the general setting from the previous section, the next question to address is thus:

$$\text{Characterize Radon measures } M \text{ on } \mathscr{X} \text{ satisfying } M\mathsf{P} \overset{\text{law}}{=} M$$

Here we note that if M is a random sample from the invariant measures for the Markov chain P, then we even have $M\mathsf{P} = M$ a.s. We thus phrase the question as:

$$\text{When does } M\mathsf{P} \overset{\text{law}}{=} M \text{ imply } M{=}M \text{ a.s.?}$$

In his 1978 paper, Liggett identified a number of examples when this is answered in the affirmative. The salient part of his conclusions is condensed into:

thm-9.13 **Theorem 9.13 [Liggett 1978]** *Let M be a random Radon measure on $\mathscr{X}$ and suppose*

(1) either P is an irreducible, aperiodic, Harris recurrent Markov chain, or

(2) P is a random walk on an Abelian group such that $\mathsf{P}(0,\cdot)$, where 0 is the identity, is not supported on a translate of a proper closed subgroup.

Then $M\mathsf{P} \overset{\text{law}}{=} M$ implies $M\mathsf{P} = M$ a.s.

Both parts use somewhat sophisticated facts from the theory of Markov chains and/or random walks on Abelian groups. The first alternative actually does not apply for us as our Markov chain — Brownian motion with a constant drift evaluated at integer multiples of some $t > 0$ — is definitely not Harris recurrent. Fortunately, the second alternative does apply and so we get:

Corollary 9.14 *For each $t > 0$, any M satisfying* (9.41) *obeys $M\mathsf{P}_t = M$ a.s.*

We will actually provide a sketch of the proof of the second part of Theorem 9.13 by simultaneously proving:

Lemma 9.15 *Any M satisfying* (9.41) *takes the form*

$$M(\mathrm{d}x\mathrm{d}h) \overset{\text{law}}{=} Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h + \widetilde{Z}^D(\mathrm{d}x) \otimes \mathrm{d}h, \tag{9.42}$$ E:9.42

where $(Z^D, \widetilde{Z}^D)$ is a pair of random Borel measures on D.

Proof. First we note that 2) above applies to the random measure

$$M^A(C) := M(A \times C) \tag{9.43}$$

and the kernel

$$\mathsf{Q}_t(h, C) := P\big(h + B_t - \tfrac{\alpha}{2}t \in C\big). \tag{9.44}$$

Indeed, this kernel is a random walk on $\mathbb{R}$ with no invariant nonempty subset at all. By assumption, the sequence $\{M^A\mathsf{P}^n : n \geq 0\}$ is stationary with respect to the left shift. The Kolmogorov Extension Theorem then permits us to realize these within a *two-sided* sequence $\{M_n^A : n \in \mathbb{Z}\}$ such that

$$M_n^A \overset{\text{law}}{=} M^A \quad \text{and} \quad M_{n+1}^A = M_n^A \mathsf{Q}_t \text{ a.s.} \tag{9.45}$$

for each $n \in \mathbb{Z}$. This makes $\{M_n^A : n \in \mathbb{Z}\}$ and instance of:

Definition 9.16 *Given a Markov chain on S with transition kernel P, a family of measures $\{\pi_n : n \in \mathbb{Z}\}$ on S is an entrance law if $\pi_{n+1} = \pi_n \mathsf{P}$ holds for all $n \in \mathbb{Z}$.*

The fact that Q_t is a smooth kernel implies that $M_n^A \ll \text{Leb}$. Hence, there are $f(n,h)$ such that $M_n^A(\mathrm{d}h) = f(n,h)\mathrm{d}h$. Moreover, these satisfy

$$f(n+1, h) = f(n, \cdot) * k_t\big(h + \tfrac{\alpha}{2}t\big) \quad \text{where} \quad k_t(h) := \frac{1}{\sqrt{2\pi t}}\, \mathrm{e}^{-\frac{h^2}{2t}}. \tag{9.46}$$ E:9.46

In particular, $f(n,h) > 0$ for all $n \in \mathbb{Z}$ and all $h \in \mathbb{R}$. A key observation made by Liggett is the content of:

Exercise 9.17 *Let X be a random variable on an Abelian group S. Prove that every entrance law $\{\pi_n : n \in \mathbb{Z}\}$ for the random walk on S with step distribution X is a stationary measure for the random walk on $\mathbb{Z} \times S$ with step distribution $(1, X)$.*

In particular, for the setting above:

Exercise 9.18 *Prove that the function $(n,h) \mapsto f(-n,h)$ is positive harmonic for the random walk on $\mathbb{Z} \times \mathbb{R}$ with step distribution $(1, \mathcal{N}(-\frac{\alpha}{2}t, t))$.*

Now we invoke, without further detail, the salient conclusion of Choquet-Deny theory: Every positive harmonic function of the random walk on $\mathbb{Z} \times \mathbb{R}$ with step distribution $(1, \mathcal{N}(-\frac{\alpha}{2}t, t))$ takes the form

$$f(n,h) = \int \lambda(\kappa)^n\, \mathrm{e}^{\kappa h}\, \nu(\mathrm{d}\kappa) \tag{9.47}$$

for some Borel measure ν on $\mathbb{R}$ and

$$\lambda(\kappa) := \mathrm{e}^{\frac{1}{2}\kappa(\kappa+\alpha)t} \tag{9.48}$$

(Another way to state this is that the extremal positive harmonic functions that are exactly those of the form $(n,h) \mapsto \lambda^n \mathrm{e}^{\kappa h}$ with λ adjusted so that (9.46) holds.)

Now note that, for any compact $C \subset \mathbb{R}$, Tonelli tells us

$$M_n^A\bigl([-1,1]\bigr) = \int_{[-1,1]} f(n,h)\,\mathrm{d}h = \int \lambda(\kappa)^n \frac{\sinh(\kappa)}{\kappa}\,\nu(\mathrm{d}\kappa) \tag{9.49}$$

where ν depends on the realization of M^A. This diverges to infinity as $n \to \infty$ or $n \to \infty$ unless

$$\nu\bigl(\{\kappa \in \mathbb{R} \colon \lambda(\kappa) \neq 1\}\bigr) = 0. \tag{9.50}$$

Since $\{M_n^A \colon n \in \mathbb{Z}\}$ is stationary, this forces ν to be of the form

$$\nu = X^A \delta_{-\alpha} + Y^A \delta_0. \tag{9.51}$$

for some non-negative random variables X^A and Y^A. Hence we get

$$M^A(\mathrm{d}h) = X^A \mathrm{e}^{-\alpha h}\mathrm{d}h + Y^A \mathrm{d}h. \tag{9.52}$$ E:9.52

But $A \mapsto M^A$ is a Borel measure and so $Z^D(A) := X^A$ and $\widetilde{Z}^D(A) := Y^A$ defines two random measures for which (9.42) holds. □

Exercise 9.19 *A slight technical caveat in the last argument is that* (9.52) *holds only a.s. with the null set depending on A. Prove that, since Borel sets in $\mathbb{R}$ are countably generated, the conclusion still holds.*

We are now ready to conclude our characterization of weak subsequential limits of the processes $\{\eta^D_{N,r_N} \colon N \geq 1\}$:

Proof of Proposition 9.7. In light of our previous reasoning, we only have to prove that the measure $\widetilde{Z}^D$ in (9.42) vanishes a.s. This follows by noting that, if $N_k \to \infty$ is a sequence such that $\eta^D_{N_k,r_{N_k}}$ converges to $\mathrm{PPP}(M)$, then

$$P\Bigl(\max_{x \in D_{N_k}} h_x^{D_{N_k}} < m_{N_k} + t\Bigr) \underset{n\to\infty}{\longrightarrow} E\Bigl(\mathrm{e}^{-M(D\times[t,\infty))}\Bigr) \tag{9.53}$$

where, as usual, $\mathrm{e}^{-\infty} := 0$. (The convergence *a priori* holds only for a dense set of t's but, since we already know that M has a density in the h variable, it extends to all t.) Now the upper-tail tightness of the maximum tells us that the left-hand side is larger than $1 - \mathrm{e}^{-at}$ for some $a > 0$. It follows that

$$M\bigl(D \times [t,\infty)\bigr) \underset{t\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{9.54}$$

This forces $\widetilde{Z}^D(D) = 0$ a.s. □