Lecture 8

Tightness of the maximum

We now move to the proof of tightness of the maximum of the DGFF stated in Theorem 7.3. The original proof due to Bramson and Zeitouni was based on a comparison with the so called modified Branching Random Walk. We substitute this part by the use of a concentric decomposition of the DGFF. This brings us much closer to what we have done for the Branching Random Walk in the previous lecture. Moreover, the concentric decomposition will be indispensable for control of the local structure of extreme-order local maxima.

8.1. Upper tail of DGFF maximum

Recall the notation m_N from (7.8) and $\widetilde{m}_n$ from (7.22). As is easy to check, for $b := 4$ and $N := 2^n$ we have

$$\sqrt{g \log 2}\, \widetilde{m}_n = m_N + O(1) \tag{8.1}$$

and so (7.15) gives us $EM_N \le m_N + O(1)$. This does not tells us much by itself (indeed, the best we can extract from this is that $P(M_N > 2m_N)$ is at most a half.) Notwithstanding, with the help of an additional argument we are able to boost this to the tightness of the upper tail of M_N above m_N:

Lemma 8.1 [Upper tail tightness] *We have*

$$\sup_{N \ge 1} E\big((M_N - m_N)_+\big) < \infty. \tag{8.2}$$

Proof. We will prove this by a variant of the Dekking-Host argument combined with the domination of the DGFF by BRW via Sudakov-Fernique. A novel idea is to use these techniques jointly for the maximum of independent copies of the DGFF h^{V_N} and the BRW ϕ^{T^b}. Abbreviate

$$\widetilde{M}_n := \sqrt{g \log 2}\, \max_{x \in L_n} \phi_x^{T^b} \tag{8.3}$$

and, considering a copy $\tilde{\phi}^{T^b}$ of ϕ^{T^b}, let $\widetilde{M}_n'$ be the corresponding quantity for $\tilde{\phi}^{T^b}$. We assume that all three fields are realized as independent on the same probability space. Apart from the already proved bound (7.16), we also readily check that

$$E\big([h_x^{V_N}-\sqrt{g\log 2}\,\phi_y^{T^b}]^2\big)\le c+(g\log 2)E\big([\tilde\phi_x^{T^b}-\phi_y^{T^b}]^2\big) \tag{8.4}$$

holds for all $x,y\in L_n$. Following the argument in the proof of Lemma 7.5, we then get (for k related to c above as there)

$$E\big(\max\{M_N,\widetilde M_n\}\big)\le E\big(\max\{\widetilde M_{n+k},\widetilde M'_{n+k}\}\big) \tag{8.5}$$

The "Dekking-Host argument" from the proof of Lemma 7.1 then bounds the right-hand side by $E\widetilde M_{n+k+1}$. For the quantity on the left-hand side we use

$$(a-b)_+=\max\{a,b\}-b \tag{8.6}$$

to get

$$E\big((M_N-\widetilde M_n)_+\big)\le E\widetilde M_{n+k+1}-E\widetilde M_n. \tag{8.7}$$

By Theorem 7.7 the difference on the right-hand side is bounded uniformly in n. Using Jensen's inequality to pass expectation over $\widetilde M_n$ inside the positive-part function, the claim follows from Theorem 7.7, (8.1) and $(a-c)_+\ge a_+-c_+$. □

Once we know that $(M_N-m_N)_+$ does not get too large with positive probability, we can now prove an analogue of Lemma 7.14 for the DGFF as well:

Lemma 8.2 *There is $\widetilde a>0$ such that*

$$\sup_{N\ge1}P\big(M_N\ge m_N+t\big)\le \mathrm{e}^{-\tilde a t},\qquad t\ge 0. \tag{8.8}$$

Proof. Fix any integer K divisible by 3 and consider the DGFF in V_{KN}. By choosing $r>0$ sufficiently large, Lemma 8.1 ensures

$$P\big(\,M_{KN}\le m_{KN}+r\big)\ge\frac12. \tag{8.9}$$

Let $\widetilde K:=K/3$ and identify $\widetilde K^2$ translates of V_{3N} inside V_{KN} such that any pair of adjacent translates is separated by a line of sites. Denote these translates $V_{3N}^{(i)}$, with $i=1,\dots,\widetilde K^2$, and abusing our earlier notation slightly, write $V_{KN}^\circ:=\bigcup_{i=1}^{\widetilde K^2}V_{3N}^{(i)}$. Moreover, let $V_N^{(i)}$ be a translate of V_N centered at the same point as $V_{3N}^{(i)}$. Using the Gibbs-Markov decomposition, we then have

$$M_{KN}\overset{\text{law}}{\ge}\max_{i=1,\dots,K^2}\max_{x\in V_N^{(i)}}\Big(h_x^{V_{3N}^{(i)}}+\varphi_x^{V_{KN},V_{KN}^\circ}\Big) \tag{8.10}$$

Consider the event

$$A_K:=\Big\{\#\Big\{i\in\{1,\dots,\widetilde K^2\}\colon \min_{x\in V_N^{(i)}}\varphi_x^{V_{KN},V_{KN}^\circ}\ge-(\log K)^{3/4}\Big\}\ge\widetilde K^2/2\Big\}. \tag{8.11}$$

Since $\mathrm{Var}(\varphi_x^{V_{KN},V_{KN}^\circ}) \le c\log K$, a combination of Borell-TIS inequality with Fernique's majorization shows that

$$\epsilon_K := \max_{i=1,\dots,K} P\Big(\min_{x\in V_N^{(i)}} \varphi_x^{V_{KN},V_{KN}^\circ} \ge (\log K)^{3/4}\Big) \tag{8.12}$$

tends rapidly to zero with $K\to\infty$, uniformly in N. As $P(A_K^{\mathrm{c}}) \le 2\epsilon_K$, we thus get

$$P\big(M_{KN} \le m_{KN}+r\big) \le 2\epsilon_K + P\Big(\max_{x\in V_N'} h_x^{V_{3N}} \le m_{KN}+r+(\log K)^{3/4}\Big)^{\widetilde{K}^2/2}, \tag{8.13}$$

where V_N' is the translate of V_N centered at the same point as V_{3N}. Now pick any $c > 2\sqrt{g}$ and note that, for K large enough and all $N\ge 3$,

$$m_{KN}+r+(\log K)^{3/4} \le m_N + c\log K \tag{8.14}$$

Invoking also $1-x\le \mathrm{e}^{-x}$, (8.13) yields

$$\frac12 - 2\epsilon_K \le \exp\Big\{-\frac12\widetilde{K}^2 P\Big(\max_{x\in V_N'} h_x^{V_{3N}} > m_N + c\log K\Big)\Big\}. \tag{8.15}$$

Using the Gibbs-Markov property, we can replace $h^{V_{3N}}$ by $h^{V_N'}$ at the cost of getting another half in front of the probability. Hereby we get

$$P\big(M_N > m_N + c\log K\big) \le c'K^{-2}. \tag{8.16}$$

The claim follows by replacing $c\log K$ by t. (This covers $t \lesssim \log N$; the opposite case is handled by a union bound.) $\square$

8.2. Concentric decomposition

Although the above conclusions seem to be quite sharp, they are not inconsistent with M_N being concentrated at values much smaller than m_N. In order to rule this out, we have to prove that $EM_N \ge m_N + O(1)$ as well. In the context of BRW, this was reduced (among other things) to calculating the asymptotic of the *conditional probability* that a given point is a maximum given that the field there is already large. In the context of the DGFF, this corresponds to

$$P\big(h^{D_N} \le m_N + t \,\big|\, h_0^{D_N} = m_N + t\big), \tag{8.17}$$

where we assumed, and we will continue below, that $0\in D_N$. For the BRW it was useful that the conditional can be reduced to (what for the DGFF is) $h^{D_N}=0$ at the cost of subtracting a suitable term from all fields. Such a strategy is possible here as well and yields:

Lemma 8.3 *Suppose $D_N \subset \mathbb{Z}^2$ is finite with $0\in D_N$. Then for all $t\in\mathbb{R}$ and $s\ge 0$,*

$$\begin{aligned} &P\big(h^{D_N} \le m_N + t + s \,\big|\, h_0^{D_N} = m_N + t\big) \\ &\qquad = P\big(h^{D_N} \le (m_N+t)(1-\mathfrak{g}^{D_N}) + s \,\big|\, h_0^{D_N} = 0\big) \end{aligned} \tag{8.18}$$

where $\mathfrak{g}^{D_N}\colon \mathbb{Z}^2 \to [0,1]$ is harmonic on $D_N \smallsetminus \{0\}$ with $\mathfrak{g}^{D_N}(0)=1$ and $\mathfrak{g}^{D_N}=0$ on D_N^{c}. In particular, the probability on the left is non-decreasing in both s and t.

Proof. The Gibbs-Markov decomposition of h^{D^N} reads

$$h^{D_N} \overset{\text{law}}{=} h^{D_N \smallsetminus \{0\}} + \varphi^{D_N, D_N \smallsetminus \{0\}}. \tag{8.19}$$

Now $\varphi^{D_N, D_N \smallsetminus \{0\}}$ has the law of the harmonic extension of h^{D_N} on $\{0\}$ to $D_N \smallsetminus \{0\}$. This means $\varphi^{D_N, D_N \smallsetminus \{0\}} = \mathfrak{g}^{D_N} h^{D_N}(0)$. Using this, the desired probability can be written as

$$P\big(h^{D_N \smallsetminus \{0\}} \leq (m_N + t)(1 - \mathfrak{g}^{D_N}) + s\big). \tag{8.20}$$

The claim then follows from the next exercise. □

Exercise 8.4 *For any finite $D \subset \mathbb{Z}^2$ with $0 \in D$,*

$$(h^D \,|\, h_0^D = 0) \overset{\text{law}}{=} h^{D \smallsetminus \{0\}}. \tag{8.21}$$

The conditioning the field to be zero is useful for the following reason:

Exercise 8.5 *Prove that*

$$h^{D_N \smallsetminus \{0\}} \underset{N \to \infty}{\overset{\text{law}}{\longrightarrow}} h^{\mathbb{Z}^2 \smallsetminus \{0\}} \tag{8.22}$$

in the sense of finite dimensional distributions.

Let us now inspect the event $h^{D_N} \leq m_N(1 - \mathfrak{g}^{D_N})$ — with t and s dropped for simplicity. The following representation using the Green function G^{D_N} will be useful

$$m_N\big(1 - \mathfrak{g}^{D_N}(x)\big) = m_N \frac{G^{D_N}(0,0) - G^{D_N}(0,x)}{G^{D_N}(0,0)}. \tag{8.23}$$

Now $m_N = 2\sqrt{g} \log N + o(\log N)$ while (for 0 deep inside D_N) $G^{D_N} = g \log N + O(1)$. With the help of the relation of the Green function to the potential kernel $\mathfrak{a}$ and its large-scale asymptotic form we then get

$$m_N\big(1 - \mathfrak{g}^{D_N}(x)\big) = \frac{2}{\sqrt{g}} \mathfrak{a}(x) + o(1) = 2\sqrt{g}\, \log|x| + O(1). \tag{8.24}$$

The restriction in the probability on the right of (8.18) is thus that the field in D_N pinned to zero at zero stays below the logarithmic cone $x \mapsto 2\sqrt{g} \log|x| + O(1)$. Notice that this is just like the restriction that the BRW on *all subtrees* along the path to the maximum stay below a linear curve; see e.g. (7.40).

In order to make the connection to the BRW derivations, we need to extract as much independence from the DGFF as possible. Obviously, the Gibbs-Markov property is the right tool to use here. We will work with a decomposition over a sequence of domains defined, for $k \geq 0$, by

$$\Delta^k := \begin{cases} \{x \in \mathbb{Z}^2 \colon |x|_\infty < 2^k\}, & \text{if } k < n, \\ D_N, & \text{if } k = n, \end{cases} \tag{8.25}$$

where n is the largest integer such that $\{x \in Z^2 \colon |x|_\infty \leq 2^{n+1}\} \subseteq D_N$. The Gibbs-Markov property now gives

$$h^{D_N} = h^{\Delta^n} \overset{\text{law}}{=} h^{\Delta^{n-1}} + h^{\Delta^n \smallsetminus \overline{\Delta^{n-1}}} + \varphi^{\Delta^n, \Delta^n \smallsetminus \partial \Delta^{n-1}} \tag{8.26}$$

where ∂D is the set of vertices on external boundary of D and $\overline{D} := D \cup \partial D$. This relation can now obviously be iterated to yield:

Lemma 8.6 *For the setting as above,*

$$h^{D_N} \overset{\text{law}}{=} \sum_{k=0}^{n} \bigl(\varphi_k + h'_k\bigr) \tag{8.27}$$

where all fields on the right are independent with

$$\varphi_k \overset{\text{law}}{=} \varphi^{\Delta^k, \Delta^k \smallsetminus \partial\Delta^{k-1}} \quad \textit{and} \quad h'_k \overset{\text{law}}{=} h^{\Delta^k \smallsetminus \overline{\Delta^{k-1}}} \tag{8.28}$$

for $k = 1, \dots, n$ and

$$\varphi_0 \overset{\text{law}}{=} h^{\{0\}} \quad \textit{and} \quad h'_0 = 0. \tag{8.29}$$

Proof. Apply induction on (8.26) watching for the provisos for $k = 0$. $\square$

The representation (8.27) is encouraging in that it break h^{D_N} into the sum of independent contributions of which one (the φ_k's) are "smooth" while the other (the h'_k's) is "rough" and large. However, to make the correspondence with the BRW closer, we need to somehow identify a Gaussian random walk in this expression. Here we use that, since φ_k has harmonic sample paths on $\Delta^k \smallsetminus \partial\Delta^{k-1}$, its values are well represented by the value at the origin. This is the content of:

Proposition 8.7 [Concentric decomposition of DGFF] *For the setting as above,*

$$h^{D_N} \overset{\text{law}}{=} \sum_{k=0}^{n} \Bigl(\bigl(1 + \mathfrak{b}_k\bigr)\varphi_k(0) + \chi_k + h'_k\Bigr), \tag{8.30}$$

where all the fields $\{\chi_k \colon k \ge 0\}$, $\{h'_k \colon k \ge 0\}$ and the random variables $\{\varphi_k(0) \colon k \ge 0\}$ are independent with the law of $\varphi_k(0)$ and h'_k as in (8.28–8.29) and with

$$\chi_k(\cdot) \overset{\text{law}}{=} \varphi_k(\cdot) - E\bigl(\varphi_k(\cdot) \,\big|\, \sigma(\varphi_k(0))\bigr) \tag{8.31}$$

and $\mathfrak{b}_k \colon \mathbb{Z}^2 \to \mathbb{R}$ defined by

$$\mathfrak{b}_k(x) := \frac{E\bigl([\varphi_k(x) - \varphi_k(0)]\varphi_k(0)\bigr)}{E\bigl(\varphi_k(0)^2\bigr)}. \tag{8.32}$$

Proof. Define χ_k on the same probability space as φ_k by the right-hand side of (8.31). Then χ_k and $\varphi_k(0)$ are uncorrelated and thus independent. Moreover, the fact that conditional expectation is a projection in L^2 ensures that $E\bigl(\varphi_k(\cdot) \,\big|\, \sigma(\varphi_k(0))\bigr)$ is a linear function of $\varphi_k(0)$. The fact that these fields have zero mean then implies

$$E\bigl(\varphi_k(x) \,\big|\, \sigma(\varphi_k(0))\bigr) = \mathfrak{f}_k(x)\varphi_k(0) \tag{8.33}$$

for some deterministic $\mathfrak{f}_k \colon \mathbb{Z}^2 \to \mathbb{R}$. Comparing covariances then yields $\mathfrak{f}_k = 1 + \mathfrak{b}_k$. Substituting

$$\varphi_k = (1 + \mathfrak{b}_k)\varphi_k(0) + \chi_k, \tag{8.34}$$

which, we note, includes the case $k = 0$, into (8.27) then gives the claim. $\square$

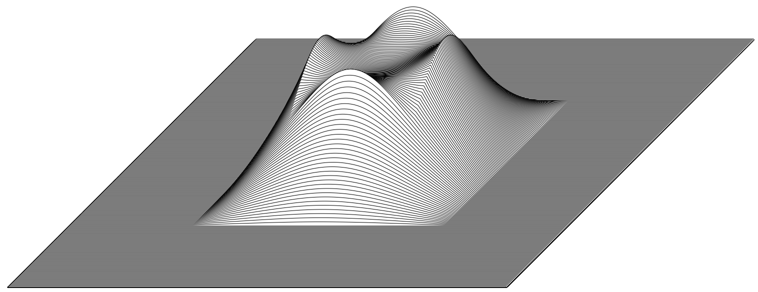

Figure 8.1: A plot of function $\mathfrak{b}_k$ on a large set Δ^{k+1}. The function equals -1 outside Δ^k and 0 at the origin. It is analytic on $\Delta^k \smallsetminus \partial\Delta^{k-1}$.

8.3. Bounding the bits and pieces

An obvious advantage of (8.30) is that it gives us a representation of DGFF as the sum of *independent* objects. However, in order to make use of that, we need estimates on the sizes of these objects as well. These will depend on the underlying set D_N but only via the smallest $k_1 \in \mathbb{N}$ such that

$$D_N \subseteq \{x \in Z^2 \colon |x|_\infty \le 2^{n+1+k_1}\} \tag{8.35}$$

with n as above. We thus assume this k_1 to be fixed; all estimates are then uniform in domains satisfying (8.35). We begin with $\varphi_k(0)$'s:

Lemma 8.8 *For each $\epsilon > 0$ there is $k_0 \ge 0$ such that*

$$\max_{k=k_0,\dots,n-1} \Bigl|\mathrm{Var}\bigl(\varphi_k(0)\bigr) - g\log 2\Bigr| < \epsilon \tag{8.36}$$

The variance of $\varphi_n(0)$ is bounded only in terms of k_1 above.

Proof (sketch). For $k < n$, $\varphi_k(0)$ admits a scaling limit to the continuum binding field $\Phi^{B_2,B_2\smallsetminus\partial B_1}(0)$, where $B_r := [-r,r]^2$. An explicit calculation with the covariances $C^{D,\widetilde{D}}$ shows $\mathrm{Var}(\Phi^{B_2,B_2\smallsetminus\partial B_1}(0)) = g\log 2$. □

Lemma 8.9 *The function $\mathfrak{b}_k$ is bounded uniformly in k. It is harmonic on $\Delta^k \smallsetminus \partial\Delta^{k-1}$ with $\mathfrak{b}_k(0) = 0$ and $\mathfrak{b}_k(\cdot) = -1$ on $(\Delta^k)^{\mathrm{c}}$. There is $c > 0$ such that for all $k \ge 0$,*

$$\bigl|\mathfrak{b}_k(x)\bigr| \le c\frac{\mathrm{dist}(x,\partial\Delta^k)}{\mathrm{dist}(0,\partial\Delta^k)}, \qquad x \in \Delta^{k-2}. \tag{8.37}$$

Proof (sketch). The harmonicity of $\mathfrak{b}_k$ follows from harmonicity of φ_k. The overall boundedness is checked by representing $\mathfrak{b}_k$ using covariances of φ_k. The bound (8.37) then follows from uniform Lipschitz continuity of the (discrete) Poisson kernels on square domains. (The bound is not claimed for $k = n$.) See Fig. 8.1. □

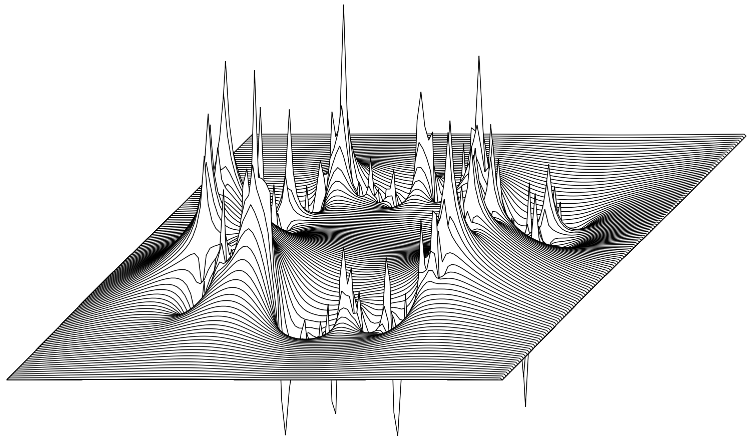

Figure 8.2: A plot of a sample of χ_k for $k := 7$.

Lemma 8.10 *For $k = 0, \dots, n$ and $\ell = 0, \dots, k-2$,*

$$E\Bigl(\max_{x\in\Delta^\ell}\bigl|\chi_\ell(x)\bigr|\Bigr) \le c2^{\ell-k} \tag{8.38}$$

and

$$P\Bigl(\Bigl|\max_{x\in\Delta^\ell}\chi_\ell(x) - E\max_{x\in\Delta^\ell}\chi_\ell(x)\Bigr| > \lambda\Bigr) \le \mathrm{e}^{-c4^{k-\ell}\lambda^2}. \tag{8.39}$$

Proof (idea). These are consequences of Fernique majorization and Borell-TIS inequality and Lipschitz property of the covariances of φ_k (which inherit to χ_k). □

The case when $\ell = k$ has intentionally been left out of the previous lemma because the function χ_k ceases to be regular near $\partial\Delta^{k-1}$; see Fig. 8.2. Here we will combine χ_k with h'_k (and χ_{k-1} to get:

Lemma 8.11 [Consequence of upper-tail estimate of M_N] *There is $a > 0$ such that each $k = 1, \dots, n$ and each $t \ge 0$,*

$$P\Bigl(\max_{x\in\Delta^k\smallsetminus\Delta^k}\bigl[\chi_{k-1}(x) + \chi_k(x) + h'_k(x)\bigr] \ge m_{2^k} + t\Bigr) \le \mathrm{e}^{-at}. \tag{8.40}$$

Proof. Recalling how the concentric decomposition was derived,

$$\varphi_{k-1} + \varphi_k + h'_k \overset{\text{law}}{=} h^{\Delta^k\smallsetminus\overline{\Delta^{k-1}}} \quad \text{on } \Delta^k \smallsetminus \Delta^{k-1} \tag{8.41}$$

Lemma 8.2 along with the Gibbs-Markov property show this field has exponential upper tail above m_{2^k}. But this field differs from the one in the statement by the term $(1 + \mathfrak{b}_k)\varphi_k(0) + (1 + \mathfrak{b}_{k-1})\varphi_{k-1}(0)$ which has even a Gaussian tail. The claim follows by a simple estimate of these contributions. □

Remark 8.12 Once we prove tightness of the maximum of the DGFF, we will in fact be able to put absolute values around the field in square brackets in (8.40). However, at this point we are able to claim only a bound on the upper tail thereof.

The fact that $h^{\Delta^n}(0) = \sum_{k=0}^n \varphi_k(0)$ now gives us a representation of the value at the prospective maximum by a Gaussian random walk. The k-th step of this walk is

$$S_k := \sum_{\ell=0}^{k-1} \varphi_\ell(0). \tag{8.42}$$

Then

$$h^{\Delta^n}(0) = 0 \quad \Leftrightarrow \quad S_{n+1} = 0. \tag{8.43}$$

This drives the following interesting exercise:

Exercise 8.13 *The DGFF on $\mathbb{Z}^2 \smallsetminus \{0\}$ can be represented as the a.s.-convergent sum*

$$h^{\mathbb{Z}^2 \smallsetminus \{0\}} \overset{\text{law}}{=} \sum_{k=0}^{\infty} \Bigl(\mathfrak{b}_k(x)\varphi_k(0) + \chi_k + h'_k\Bigr), \tag{8.44}$$

where the objects on the right are independent and with laws as above. See Exercise 8.5.

We note in passing that the above random walk can be thought of as somehow corresponding to circle averages of the CGFF; see Exercise 1.29. The best connection that we can see is that the random walk will indeed pretty much determine the behavior of the DGFF at a point where the field is large.

8.4. Random walk representation

We will now move to the discussion towards the proof of the lower bound on EM_N. A key technical step in this will be the proof of:

Proposition 8.14 *For all $\epsilon \in (0,1)$ there is $c = c(\epsilon) > 0$ such that for all $N > 2$ and all sets $D_N \subset \mathbb{Z}^2$ satisfying*

$$[-\epsilon N, \epsilon N]^2 \cap \mathbb{Z}^2 \subseteq D_N \subseteq [-\epsilon^{-1}N, \epsilon^{-1}N]^2 \cap \mathbb{Z}^2 \tag{8.45}$$

we have

$$P\bigl(\, h^{D_N} \le m_N \,\big|\, h_0^{D_N} = m_N\bigr) \ge \frac{c}{\log N}. \tag{8.46}$$

In order to prove this, we will need to control the growth of the various terms on the right-hand side of (8.27). We will use this using a *control variable K* that we define next:

Definition 8.15 [Control variable] *For k, ℓ integers denote*

$$\Theta_k(\ell) := \bigl[\log(k \vee (\ell \wedge (n-\ell)))\bigr]^2. \tag{8.47}$$

Then define K be the smallest positive integer k such that for all $\ell = 0, \dots, n$

(1) $|\varphi_\ell(0)| \le \Theta_k(\ell)$

(2) for all $r = 1, \dots, \ell-2$,

$$\max_{x\in\Delta^r} \bigl|\chi_\ell(x)\bigr| \le 2^{(r-\ell)/2}\Theta_k(\ell) \tag{8.48}$$

(3)

$$\max_{x\in\Delta^\ell\smallsetminus\Delta^{\ell-1}} \bigl[\chi_\ell(x)+\chi_{\ell-1}(x)+h'_\ell(x)\bigr] \le m_{2^\ell}+\Theta_k(\ell) \tag{8.49}$$

We call K the control variable.

Based on the above lemmas, we readily check that, for some $c>0$,

$$P(K=k) \le \mathrm{e}^{-c(\log k)^2}, \qquad k\ge 1, \tag{8.50}$$

which indicates that the control that the control variable provides will be good. Unfortunately, we will need to control the growth of the relevant variables on the background of events whose probability will decay to zero as $n\to\infty$. The key step is to link the event in Proposition 8.14 to the behavior of the above random walk. This is the content of:

Lemma 8.16 [Reduction to random walk event] *Assume h^{D_N} is realized as the sum on the right of* (8.30)*. There is a numerical constant $C>0$ such that uniformly in the above setting, the following holds for each $k=0,\dots,n$:*

$$\begin{aligned}\{h_0^{D_N}=0\}\cap\bigl\{h^{D_N}\le m_N(1-\mathfrak{g}^{D_N}) \text{ on } \Delta^k\smallsetminus\Delta^{k-1}\bigr\}&\\ \supseteq \{S_{n+1}=0\}\cap\bigl\{S_k\ge C[1+\Theta_K(k)]\bigr\}.&\end{aligned} \tag{8.51}$$

Proof. Fix k as above and let $x\in\Delta^k\smallsetminus\Delta^{k-1}$. In light of (8.43), on $\{h_0^{D_N}=0\}$ we can drop the "1" on the right-hand side of (8.30) without changing the result. Noting that $b_\ell(x)=-1$ for $\ell<k$, on this event we then get

$$\begin{aligned}h^{D_N}(x)-m_{2^k} = -S_k &+ \sum_{\ell=k}^n \mathfrak{b}_\ell(x)\varphi_\ell(0)\\ &+\sum_{\ell=k+1}^n \chi_\ell(x) + \bigl[\chi_{k-1}(x)+\chi_k(x)+h'_k(x)-m_{2^k}\bigr].\end{aligned} \tag{8.52}$$

The bounds in the definition of the control variable permit us to bound all terms but $-S_k$ by $C\Theta_K(k)$ from above, for $C>0$ a constant independent of k and x. Adjusting $\hat c$ if necessary, a careful use of the representation (8.23) shows

$$m_N\bigl(1-\mathfrak{g}^{D_N}(x)\bigr) \ge m_{2^k}-C. \tag{8.53}$$

Hence

$$m_N(1-\mathfrak{g}^{D_N})-h^{D_N}(x) \ge m_{2^k}-h^{D_N}(x)-C \ge S_k - C[1+\Theta_K(k)]. \tag{8.54}$$

This now readily yields the claim. □

Moving over to the probability in the statement of Proposition 8.14, we first write

$$P\big(h^{D_N} \le m_N \,\big|\, h_0^{D_N} = m_N\big) = P\big(h^{D_N\smallsetminus\{0\}} \le m_N(1-\mathfrak{g}^{D_N})\big) \tag{8.55}$$

Next we observe:

Exercise 8.17 *Prove that, for any finite $V \subset \mathbb{Z}^2$, the DGFF on V is positively associated in the sense of FKG. Explicitly, show that any two increasing integrable functions of h^V are positively correlated.*

We are now ready to give:

Proof of Proposition 8.14. We now fix $k \in \{1,\dots,n\}$ and note that the relevant probability (recast for the DGFF in $D_N \smallsetminus \{0\}$ and adjusted via Lemma 8.3) can be bounded via

$$P\big(h^{D_N\smallsetminus\{0\}} \le m_N(1-\mathfrak{g}^{D_N})\big) \ge P(A^1_{n,k})P(A^2_{n,k})P(A^3_{n,k}) \tag{8.56}$$

where

$$\begin{aligned}
A^1_{n,k} &:= \big\{h^{D_N\smallsetminus\{0\}} \le m_N(1-\mathfrak{g}^{D_N}) \text{ on } \Delta^k\big\}\\
A^2_{n,k} &:= \big\{h^{D_N\smallsetminus\{0\}} \le m_N(1-\mathfrak{g}^{D_N}) \text{ on } \Delta^{n-k}\smallsetminus\Delta^k\big\}\\
A^3_{n,k} &:= \big\{h^{D_N\smallsetminus\{0\}} \le m_N(1-\mathfrak{g}^{D_N}) \text{ on } \Delta^{n}\smallsetminus\Delta^{n-k}\big\}
\end{aligned} \tag{8.57}$$

We now observe that, for any k fixed, we have

$$\inf_{n\ge1} P(A^1_{n,k}) > 0 \tag{8.58}$$

due to the fact that $h^{D_N\smallsetminus\{0\}}$ tends in law to $h^{\mathbb{Z}^2\smallsetminus\{0\}}$ (see Exercise 8.5) while the bound on its values tends to $\frac{2}{\sqrt{g}}\mathfrak{a}$, see (8.24). Noting similarly that

$$m_N(1-\mathfrak{g}^{D_N}) \ge m_N - c \quad \text{on } \Delta^n\smallsetminus\Delta^{n-k} \tag{8.59}$$

with c depending only on k,

$$P(A^3_{n,k}) \ge P\big(M_N \le m_N - c\big) \tag{8.60}$$

which is uniformly positive in $N \ge 1$ by a simple rewrite of the proof of Lemma 8.2. (The value of m_N there could be changed by a constant without changing the argument.) Hence, it suffices to bound $P(A^2_{n,k})$. Using Lemma 8.16 and the fact that $k \mapsto \Theta_k(\ell)$ is non-decreasing, we bound this as

$$\begin{aligned}
P(A^2_{n,k}) &\ge P\bigg(\{K\le k\}\cap\bigcap_{\ell=k+1}^{n-k-1}\{S_\ell \ge C[1+\Theta_k(\ell)]\}\,\bigg|\,S_{n+1}=0\bigg)\\
&\ge P\bigg(\bigcap_{\ell=k+1}^{n-k-1}\{S_\ell \ge C[1+\Theta_k(\ell)]\}\cap\bigcap_{\ell=0}^{n}\{S_\ell\ge -1\}\,\bigg|\,S_{n+1}=0\bigg)\\
&\qquad\qquad - P\bigg(\{K>k\}\cap\bigcap_{\ell=0}^{n}\{S_\ell\ge-1\}\,\bigg|\,S_{n+1}=0\bigg)
\end{aligned} \tag{8.61}$$

Next we note:

Lemma 8.18 [Entropic repulsion] *There is a constant $c > 0$ such that for all $n \ge 1$ and all $k = 1, \dots, \lfloor n/2 \rfloor$*

$$P\biggl(\bigcap_{\ell=k+1}^{n-k-1} \{S_\ell \ge C[1+\Theta_k(\ell)]\} \biggm| \bigcap_{\ell=0}^{n} \{S_\ell \ge -1\} \cap \{S_{n+1} = 0\} \biggr) \ge c \tag{8.62}$$

Lemma 8.19 *There is a constant $c' > 0$ such that for all $n \ge 1$ and all $k = 1, \dots, \lfloor n/2 \rfloor$,*

$$P\biggl(\{K > k\} \cap \bigcap_{\ell=0}^{n} \{S_\ell \ge -1\} \biggm| S_{n+1} = 0 \biggr) \le \frac{1}{n} \mathrm{e}^{-c'(\log k)^2} \tag{8.63}$$

We will not prove these lemmas here as that would take us on a detour to the area of "random walks above polylogarithmic curves" or "Inhomogenous Ballot Theorem" that we have no time for. Bramson's seminal work (Commun. Pure Appl. Math. 31 (1978), no. 5, 531–581) addresses these and so does the joint paper of the lecturer with O. Louidor (arXiv:1606.00510).

Denoting by $\{B_t \colon t \ge 0\}$ the standard Brownian motion and letting $\sigma_n^2 := \mathrm{Var}(S_{n+1})$, we can embed the random walk into Brownian motion to get

$$P\biggl(\bigcap_{\ell=0}^{n} \{S_\ell \ge -1\} \biggm| S_{n+1} = 0 \biggr) \ge P^0\Bigl(B \ge -1 \text{ on } [0, \sigma_n^2] \Bigm| B_{\sigma_n^2} = 0 \Bigr). \tag{8.64}$$

As σ_n^2 is proportional to n, the Reflection Principle bounds the last probability by c''/n for some $c'' > 0$. Putting these together and choosing k sufficiently large, we thus get $P(A_{n,k}) \ge c/n$. Since $n \approx \log_2 N$, this yields (8.46) via (8.56). □

8.5. Tightness of DGFF maximum: lower bound

We will now harvest the fruit of our hard labor in the previous sections and prove tightness of the maximum of DGFF. First we claim:

Lemma 8.20 *For the DGFF in V_N, we have*

$$\inf_{N \ge 1} P\bigl(M_N \ge m_N\bigr) > 0. \tag{8.65}$$

Proof. Let $V'_{N/2}$ be the square of side $N/2$ centered at the same point as V_N. For each $x \in V'_{N/2}$ and denoting $D_N := -x + V_N$, we have

$$\begin{aligned} P\bigl(h_x^{V_N} \ge m_N, h^{V_N} \le h_x^{V_N}\bigr) &= P\bigl(h_0^{D_N} \ge m_N,\, h^{D_N} \le h_0^{D_N}\bigr) \\ &= \int_0^\infty P\bigl(h_0^{D_N} - m_N \in \mathrm{d}s\bigr)\, P\bigl(h^{D_N} \le m_N + s \,\big|\, h_0^{D_N} = m_N + s\bigr) \end{aligned} \tag{8.66}$$

Rewriting the conditional probability using the DGFF on $D_N \smallsetminus \{0\}$, Proposition 8.14 yields for any $s \ge 0$ that

$$\begin{aligned} &P\bigl(h^{D_N} \le m_N + s \,\big|\, h_0^{D_N} = m_N + s\bigr) \\ &\qquad\qquad \ge P\bigl(h^{D_N} \le m_N \,\big|\, h_0^{D_N} = m_N\bigr) \ge \frac{c}{\log N}. \end{aligned} \tag{8.67}$$

Hence we get

$$P\big(h_x^{V_N} \geq m_N,\, h^{V_N} \leq h_x^{V_N}\big) \geq \frac{c}{\log N} P\big(h_x^{V_N} \geq m_N\big). \tag{8.68}$$

A calculation now shows $P\big(h_x^{V_N} \geq m_N\big) \geq c'(\log N)N^{-2}$. The constants c and c' work uniformly for all $x \in V'_{N/2}$. Invoking

$$P(M_N \geq m_N) \geq \sum_{x \in V'_{N/2}} P\big(h_x^{V_N} \geq m_N,\, h^{V_N} \leq h_x^{V_N}\big) \tag{8.69}$$

the claim thus follows the fact that $|V'_{N/2}|$ has order N^2 vertices. □

Now we claim:

Lemma 8.21 [Tightness of lower tail] *There is $a > 0$ and $t_0 > 0$ such that*

$$\sup_{N \geq 1} P\big(M_N < m_N - t\big) \leq \mathrm{e}^{-at^2}, \quad t > t_0. \tag{8.70}$$

We remark that this bound is not sharp even as far its structure is concerned. Indeed, the lower tails of M_N are known to have a doubly exponential tail. However, the proof of the above is easier and suffices for our needs.

Proof. Consider the box V_{3N} and let V'_N be the translate of V_N centered at the same point as V_{3N}. Define

$$\widetilde{M}_N := \max_{x \in V'_N} h_x^{V_{3N}}. \tag{8.71}$$

Now consider the box V_{18N} and let V'_{6N} be the translate of V_{6N} centered at the same point as V_{18N}. Removing the "axes of symmetry" from V'_{6N}, we get four translates $V_{3N}^{(1)}, \dots, V_{3N}^{(4)}$ of V_{3N}. Letting

$$V_{18N}^{\circ} := V_{18N} \smallsetminus \bigcup_{i=1}^{4} \partial V_{3N}^{(i)} \tag{8.72}$$

the Gibbs-Markov property then permits us to realize $h^{V_{18N}}$ as $h^{V_{18N}^{\circ}} + \varphi^{V_{18N}, V_{18N}^{\circ}}$.

Let $\widehat{M}_N^{(i)}$, for $i = 1, \dots, 4$, be the maximum of $h^{V_{18N}^{\circ}}$ in the translate of V_N centered at the same point as $V_{3N}^{(i)}$. Let $\widehat{\varphi}_i$ denote the minimum of $\varphi^{V_{18N}, V_{18N}^{\circ}}$ on the same translate. For any $\lambda \in \mathbb{R}$, we then have

$$\big\{\widehat{M}_{18N} \leq \lambda\big\} \subseteq \bigcap_{i=1}^{4} \big\{\widehat{M}_N^{(i)} + \widehat{\varphi}_i \leq \lambda\big\}. \tag{8.73}$$

A combined use of Fernique majorization and Borell-TIS inequality imply that each $\hat{\varphi}_i$, and thus also their minimum, has a Gaussian tail. Using this above, from the fact that the $\widehat{M}_N^{(i)}$'s are independent and equidistributed we get

$$P\big(\widehat{M}_{18N} \leq \lambda\big) \leq \mathrm{e}^{-cr^2} + P\big(\widehat{M}_N \leq \lambda + r\big)^4 \tag{8.74}$$

as soon as, say, $r \geq 1$.

Suppose we know that, for some $t_0 > 0$ and $a > 0$,

$$P\bigl(\widehat{M}_N \leq m_N - t\bigr) \leq \mathrm{e}^{-at^2}, \quad t > t_0. \tag{8.75}$$

Then (8.74) along with the fact that $m_{18N} \leq m_N + g\log(18)$ yields

$$\begin{aligned} P\bigl(\widehat{M}_{18N} \leq m_{18N} - t\bigr) &\leq \mathrm{e}^{-cr^2} + P\bigl(\widehat{M}_N \leq m_{18N} - t + r\bigr)^4 \\ &\leq \mathrm{e}^{-cr^2} + \mathrm{e}^{-4a[t-r-g\log(18)]^2} \end{aligned} \tag{8.76}$$

Setting $r := t/3$ (which requires $t \geq 3$) and assuming that also $t > 6g\log(18)$ then bounds the right hand side by $\mathrm{e}^{-ct^2/9} + \mathrm{e}^{2at^2}$. This is less than e^{-at^2} provided we choose a so small that $c/9 > 2a$ and t_0 such that $\mathrm{e}^{at_0} \geq 2$. Since the bound (8.75) obviously holds for $N := 18$ and a small enough, we get it for any power of 18.

To get the desired claim from (8.75), we first note that $M_N \geq \widehat{M}_N$ and then observe that $N \mapsto M_N$ is stochastically increasing (and so proving the claim along powers of 18 is sufficient). □

From here we now get:

Proof of Theorem 7.3. Thanks to Lemmas 8.2 and 8.21, we have $EM_N = m_N + O(1)$. Lemma 7.1 then ensures that $\{M_N - m_N \colon N \geq 1\}$ is tight. □