Lecture 7

Connection to Branching Random Walk

The aim of this lecture is to prepare the grounds for the proof of tightness of the maximum of the DGFF. We will begin with a recount of the so called Dekking-Host argument which yields, rather seamlessly, tightness along certain subsequences. Going beyond this will require development of a connection to Branching Random Walk and proving sharp concentration for the maximum thereof. We then show how this can be used to curb the upper tail of the maximum of the DGFF. The lower tail will be dealt with in the next lecture.

7.1. Dekking-Host argument for DGFF

Understanding the law of the maximum of the DGFF has been one of the holy grails of this whole subject area. We have already shown that the maximum M_N of the DGFF in box of side N,

$$M_N := \max_{x\in V_N} h_x^{V_N}\,, \tag{7.1}$$

grows as $M_N \sim 2\sqrt{g}\log N$ in probability, with the same growth rate for EM_N. The natural questions then are:

(1) what is the precise growth rate of EM_N; i.e., the lower order corrections?

(2) what is the size of the fluctuations, i.e., the growth rate of $M_N - EM_N$?

As observed by Bolthausen, Deuschel and Zeitouni in 2011, an argument that goes back to a paper by Dekking and Host from 1991 shows that for the DGFF that these seemingly unrelated questions are tied together. This is quite apparent from:

lemma-DH **Lemma 7.1 [Dekking-Host argument]** *For M_N as above and any $N\ge 2$,*

$$E\bigl|M_N - EM_N\bigr| \le 2\bigl(EM_{2N} - EM_N\bigr). \tag{7.2}$$ E:7.2

Proof. We will use an idea underlying the solution to the second part of Exercise 3.4. Consider the box V_{2N} and note that it embeds four translates $V_N^{(1)},\dots,V_N^{(4)}$ of V_N

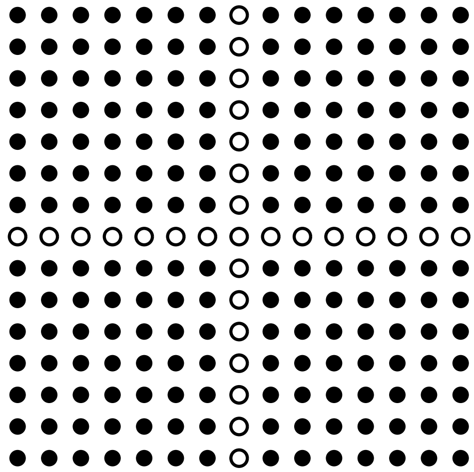

fig-box

Figure 7.1: The partition of box V_{2N} (both kinds of bullets) into four translates of V_N for $N := 8$ and two line of sites (empty bullets) in the middle. The set V_{2N}° is the collection of all full bullets.

such that any pair of these translates is separated by a line of sites in-between; see Fig. 7.1. Denoting the union of the four transates by V_{2N}°, the Gibbs-Markov property tells us that the field

$$h^{V_{2N}} := h^{V_{2N}^\circ} + \varphi^{V_{2N},V_{2N}^\circ}, \quad \text{with} \quad h^{V_{2N}^\circ} \perp\!\!\!\perp \varphi^{V_{2N},V_{2N}^\circ}, \tag{7.3}$$

has the law of DGFF in V_{2N}. Writing M_{2N} for the maximum of the field on the left, letting X denote the (a.s.-unique) vertex where $h^{V_{2N}^\circ}$ achieves its maximum and abbreviating

$$M_N^{(i)} := \max_{x\in V_N^{(i)}} h_x^{V_{2N}^\circ}, \quad i = 1,\dots,4, \tag{7.4}$$

it follows that

$$EM_{2N} = E\bigl(\max_{i=1,\dots,4} M_N^{(i)} + \varphi_X^{V_{2N},V_{2N}^\circ}\bigr)\,. \tag{7.5}$$

But $\varphi^{V_{2N},V_{2N}^\circ}$ is independent of $h^{V_{2N}^\circ}$ and so of X as well. Hence, $E\varphi_X^{V_{2N},V_{2N}^\circ} = 0$. Moreover, $\{M_N^{(i)}\colon i = 1,\dots,4\}$ depend on independent DGFFs and are thus independent as well. Bounding the maximum of four terms by that of just the first two, we get

$$M_N' \overset{\text{law}}{=} M_N,\, M_N' \perp\!\!\!\perp M_N \quad \Rightarrow \quad EM_{2N} \geq E\max\{M_N, M_N'\} \tag{7.6}$$

Now use that $2\max\{a,b\} = a + b + |a-b|$ to conclude

$$E\bigl|M_N - M_N'\bigr| = 2E\max\{M_N, M_N'\} - E(M_N + M_N') \leq 2EM_{2N} - 2EM_N. \tag{7.7}$$

The claim follows by using Jensen's inequality to pass expectation over M_N' inside the absolute value. $\square$

From the growth rate of EM_N we then readily conclude:

Corollary 7.2 [Tightness along subsequences] *There is a deterministic sequence* $\{N_k\colon k\geq 1\}$ *of integers with* $N_k\to\infty$ *such that* $\{M_{N_k}-EM_{N_k}\colon k\geq 1\}$ *is tight.*

Proof. Denote $a_n := EM_{2^n}$. The bound (7.2) shows that $\{a_n\}$ is non-decreasing. The fact that $EM_N\leq c\log N$ (proved earlier by elementary first-moment calculations) reads as $a_n\leq c'n$ for $c':=c\log 2$. The increments of an increasing sequence with at most linear growth cannot diverge to infinity, so there must be an increasing sequence $\{n_k\colon k\geq 1\}$ such that $a_{n_k+1}-a_{n_k}\leq 2c'$. Setting $N_k:=2^{n_k}$, this implies $E|M_{N_k}-EM_{N_k}|\leq 2c'$, which by Markov's inequality gives the stated tightness. $\square$

Unfortunately, compactness along (existential) subsequences is the best one can infer from the leading order asymptotic of EM_N. If we want to get any better along the same line of reasoning, we need to control the asymptotic of EM_N up to terms of order unity. This was achieved in:

thm-7.3 **Theorem 7.3 [Bramson and Zeitouni, 2012]** *Denote*

$$m_N := 2\sqrt{g}\,\log N-\frac{3}{4}\sqrt{g}\,\log\log N\,. \tag{7.8}$$ E:7.8

Then

$$\sup_{N\geq 1}\bigl|EM_N-m_N\bigr|<\infty. \tag{7.9}$$

As a consequence, $\{M_N-m_N\colon N\geq 1\}$ *is tight.*

The rest of this lecture will be spent on proving this theorem using, however, a somewhat different (and in the eyes of the lecturer, easier) approach than the one used by Bramson and Zeitouni.

7.2. Upper bound by Branching Random Walk

The Gibbs-Markov decomposition used in the proof of Lemma 7.1 can further be iterated as follows: Consider a box V_N of side $N=2^n$ for some $n\in\mathbb{N}$. The square V_N then contains four translates of $V_{2^{n-1}}$ separated by a "cross" of sites in-between. Each of these squares then contain four translates of $V_{2^{n-2}}$, etc. Letting, inductively, $V_n^{(i)}$ denote the union of the resulting 4^i translates of $V_{2^{n-i}}$ contained in the squares constituting $V_n^{(i-1)}$, see Fig. 7.2, with $V_N^{(0)}:=V_N$, we can then write

$$\begin{aligned}
h^{V_N} &\overset{\text{law}}{=} h^{V_N^{(1)}}+\varphi^{V_N^{(0)},V_N^{(1)}}\\
&\overset{\text{law}}{=} h^{V_N^{(2)}}+\varphi^{V_N^{(0)},V_N^{(1)}}+\varphi^{V_N^{(1)},V_N^{(2)}}\\
&\;\;\vdots \qquad\quad \ddots \qquad\qquad \ddots\\
&\overset{\text{law}}{=} \varphi^{V_N^{(0)},V_N^{(1)}}+\dots+\varphi^{V_N^{(n-1)},V_N^{(n)}}
\end{aligned} \tag{7.10}$$ E:7.10

where $V_N^{(n)}=\emptyset$ and so $\varphi^{V_N^{(n-1)},V_N^{(n)}}=h^{V_N^{(n-1)}}$. As seen in Fig. 7.2, the latter field is just a collection of independent Gaussians, one for each element of $V_N^{(n-1)}$.

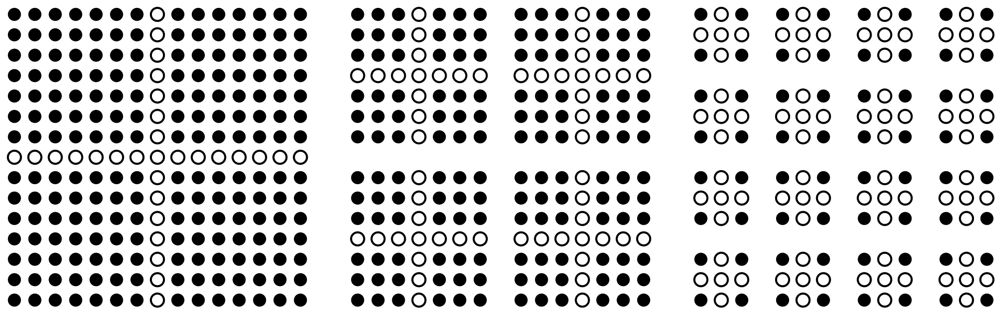

fig-box-hierarchy

Figure 7.2: The sets $V_N^{(1)}$, $V_N^{(2)}$ and $V_N^{(3)}$ corresponding to the hierarchical representation of the DGFF on V_N with $N := 16$. The empty bullets mark the sets of vertices that are being removed to define $V_N^{(i)}$. Boundary vertices (where the fields are set to zero by default) are not depicted otherwise. The binding field $\varphi^{V_N^{(1)},V_N^{(2)}}$ is independent on each of the four squares constituting $V_N^{(1)}$, but is neither constant nor independent on the squares constituting $V_N^{(2)}$.

All fields on the right-hand side of (7.10) are independent. Moreover, $\varphi^{V_N^{(i)},V_N^{(i+1)}}$ is actually a concatenation of independent (and identically distributed) fields, one for each of 4^i copies of $V_{2^{n-i}}$ constituting $V_N^{(i)}$. The point is that the values of $\varphi^{V_N^{(i-1)},V_N^{(i)}}$ are not constant on the copies of $V_{2^{n-i-1}}$ constituting $V_N^{(i+1)}$. It it were constant and independent as stated, we would get a representation of the DGFF by means of a Branching Random Walk which we will introduce next.

For an integer $b \geq 2$, consider a b-ary tree T^b which is a connected graph without cycles where each vertex except one denoted by $\varnothing$ has exactly $b+1$ neighbors. The distinguished vertex $\varnothing$ is called the root; we require that the degree of the root is b. We will write L_n for the vertices at distance n from the root — these are the *leaves* at depth n.

Every vertex $x \in L_n$ can be identified with the sequence $(x_1, \dots, x_n) \in \{1, \dots, b\}^n$ where x_i can be thought of as an instruction which "turn" to take at the i-th step on the (unique) path from the root to x. Note that the specific case of $b = 4$ can be identified with a binary decomposition of $\mathbb{Z}^2$ as follows: Write $x \in \mathbb{Z}^2$ with non-negative coordinates in vector notation as

$$x = \Bigl(\sum_{i\geq 0} \sigma_i 2^i, \sum_{i\geq 0} \tilde{\sigma}_i 2^i\Bigr), \tag{7.11}$$

where $\sigma_i, \sigma_i \in \{0,1\}$ for all $i \geq 0$. Setting

$$x_i = 2\sigma_{n-i+1} + \tilde{\sigma}_{n-i+1} + 1 \tag{7.12}$$

we can then identify V_{2^n} with a (proper) subset of L_n.

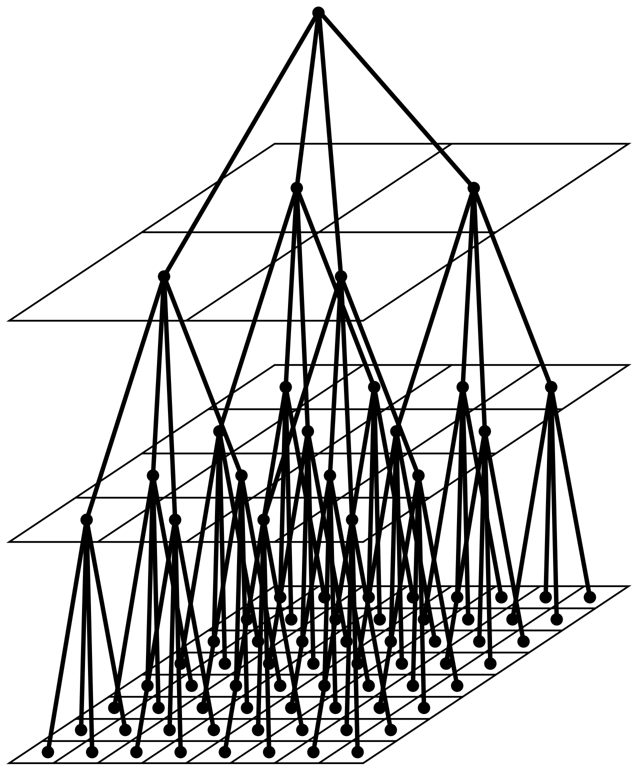

fig-pyramid

Figure 7.3: An schematic picture of how the values of BRW on T^4 are naturally interpreted as a field on $\mathbb{Z}^2$. There is a close link to the Gibbs-Markov decomposition of the DGFF from (7.10).

Definition 7.4 [Branching random walk] *Given integers $b \geq 2$, $n \geq 1$ and a random variable Z, let $\{Z_x \colon x \in T^b\}$ be i.i.d. copies of Z indexed by the vertices of T^b. The* Branching Random Walk (BRW) on T^b of depth n with step distribution Z *is then the family of random variables $\{\varphi_x^{T^b} \colon x \in L_n\}$ where for $x = (x_1, \dots, x_n) \in L_n$ we set*

$$\phi_x^{T^b} := \sum_{k=0}^{\infty} Z_{(x_1,\dots,x_k)}, \tag{7.13}$$

with the $k = 0$ term associated with the root value $Z_\varnothing$.

The specific case of interest for us is the *Gaussian Branching Random Walk* where we take Z normal. The value of the BRW at a given point $x \in L_n$ is then very much like (7.10) — the sum of the independent Gaussians along the unique path from the root to x. As already noted, the correspondence is not perfect (also because L_n has more active vertices than V_{2^n}). However, we can still use it fruitfully to get:

lemma-UB-BRW

Lemma 7.5 [Upper bound of DGFF by BRW] *Consider a BRW ϕ^{T^4} on a 4-ary tree T_n^4 with step distribution $\mathcal{N}(0,1)$ and identify V_N for $N := 2^n$ with a subset of L_n as above.*

There is $c > 0$ such that for each $n \geq 1$ and each $x, y \in L_n$,

$$E\big([h_x^{V_N} - h_y^{V_N}]^2\big) \leq c + (g\log 2)\, E\Big(\big[\phi_x^{T^4} - \phi_y^{T^4}\big]^2\Big) \tag{7.14}$$ E:7.13

In particular, there is $k \in \mathbb{N}$ such that for each $n \geq 1$ (and $N := 2^n$),

$$E\Big(\max_{x \in V_N} h_x^{V_N}\Big) \leq \sqrt{g\log 2}\, E\Big(\max_{x \in L_{n+k}} \phi_x^{T^4_{n+k}}\Big). \tag{7.15}$$ E:7.14

Proof. Since $V \mapsto E([h_x^V - h_y^V]^2)$ is non-decreasing, the representation of the Green function from Lemma 1.19 along with the asymptotic for the potential kernel from Lemma 1.21 shows that, for some constant $\tilde{c} > 0$ and all $x, y \in V_N$,

$$E\big([h_x^{V_N} - h_y^{V_N}]^2\big) \leq \tilde{c} + 2g\log|x - y|. \tag{7.16}$$ E:7.15

Denoting by $d_n(x, y)$ the *ultrametric distance* between for $x, y \in L_n$, i.e., the distance on T^b from x to the nearest common ancestor with y, we have

$$E\Big(\big[\phi_x^{T^4} - \phi_y^{T^4}\big]^2\Big) = 2d_n(x, y). \tag{7.17}$$ E:7.16

We now pose:

Exercise 7.6 *There is $\tilde{c}' \in (0, \infty)$ such that for each $n \geq 1$ and each $x, y \in L_n$,*

$$|x - y| \leq \tilde{c}'\, 2^{d_n(x,y)} \tag{7.18}$$

Combining this with (7.16–7.17), we then get (7.14).

To get (7.15), let $k \in \mathbb{N}$ be so large that c in (7.14) obeys $c \leq k\,g\log 2$. Now, for each $x = (x_1, \dots, x_d) \in L_n$ let $\theta(x) := (x_1, \dots, x_n, 1, \dots, 1) \in L_{n+k}$. Then (7.14) can, with the help of (7.17), be recast as

$$E\big([h_x^{V_N} - h_y^{V_N}]^2\big) \leq (g\log 2)\, E\Big(\big[\phi_{\theta(x)}^{T^4} - \phi_{\theta(y)}^{T^4}\big]^2\Big), \qquad x, y \in L_n. \tag{7.19}$$

The Sudakov-Fernique inequality then gives

$$E\Big(\max_{x \in V_N} h_x^{V_N}\Big) \leq \sqrt{g\log 2}\, E\Big(\max_{x \in L_n} \phi_{\theta(x)}^{T^4}\Big). \tag{7.20}$$

The claim now follows by extending the maximum to all vertices in L_{n+k}. □

7.3. Maximum of Gaussian Branching Random Walk

In order to use Lemma 7.5 to bound the expected maximum of the DGFF, we need to control the maximum of the BRW. This is a classical subject with strong connections to large deviation theory. (Indeed, as there are b^n branches of the tree the maximum will be carried by unusual events whose probability decays exponentially with n.) For Gaussian BRW, we can instead rely on explicit calculations and so the asymptotic is completely explicit as well:

thm-7.7 **Theorem 7.7 [Maximum of Gaussian BRW]** *For $b \geq 2$, let $\{\phi_x^{T^b} \colon x \in x \in T^b\}$ be the Branching Random Walk on b-ary tree with step distribution $\mathcal{N}(0,1)$. Then*

$$E\Bigl(\max_{x\in L_n} \phi_x^{T^b}\Bigr) = \sqrt{2\log b}\, n - \frac{3}{2\sqrt{2\log b}} \log n + O(1), \tag{7.21}$$ E:7.20

where $O(1)$ is a quantity that remains bounded as $n \to \infty$.

Denote

$$\widetilde{m}_n := \sqrt{2\log b}\, n - \frac{3}{2\sqrt{2\log b}} \log n \tag{7.22}$$ E:7.22

The proof starts by showing that the expected maximum is $\geq \widetilde{m}_n + O(1)$. This is achieved via a second moment estimate of the kind we saw in our discussion of the intermediate level set of the DGFF. However, as we are dealing with the absolute maximum, a truncation is necessary. Thus, for $x = (x_1, \dots, x_n) \in L_n$, let

$$G_n(x) := \bigcap_{k=1}^{n} \Bigl\{ \phi^{T^b}_{(x_1,\dots,x_k)} \geq \frac{k}{n+1}\widetilde{m}_n - \bigl(k \wedge (n-k)\bigr)^{3/4} - 1 \Bigr\} \tag{7.23}$$

be the "good" event enforcing linear growth of the values of the BRW on the unique path from the root to x. Now define

$$\Gamma_n := \bigl\{ x \in L_n \colon \phi_x^{T^b} \geq \widetilde{m}_n,\, G_n(x) \text{ occurs} \bigr\} \tag{7.24}$$

as the analogue of the truncated level set from our discussion of intermediate levels of the DGFF. We now claim:

Lemma 7.8 *For the setting as above,*

$$\inf_{n\geq 1} E|\Gamma_n| > 0 \tag{7.25}$$ E:7.24

while

$$\sup_{n\geq 1} E\bigl(|\Gamma_n|^2\bigr) < \infty. \tag{7.26}$$ E:7.25

We will only prove the first part as it quite instructive while leaving the second part as a technical exercise:

Proof of (7.25). Fix $x \in L_n$ and, for $k = 0, \dots, n$, abbreviate $Z_0 := Z_{(x_1,\dots,x_k)}$. Also denote $\theta_n(k) := (k \wedge (n-k))^{3/4} + 1$. Then

$$\begin{aligned} &P\bigl(\phi_x^{T^b} \geq \widetilde{m}_n,\, G_n(x) \text{ occurs}\bigr) \\ &\quad = P\Biggl(\{Z_1 + \dots + Z_n \geq \widetilde{m}_n\} \cap \bigcap_{k=0}^{n-1} \Bigl\{ Z_0 + \dots + Z_k \geq \frac{k}{n+1}\widetilde{m}_n - \theta_n(k) \Bigr\} \Biggr) \end{aligned} \tag{7.27}$$

Conditioning i.i.d. Gaussians on their total sum can be reduced to shifting these Gaussians by the arithmetic mean of their values. Denoting

$$\mu_n(\mathrm{d}s) := P(Z_0 + \dots + Z_n - \widetilde{m}_n \in \mathrm{d}s) \tag{7.28}$$ E:7.27

this allows us to express the desired probability as

$$\int_0^\infty \mu_n(\mathrm{d}s) P\biggl(\bigcap_{k=0}^{n-1}\Bigl\{Z_0+\dots+Z_k \ge -\frac{k}{n+1}s-\theta_n(k)\Bigr\}\,\Bigg|\, Z_0+\dots+Z_n=0\biggr) \tag{7.29}$$ E:7.29

Realizing Z_k as the increment of the standard Brownian motion on $[k, k+1)$ and recalling that the Brownian Bridge from $[0, r]$ is the standard Brownian motion conditioned to vanish at time r, the giant probability on the right is bounded from below by the probability that the standard Brownian bridge on $[0, n+1]$ stays above -1 for all times in $[0, n+1]$. Here we observe:

Exercise 7.9 *Let $\{B_t\colon t \ge 0\}$ be the standard Brownian motion started from* 0*. Prove that for all $a > 0$ and all $r > 0$,*

$$P^0\bigl(B_t \ge -a \,|\, B_r = 0\bigr) = 1 - \exp\{-2a^2 r^{-1}\}. \tag{7.30}$$

Hence, the giant probability in (7.29) is at least a constant times $1/n$. A calculation shows that, for some constant $c > 0$,

$$\mu_n\bigl([0,\infty)\bigr) \ge \frac{c}{\sqrt{n}}\,\mathrm{e}^{-\frac{\widetilde{m}_n^2}{2(n+1)}} = c\,\mathrm{e}^{O(n^{-1}\log n)} b^{-n} n \tag{7.31}$$

thanks to our choice of $\widetilde{m}_n$. The linear term in n cancels the $1/n$ arising from the Brownian-bridge estimate and so we conclude that the desired probability is at least a constant times b^{-n}. Summing over all $x \in L_n$, we get (7.25). $\square$

Exercise 7.10 *Prove* (7.26)*. (Note: It is here that we need the term $(k \wedge (n-k))^{3/4}$ in the definition of G_n. Any power larger than* 1/2 *will do.)*

As a consequence we get:

Corollary 7.11 *Using the notation $\widetilde{m}_n$ as above,*

$$\inf_{n\ge 1} P\Bigl(\max_{x\in L_n} \phi_x^{T^b} \ge \widetilde{m}_n\Bigr) > 0. \tag{7.32}$$ E:7.30

Proof. The probability is bounded below by $P(|\Gamma_n| > 0)$ which is bounded from below by the ratio of the first moment squared and the second moment; see (2.13). By (7.25–7.26), the said ratio is bounded away from zero uniformly in $n \ge 1$. $\square$

Next we boost this lower bound to an exponential tail estimate. (Note that we could have perhaps done this already at the level of the above moment calculation but only at the cost of making that calculation yet more complicated.) Our method of proof will be restricted to $b > 2$ and so this is what we will assume in the statement:

lemma-7.11 **Lemma 7.12** *For each integer $b > 2$ there is $a = a(b) > 0$ such that*

$$\sup_{n\ge 1} P\Bigl(\max_{x\in L_n} \phi_x^{T^b} < \widetilde{m}_n - t\Bigr) \le \mathrm{e}^{-at}, \quad t > 0. \tag{7.33}$$ E:7.31

In particular, "$\ge$" holds in (7.21)*.*

Proof. The proof will be based on a percolation argument. Recall that the threshold for site percolation on T^b is $p_c(b) = 1/b$. Thus, for $b > 2$ there is $\epsilon > 0$ such that the set $\{x \in T^b \colon Z_x \ge \epsilon\}$ contains an infinite connected component $\mathcal{C}$; we take the one which is closest to the origin.

Exercise 7.13 *Show that there are $\theta > 1$ and $c > 0$ such that for all $r \ge 1$,*

$$P\Big(\exists n \ge r \colon |\mathcal{C} \cap L_n| < \theta^n\Big) \le \mathrm{e}^{-cr} \tag{7.34}$$ E:7.33

We will now show that this implies:

$$P\big(|\{x \in L_k \colon \phi_x^{T^b} \ge 0\}| < \theta^k\big) \le \mathrm{e}^{-ck}, \quad k \ge 1, \tag{7.35}$$ E:7.34

for some $c > 0$. First, a crude first moment estimate shows

$$P\big(\min_{x \in L_r} \phi_x^{T^b} \le -2\sqrt{\log b}\, r\big) \le cb^{-r} \tag{7.36}$$

Taking $r := \delta n$, on the event when the above minimum is at least $-2\sqrt{\log b}\, r$ and that $\mathcal{C} \cap L_r \neq \emptyset$, we then have

$$\phi_x^{T^b} \ge -2\sqrt{\log b}\, \delta n + \epsilon(n - \delta n), \qquad x \in \mathcal{C} \cap L_n. \tag{7.37}$$

This is positive as soon as $\epsilon(1-\delta) > 2\sqrt{\log b}\ \delta$. Hence (7.35) follows via (7.34).

Moving to the proof of (7.33), let $k \in \mathbb{N}$ be largest such that $\widetilde{m}_n - t \le \widetilde{m}_{n-k}$. Denote $A_k := \{x \in L_k \colon \phi_x^{T^b} \ge 0\}$. On the event in (7.33), the maximum of the BRW of depth $n-k$ started at any vertex in A_k must be less than $\widetilde{m}_{n-k}$. Conditional on A_k, this has probability $(1-c)^{|A_k|}$, where c is the infimum in (7.32). On the event that $|A_k| \ge \theta^k$, this decays double exponentially with k. So the probability in (7.33) is dominated by that in (7.35). The claim follows by noting that $t \approx 2\sqrt{\log b}\ k$. $\square$

We are now ready to address the upper bound as well:

lemma-7.12 **Lemma 7.14** *There is $\tilde{a} = \tilde{a}(b) > 0$ such that*

$$\sup_{n \ge 1} P\Big(\max_{x \in L_n} \phi_x^{T^b} < \widetilde{m}_n + t\Big) \le \mathrm{e}^{-\tilde{a}t}, \quad t > 0. \tag{7.38}$$

Proof. Fix $x \in L_n$ and again write $Z_k := Z_{(x_1,\dots,x_k)}$. Each vertex $(x_1,\dots,x_k)$ has $b-1$ "children" $y_1,\dots,y_{b-1}$ besides the one on the path from the root to x. Letting $\widetilde{M}_\ell^{(i)}$ denote the maximum of the BRW of depth ℓ rooted at y_i and writing

$$\widehat{M}_\ell := \max_{i=1,\dots,b-1} \widetilde{M}_{\ell-1}^{(i)}, \tag{7.39}$$ E:7.39a

the desired probability can be cast as

$$\int_t^\infty \mu_n(\mathrm{d}s) P\bigg(\bigcap_{k=0}^{n-1}\{Z_0 + \dots + Z_k + \widehat{M}_{n-k} \le \widetilde{m}_n + s\}\ \bigg|\ Z_0 + \dots + Z_n = \widetilde{m}_n + s\bigg) \tag{7.40}$$ E:7.40a

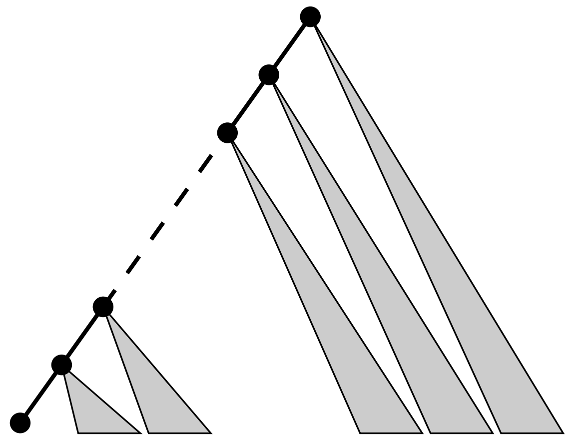

fig-tree

Figure 7.4: The picture demonstrating the geometric setup for the representation in (7.40). The bullets mark the vertices on the path from the root (top vertex) to x (vertex on the left). The union of the relevant subtrees of these vertices are marked by shaded triangles. The maximum of the field in the subtrees of ℓ-th vertex on the path is the quantity in (7.39).

where μ_n is as in (7.28) and where $Z_0, \dots, Z_k, \widehat{M}_1, \dots, \widehat{M}_n$ are independent with their respective distributions. Shifting the normals by arithmetic mean of their sum, the giant probability equals

$$P\biggl(\bigcap_{k=0}^{n-1}\Bigl\{Z_0 + \dots + Z_k + \widehat{M}_{n-k} \le \frac{n-k}{n+1}(\widetilde{m}_n + s)\Bigr\}\,\Bigg|\, Z_0 + \dots + Z_n = 0\biggr) \qquad (7.41)$$ E:7.37

The quantity on the right-hand side of the inequality in this probability is at most $\widetilde{m}_{n-k} + \theta_n(k)$, where $\theta_n(k) := (k \wedge (n-k))^{3/4}$. Introducing

$$\Theta_n := \max_{k=0,\dots,n-1}\bigl[\widetilde{m}_{n-k} - \widehat{M}_{n-k} - \theta_n(k)\bigr]_+. \qquad (7.42)$$

the probability in (7.41) is then bounded by

$$P\biggl(\bigcap_{k=0}^{n-1}\Bigl\{Z_0 + \dots + Z_k \le \Theta_n + 2\theta_n(k) + s\Bigr\}\,\Bigg|\, Z_0 + \dots + Z_n = 0\biggr) \qquad (7.43)$$ E:7.43

Here we again note:

Exercise 7.15 [Inhomogenous ballot problem] *Let $S_k := Z_0 + \dots + Z_k$ be the Gaussian random walk. Prove that that there is $c \in (0, \infty)$ such that for any $a \ge 1$ and any $n \ge 1$,*

$$P\biggl(\bigcap_{k=0}^{n-1}\Bigl\{S_k \le a + 2\theta_n(k)\Bigr\}\,\Bigg|\, S_n = 0\biggr) \le c\frac{a^2}{n}. \qquad (7.44)$$

(This is quite hard. Check the Appendix of arXiv:1606.00510 for ideas and references.)

This bounds the probability in (7.43) by a constant times $n^{-1}E([\Theta_n+s]^2)$. The second moment of Θ_n is bounded uniformly in $n \geq 1$ via Lemma 7.12. The probability is (for $s \geq 1$) thus at most a constant times s^2/n. Since

$$\mu_n\bigl([s,s+1]\bigr) \leq c\,\mathrm{e}^{-cs}\,nb^{-n} \tag{7.45}$$

uniformly in n, the claim follows. □

This now quickly concludes:

Proof of Theorem 7.7. Combining Lemmas 7.12–7.14, the maximum has exponential tails away from $\widetilde{m}_n$, uniformly in $n \geq 1$. This yields the claim. □