Lecture 4

Intermediate level sets: nailing the limit

The content of this lecture is to finish the proof of Theorem 2.7 as well as of the results that follow thereafter. This amounts to proving a number of properties of the Z^D_λ-measures from Proposition 3.11 that will ultimately characterize these measures uniquely. This will imply the conformal invariance as well as characterization of Z^D_λ as a Liouville Quantum Gravity. Finally, we will also comment on necessary changes to the argument in the case when truncations are required for the second-moment calculations to work.

4.1. Gibbs-Markov property in scaling limit

We have shown so far that every subsequential limit of the measures $\{\eta^D_N\colon N\ge 1\}$ takes the form $Z^D_\lambda(\mathrm{d}x)\otimes \mathrm{e}^{-\alpha\lambda h}\mathrm{d}h$ for *some* random Borel measure Z^D_λ on D. Our aim is now to prove a number of properties that the Z^D_λ-measures satisfy the most important of which is the behavior under restriction to subdomain. This will be a direct consequence of the Gibbs-Markov decomposition of the DGFF. However, as the Z^D_λ-measure appears only in the scaling limit, we have to first describe the scaling-limit of the Gibbs-Markov decomposition itself.

The key point we wish to make is that, although the DGFF has no pointwise scaling limit, the binding field does. This is facilitated (and basically implied) by the fact that the binding field has harmonic sample paths. It is no surprise that so will the limit binding field as well. To define the relevant objects, let $\widetilde D, D\in\mathfrak{D}$ be two domains satisfying $\widetilde D\subseteq D$. For each $x,y\in\widetilde D$, set

$$C^{D,\widetilde D}(x,y):=g\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log|y-z|-g\int_{\partial\widetilde D}\Pi^{\widetilde D}(x,\mathrm{d}z)\log|y-z| \tag{4.1}$$

Given any admissible approximating sequences $\{D_N\colon N\ge 1\}$ and $\{\widetilde D_N\colon N\ge 1\}$ of domains D and $\widetilde D$, respectively, of which we also assume that $\widetilde D_N\subseteq D_N$ for each $N\ge 1$, we now observe:

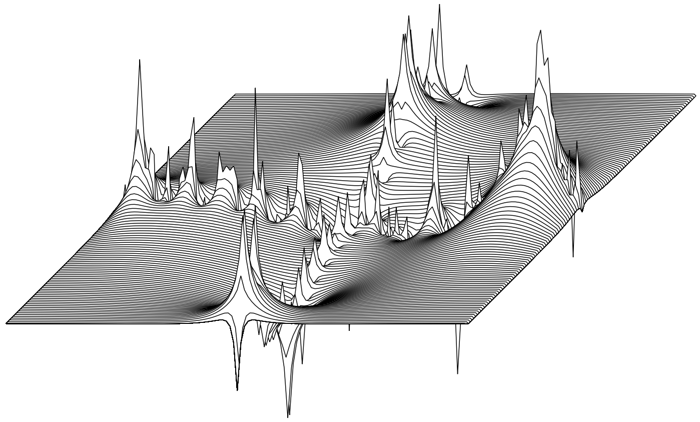

Figure 4.1: A sample of $\varphi_N^{D,\widetilde{D}}$ where $D := (-1,1)^2$ and $\widetilde{D}$ obtained from D by removing points on the coordinate axes.

Lemma 4.1 [Convergence of covariances] *Locally uniformly in $x, y \in \widetilde{D}$,*

$$G^{D_N}\big(\lfloor xN \rfloor, \lfloor yN \rfloor\big) - G^{\widetilde{D}_N}\big(\lfloor xN \rfloor, \lfloor yN \rfloor\big) \underset{N\to\infty}{\longrightarrow} C^{D,\widetilde{D}}(x,y). \tag{4.2}$$

We leave it to the reader to solve:

Exercise 4.2 *Prove Lemma 4.1 while noting that this includes uniform convergence on the diagonal $x = y$. Hint: Use the representation in Lemma 1.19.*

From here we get:

Lemma 4.3 *For any D and $\widetilde{D}$ as above, $x, y \mapsto C^{D,\widetilde{D}}(x,y)$ is a positive, positive semi-definite kernel on $\widetilde{D} \times \widetilde{D}$. In particular, there is a Gaussian process $x \mapsto \Phi^{D,\widetilde{D}}(x)$ on $\widetilde{D}$ with zero mean and covariance*

$$\mathrm{Cov}\big(\Phi^{D,\widetilde{D}}(x), \Phi^{D,\widetilde{D}}(y)\big) = C^{D,\widetilde{D}}(x,y), \qquad x, y \in \widetilde{D}. \tag{4.3}$$

Proof. Let $U \subset V$ be non-empty and finite. The Gibbs-Markov decomposition implies

$$\mathrm{Cov}(\varphi_x^{V,U}, \varphi_y^{V,U}) = G^V(x,y) - G^U(x,y). \tag{4.4}$$

Hence, $x, y \mapsto G^V(x,y) - G^U(x,y)$ is positive semi-definite on $U \times U$. In light of (4.2), this extends to $C^{D,\widetilde{D}}$ on $\widetilde{D} \times \widetilde{D}$ by a limiting argument. Standard arguments then imply existence of the Gaussian process $\Phi^{D,\widetilde{D}}$. $\square$

We will at times call the process $\Phi^{D,\widetilde{D}}$ the *continuum binding field*. To justify this name, let us abbreviate $\varphi_x^{D_N,\widetilde{D}_N}$ as $\varphi_N^{D,\widetilde{D}}(x)$ and observe:

Lemma 4.4 [Coupling of binding fields] *The sample paths of $\Phi^{D,\widetilde{D}}$ are a.s. continuous on $\widetilde{D}$. Moreover, for each $\delta > 0$ and each $N \geq 1$ there is a coupling of $\varphi_N^{D,\widetilde{D}}$ and $\Phi^{D,\widetilde{D}}$ such that*

$$\sup_{\substack{x\in\widetilde{D}\\ \operatorname{dist}(x,\partial\widetilde{D})>\delta}} \bigl|\,\Phi^{D,\widetilde{D}}(x) - \varphi_N^{D,\widetilde{D}}(x/N)\bigr| \underset{N\to\infty}{\longrightarrow} 0, \qquad \text{in probability.} \tag{4.5}$$

Proof modulo regularity of the fields. The convergence of covariances from Lemma 4.1 implies $\varphi_N^{D,\widetilde{D}}(x/N) \to \Phi^{D,\widetilde{D}}(x)$ in law for each $x \in \widetilde{D}$, so the point is to extend this to the convergence of these fields as functions. Fix $\delta > 0$ and denote

$$\widetilde{D}^\delta := \bigl\{x \in \mathbb{C}\colon\ \operatorname{dist}(x, \partial\widetilde{D}) > \delta\bigr\}\,. \tag{4.6}$$

Fix $r > 0$ small and let $x_1, \dots, x_k$ be an r-net in $\widetilde{D}^\delta$. As convergence of the covariances implies convergence in law, and convergence in law on $\mathbb{R}^n$ can be realized as convergence in probability, for each $N \geq 1$ there is a coupling of $\varphi_N^{D,\widetilde{D}}$ and $\Phi^{D,\widetilde{D}}$ such that

$$P\Bigl(\max_{i=1,\dots,k} \bigl|\,\Phi^{D,\widetilde{D}}(\lfloor Nx_i\rfloor) - \varphi_N^{D,\widetilde{D}}(x_i)\bigr| > \epsilon\Bigr) \underset{N\to\infty}{\longrightarrow} 0. \tag{4.7}$$

The claim will then follow if we can show that

$$\lim_{r\downarrow 0}\ \limsup_{N\to\infty}\ P\Biggl(\sup_{\substack{x,y\in\widetilde{D}^\delta\\ |x-y|<r}} \bigl|\Phi^{D,\widetilde{D}}(x) - \Phi^{D,\widetilde{D}}(y)\bigr| > \epsilon\Biggr) = 0 \tag{4.8}$$

and similarly for $\Phi^{D,\widetilde{D}}(\cdot)$ replaced by $\varphi_N^{D,\widetilde{D}}(\lfloor N\cdot\rfloor)$. This, along with continuity of $\Phi^{D,\widetilde{D}}$, will follow from the regularity estimates on Gaussian processes to be proved in forthcoming lectures. $\square$

For future reference, we suggest that the reader solve:

Exercise 4.5 *Prove (by comparing covariances) that $\Phi^{D,\widetilde{D}}$ is the projection of the CGFF on D onto the subspace of functions in $\mathsf{H}_0^1(D)$ that are harmonic on $\widetilde{D}$.*

We will also need to note that the binding field has a nesting property:

Exercise 4.6 [Nesting property] *Show that if $U \subset V \subset W$, then*

$$\varphi^{W,U} \overset{\text{law}}{=} \varphi^{W,V} + \varphi^{V,U} \quad \textit{with} \quad \varphi^{W,V} \perp\!\!\!\perp \varphi^{V,U}. \tag{4.9}$$

Similarly, if $D'' \subset D' \subset D$ are admissible with $\operatorname{Leb}(D \smallsetminus D'') = 0$, *then*

$$\Phi^{D,D''} = \Phi^{D,D'} + \Phi^{D',D''} \quad \textit{with} \quad \Phi^{D,D'} \perp\!\!\!\perp \Phi^{D',D''}\,. \tag{4.10}$$

The binding fields behave nicely under conformal maps of the underlying domains. Indeed, we have:

Exercise 4.7 *Show that under a conformal map $f\colon D \to f(D)$, we have*

$$C^{f(D),f(\widetilde{D})}\big(f(x),f(y)\big) = C^{D,\widetilde{D}}(x,y), \qquad x,y \in \widetilde{D}, \tag{4.11}$$

for any $\widetilde{D} \subset D$. Prove that this implies

$$\Phi^{f(D),f(\widetilde{D})} \circ f \overset{\text{law}}{=} \Phi^{D,\widetilde{D}}. \tag{4.12}$$

4.2. Properties of Z_λ^D-measures

We are now ready to move to the discussion of the properties of (the laws of) the Z_λ^D-measures. These will often relate the measures in two distinct domains which requires that the subsequential limit be taken for both of these domains at the same time. Applying Cantor's diagonal argument, we can in fact use the same subsequence of any *countable* collection $\mathfrak{D}_0 \subset \mathfrak{D}$ of admissible domains and assume that a subsequential limit η^D, and thus also the measure Z_λ^D, has been extracted for each $D \in \mathfrak{D}_0$. All of our subsequent statements will now be restricted to the domains in $\mathfrak{D}_0$. We will later assume that $\mathfrak{D}_0$ contains all finite unions of dyadic open and/or half-open squares.

Some of the properties of the measures $\{Z_\lambda^D\colon D \in \mathfrak{D}_0\}$ are quite elementary consequences of the above derivations and so we relegate them to:

Exercise 4.8 [Easy properties] *Prove that:*

(1) for each $D \in \mathfrak{D}_0$, the measure Z_λ^D is supported on D; i.e., $Z_\lambda^D(\partial D) = 0$ a.s.,

(2) if $A \subset D \in \mathfrak{D}_0$ is measurable with $\mathrm{Leb}(A) = 0$, then $Z_\lambda^D(A) = 0$ a.s.,

(3) if $D, \widetilde{D} \in \mathfrak{D}_0$ obey $D \cap \widetilde{D} = \emptyset$, then

$$Z_\lambda^{D\cup\widetilde{D}}(\mathrm{d}x) \overset{\text{law}}{=} Z_\lambda^D(\mathrm{d}x) + Z_\lambda^{\widetilde{D}}(\mathrm{d}x), \tag{4.13}$$

with the measures Z_λ^D and $Z_\lambda^{\widetilde{D}}$ on the right regarded as independent, and

(4) the law of Z_λ^D is translation invariant; i.e.,

$$Z_\lambda^{a+D}(a + \mathrm{d}x) \overset{\text{law}}{=} Z_\lambda^D(\mathrm{d}x) \tag{4.14}$$

for each $a \in \mathbb{C}$ such that $D, a + D \in \mathfrak{D}_0$.

As already mentioned, a key point for us is to prove the behavior under restriction to a subdomain. We formulate this as follows:

Proposition 4.9 [Gibbs-Markov for Z_λ^D-measures] *For any $D, \widetilde{D} \in \mathfrak{D}_0$ satisfying $\widetilde{D} \subseteq D$ and $\mathrm{Leb}(D \smallsetminus \widetilde{D}) = 0$,*

$$Z_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha\lambda\Phi^{D,\widetilde{D}}(x)}\, Z_\lambda^{\widetilde{D}}(\mathrm{d}x), \tag{4.15}$$

where $\Phi^{D,\widetilde{D}} \perp\!\!\!\perp Z_\lambda^{\widetilde{D}}$ with the law as above.

Note that, by Exercise 4.8(2), both sides of the expression assign zero mass to $D \smallsetminus \widetilde{D}$ a.s. It therefore does not matter that $\Phi^{D,\widetilde{D}}$ is not really defined there.

Proof. Suppose $D, \widetilde{D} \in \mathfrak{D}_0$ obey $\widetilde{D} \subseteq D$ and $\text{Leb}(D \smallsetminus \widetilde{D}) = 0$. The Gibbs-Markov decomposition of the DGFF yields

$$h^{D_N} \overset{\text{law}}{=} h^{\widetilde{D}_N} + \varphi_N^{D,\widetilde{D}}. \tag{4.16}$$

Hence, if $f\colon \overline{D} \times \mathbb{R} \to \mathbb{R}$ is continuous with compact support in $\widetilde{D} \times \mathbb{R}$, then

$$\langle \eta_N^D, f \rangle \overset{\text{law}}{=} \langle \eta_N^{\widetilde{D}}, f_\varphi \rangle \tag{4.17}$$

where

$$f_\varphi(x,h) := f\bigl(x, h + \varphi_N^{D,\widetilde{D}}(\lfloor xN \rfloor)\bigr) \tag{4.18}$$

with $\varphi_N^{D,\widetilde{D}}$ independent of $\eta_N^{\widetilde{D}}$ on the right-hand side of (4.17). Invoking the coupling of $\varphi_N^{D,\widetilde{D}}$ to $\Phi^{D,\widetilde{D}}$ from Lemma 4.4, along with continuity and restriction on the support of f, we thus have

$$\langle \eta_N^{\widetilde{D}}, f_\varphi \rangle = \langle \eta_N^{\widetilde{D}}, f_\Phi \rangle + o(1) \tag{4.19}$$

where $o(1) \to 0$ in probability (as $N \to \infty$) and where

$$f_\Phi(x,h) := f\bigl(x, h + \Phi^{D,\widetilde{D}}(x)\bigr) \tag{4.20}$$

with $\Phi^{D,\widetilde{D}}$ independent of $\eta_N^{\widetilde{D}}$ on the right-hand side of (4.19). Lemma 4.4 ensures that $x \mapsto \Phi^{D,\widetilde{D}}(x)$ is continuous on $\widetilde{D}$ a.s. and so for any simultaneous subsequential limits η^D of $\{\eta_N^D\colon N \ge 1\}$ and $\eta^{\widetilde{D}}$ of $\{\eta_N^{\widetilde{D}}\colon N \ge 1\}$ we thus obtain

$$\langle \eta^D, f \rangle \overset{\text{law}}{=} \langle \eta^{\widetilde{D}}, f_\Phi \rangle\,, \tag{4.21}$$

where $\Phi^{D,\widetilde{D}}$ (implicitly contained in f_Φ) is independent of $\eta^{\widetilde{D}}$ on the right-hand side. But the representation from Proposition 3.11 now permits us to write

$$\begin{aligned}
\int_{D\times\mathbb{R}} Z_\lambda^D(\text{d}x)\, \text{e}^{-\alpha\lambda h}\text{d}h\; f(x,h) &= \langle \eta^D, f \rangle \\
&\overset{\text{law}}{=} \langle \eta^{\widetilde{D}}, f_\Phi \rangle \\
&= \int_{D\times\mathbb{R}} Z_\lambda^{\widetilde{D}}(\text{d}x)\text{e}^{-\alpha\lambda h}\text{d}h\; f\bigl(x, h + \Phi^{D,\widetilde{D}}(x)\bigr) \\
&= \int_{D\times\mathbb{R}} Z_\lambda^{\widetilde{D}}(\text{d}x)\text{e}^{-\alpha\lambda h}\text{d}h\; \text{e}^{\alpha\lambda\Phi^{D,\widetilde{D}}(x)}\, f(x,h).
\end{aligned} \tag{4.22}$$

As this holds for any continuous $f\colon D \times \mathbb{R} \to \mathbb{R}$ with support in $\widetilde{D}$, the desired claim follows. $\square$

4.3. Representation via Gaussian multiplicative chaos

We will now show that the above properties determine the laws of $\{Z_\lambda^D\colon D\in\mathfrak{D}_0\}$ uniquely. To this end we restrict our attention to dyadic open squares, i.e., those of the form

$$2^{-n}z+(0,2^{-n})^2 \quad \text{for} \quad z\in\mathbb{Z}^2 \text{ and } n\ge 0. \tag{4.23}$$

For a fixed $m\in\mathbb{Z}$, set $S:=(0,2^{-m})^2$ and let $\{S_{n,i}\colon i=1,\dots,4^n\}$ be an enumeration of the dyadic squares of side $2^{-(n-m)}$ that have a non-empty intersection with S. (See Fig. 3.1 for an illustration of this setting.) Recall that we assumed that $\mathfrak{D}_0$ contains all these squares which makes the Z_λ^D-measure defined on all of them.

The Gibbs-Markov decomposition then gives

$$Z_\lambda^S(\mathrm{d}x) \overset{\text{law}}{=} \sum_{i=1}^{4^n} \mathrm{e}^{\alpha\lambda\Phi^{S,\tilde S^n}}\, Z_\lambda^{S_{n,i}}(\mathrm{d}x), \tag{4.24}$$

where we abbreviated

$$\tilde S^n := \bigcup_{i=1}^{4^n} S_{n,i} \tag{4.25}$$

and where the measures $\{Z_\lambda^{S_{n,i}}\colon i=1,\dots,4^n\}$ and the field $\Phi^{S,\tilde S^n}$ on the right-hand side are regarded as independent. The problem with expression (4.24) is that it gives the Z_λ-measure in one set in terms of Z_λ-measures in other sets. (In the language of the field, the Z_λ measure is a fixed point of a certain *smoothing transformation*.) To amend this, we formally replace the $Z_\lambda^{S_{n,i}}$-measures by their expectation and define a new measure

$$Y_n^S(\mathrm{d}x) := \hat c \sum_{i=1}^{4^n} 1_{S_{n,i}}(x)\, \mathrm{e}^{\alpha\lambda\Phi^{S,\tilde S^n}}\, \psi_\lambda^{\tilde S^n}(x)(x)\,\mathrm{d}x. \tag{4.26}$$

where $\hat c$ is as in (3.58). This is connected to the expectation applied on (4.24) because, in light of the fact that

$$\psi_\lambda^{\tilde S^n}(x) = \psi_\lambda^{S_{n,i}}(x) \quad \text{for} \quad x\in S_{n,i} \tag{4.27}$$

we have

$$Y_n(A) = E\big[Z_\lambda^S(A)\,\big|\,\sigma(\Phi^{S,\tilde S^n})\big] \tag{4.28}$$

for any Borel $A\subset\mathbb{C}$. The next point to observe is that these measures can be interpreted in terms of *Gaussian multiplicative chaos* which, in particular, implies existence of their $n\to\infty$ limit:

Lemma 4.10 *There is a random Borel measure Y_∞ such that for each measurable $A\subset\mathbb{C}$,*

$$Y_n^S(A) \xrightarrow[n\to\infty]{\text{law}} Y_\infty^S(A). \tag{4.29}$$

Proof. Denoting $S^0 := S$, the nesting property of the binding field allows us to represent all fields $\{\Phi^{S,\tilde{S}^n}\colon n \geq 1\}$ on the same probability space via

$$\Phi^{S,\tilde{S}^n} := \sum_{k=0}^{n-1} \Phi^{S^k,S^{k+1}} \tag{4.30}$$

where the fields $\{\Phi^{S^k,S^{k+1}}\colon k \geq 0\}$ are independent with the appropriate laws. In this representation, the measures Y_n are defined all on the same probability space and so we can actually prove the stated convergence in almost-sure sense. Since (2.50) and (4.1) imply

$$\widetilde{D} \subseteq D \quad \Rightarrow \quad \psi_\lambda^{\widetilde{D}}(x) = \psi_\lambda^D(x) \mathrm{e}^{\frac{1}{2}\alpha^2\lambda^2 \mathrm{Var}[\Phi^{D,\widetilde{D}}(x)]}, \quad x \in \widetilde{D}, \tag{4.31}$$

we can now rewrite (4.26) as

$$Y_n^S(\mathrm{d}x) := \hat{c}\, \psi_\lambda^S(x) \sum_{i=1}^{4^n} 1_{S_{n,i}}(x)\, \mathrm{e}^{\alpha\lambda\Phi^{S,\tilde{S}^n} - \frac{1}{2}\alpha^2\lambda^2 \mathrm{Var}[\Phi^{D,\widetilde{D}}(x)]}\, \mathrm{d}x \tag{4.32}$$

This casts Y_n^S in the form we encountered when defining the Gaussian Multiplicative Chaos. Adapting the proof of Lemma 2.16, for any Borel $A \subset C$ we get

$$Y_n^S(A) \underset{n\to\infty}{\longrightarrow} Y_\infty^S(A), \quad \text{a.s.} \tag{4.33}$$

where Y_∞^S is a random Borel measure on D. The claim follows. $\square$

We now claim:

Proposition 4.11 [Characterization of Z_λ^D measure] *For each dyadic square $S \subset C$ and any bounded and continuous function $f\colon S \to [0,\infty)$, we have*

$$E\big(\mathrm{e}^{-\langle Z_\lambda^S, f\rangle}\big) = E\big(\mathrm{e}^{-\langle Y_\infty^S, f\rangle}\big) \tag{4.34}$$

In particular,

$$Z_\lambda^S(\mathrm{d}x) \overset{\text{law}}{=} Y_\infty^S(\mathrm{d}x). \tag{4.35}$$

Proof of "$\geq$" in (4.34). Writing Z_λ^S via (4.24) and invoking conditional expectation given $\Phi^{S,\tilde{S}^n}$ with the help of (4.28), the conditional Jensen's inequality shows

$$\begin{aligned} E\big(\mathrm{e}^{-\langle Z_\lambda^S, f\rangle}\big) &= E\Big(E\big(\mathrm{e}^{-\langle Z_\lambda^S, f\rangle} \,\big|\, \sigma(\Phi^{S,\tilde{S}^n})\big)\Big) \\ &\geq E\big(\mathrm{e}^{-E[\langle Z_\lambda^S, f\rangle \,|\, \sigma(\Phi^{S,\tilde{S}^n}]}\big) = E\big(\mathrm{e}^{-\langle Y_n^S, f\rangle}\big)\,. \end{aligned} \tag{4.36}$$

By Lemma 4.10, $E(\mathrm{e}^{-\langle Y_n^S, f\rangle}) \to E(\mathrm{e}^{-\langle Y_\infty^S, f\rangle})$ as $n \to \infty$ and so we get "$\geq$" in (4.34). $\square$

For the proof of the opposite inequality in (4.34) we first note:

Lemma 4.12 [Reverse Jensen's inequality] *If $X_1, \dots, X_n$ are non-negative independent random variables, then for each $\epsilon > 0$,*

$$E\biggl(\exp\Bigl\{-\sum_{i=1}^n X_i\Bigr\}\biggr) \le \exp\Bigl\{-\mathrm{e}^{-\epsilon}\sum_{i=1}^n E(X_i\,;\,X_i \le \epsilon)\Bigr\}. \tag{4.37}$$

Proof. In light of assumed independence, it suffices to prove this for $n = 1$. This is checked by bounding $E(\mathrm{e}^{-X}) \le E(\mathrm{e}^{-\widetilde{X}})$, where $\widetilde{X} := X1_{\{X\le\epsilon\}}$, writing

$$-\log E(\mathrm{e}^{-\widetilde{X}}) = \int_0^1 \mathrm{d}s\; \frac{E(\widetilde{X}\mathrm{e}^{-s\widetilde{X}})}{E(\mathrm{e}^{-s\widetilde{X}})} \tag{4.38}$$

and invoking $E(\widetilde{X}\mathrm{e}^{-s\widetilde{X}}) \ge \mathrm{e}^{-\epsilon}E(\widetilde{X})$ and $E(\mathrm{e}^{-s\widetilde{X}}) \le 1$. □

We are now ready to give:

Proof of "$\le$" in (4.34). Pick n large and assume Z^S_λ is again represented via (4.24). We first invoke an additional truncation: Fiven $\delta > 0$, let $S^\delta_{n,i}$ be the translate of $(\delta 2^{-(n-m)}, (1-\delta)2^{-(n-m)})$ centered at the same point as $S_{n,i}$. Denote

$$\tilde{S}^n_\delta := \bigcup_{i=1}^{4^n} S^\delta_{n,i} \quad\text{and}\quad f_{n,\delta}(x) := f(x)1_{\tilde{S}^n_\delta}(x). \tag{4.39}$$

Denoting also

$$X_i := \int_{S_{n,i}} f_{n,\delta}(x)\,\mathrm{e}^{\alpha\lambda\Phi^{S,\tilde{S}^n}}\,Z^{S_{n,i}}_\lambda(\mathrm{d}x) \tag{4.40}$$

from $f \ge f_{n,\delta}$ we then have

$$E\bigl(\mathrm{e}^{-\langle Z^S_\lambda, f\rangle}\bigr) \le E\bigl(\mathrm{e}^{-\langle Z^S_\lambda, f_{n,\delta}\rangle}\bigr) = E\biggl(\exp\Bigl\{-\sum_{i=1}^n X_i\Bigr\}\biggr). \tag{4.41}$$

Conditioning on $\Phi^{S,\tilde{S}^n}$, the bound (4.37) yields

$$E\bigl(\mathrm{e}^{-\langle Z^S_\lambda, f\rangle}\bigr) \le E\biggl(\exp\Bigl\{-\mathrm{e}^{-\epsilon}\sum_{i=1}^{4^n} E\bigl(X_i 1_{\{X_i\le\epsilon\}}\,\big|\,\sigma(\Phi^{S,\tilde{S}^n})\bigr)\Bigr\}\biggr) \tag{4.42}$$

Since (4.28) shows

$$\sum_{i=1}^{4^n} E\bigl(X_i\,\big|\,\sigma(\Phi^{S,\tilde{S}^n})\bigr) = \langle Y^S_n, f_{n,\delta}\rangle \tag{4.43}$$

we will need:

Lemma 4.13 *Assume $\lambda \in (0, 1/\sqrt{2})$. Then for each $\epsilon > 0$*

$$\lim_{n\to\infty}\sum_{i=1}^{4^n} E\bigl(X_i;\, X_i > \epsilon\bigr) = 0\,. \tag{4.44}$$

Proof. For this we note

$$\begin{aligned}\sum_{i=1}^{4^n} E\big(X_i;\, X_i > \epsilon\big) &\le \frac{1}{\epsilon}\sum_{i=1}^{4^n} E(X_i^2)\\ &\le \frac{\|f\|^2}{\epsilon}\sum_{i=1}^{4^n} E\bigg(\int_{S_{n,i}^\delta\times S_{n,i}^\delta} E\big(\mathrm{e}^{\alpha\lambda[\Phi^{S,\tilde S^n}(x)+\Phi^{S,\tilde S^n}(x)]}\big) Z_\lambda^{S_{n,i}}(\mathrm{d}x) Z_\lambda^{S_{n,i}}(\mathrm{d}y)\bigg).\end{aligned}\tag{4.45}$$

Denote $L := 2^n$. In light of the fact that, for some constant c independent of n,

$$\mathrm{Var}(\Phi^{S,\tilde S^n}(x)) = g\log\frac{r_S(x)}{r_{S_{n,i}}(x)} \le g\log(L) + c \tag{4.46}$$

holds uniformly in $x \in \tilde S^n_\delta$, the right-hand side of (4.45) is bounded by $\|f\|^2$ times

$$c' \mathrm{e}^{4\frac12\alpha^2\lambda^2\log(L)}\sum_{i=1}^{4^n} E\big[Z_\lambda^{S_{n,i}}(S_{n,i})^2\big] \le c'' L^{8\lambda^2+2-(4+4\lambda^2)} = c'' L^{-2(1-2\lambda^2)}, \tag{4.47}$$

where we also invoked (3.60) and the fact that there are L^2 terms in the sum. This tends to zero as $L\to\infty$ whenever $\lambda < 1/\sqrt{2}$ thus proving (4.44). □

Continuing in the proof of Proposition 4.11, from (4.42) and (4.44) we now get

$$E\big(\mathrm{e}^{-\langle Z_\lambda^S, f\rangle}\big) \le \limsup_{n\to\infty} E\big(\mathrm{e}^{-\mathrm{e}^{-\epsilon}\langle Y_n^S, f_{n,\delta}\rangle}\big) \tag{4.48}$$

But

$$\langle Y_n^S, f_{n,\delta}\rangle = \langle Y_n^S, f\rangle - \|f\| Y_n^S(S\smallsetminus \tilde S^n_\delta) \tag{4.49}$$

and so

$$E\big(\mathrm{e}^{-\mathrm{e}^{-\epsilon}\langle Y_n^S, f_{n,\delta}\rangle}\big) \le \mathrm{e}^{\epsilon\|f\|}\, E\big(\mathrm{e}^{-\mathrm{e}^{-\epsilon}\langle Y_n^S, f\rangle}\big) + P\big(Y_n^S(S\smallsetminus\tilde S^n_\delta) > \epsilon\big) \tag{4.50}$$

A calculation based on (4.32) shows

$$P\big(Y_n^S(S\smallsetminus\tilde S^n_\delta) > \epsilon\big) \le c\epsilon^{-1}\mathrm{Leb}(S\smallsetminus\tilde S^n_\delta) \le c'\epsilon^{-1}\delta. \tag{4.51}$$

Hence

$$E\big(\mathrm{e}^{-\langle Z_\lambda^S, f\rangle}\big) \le \limsup_{\epsilon\downarrow 0}\,\limsup_{\delta\downarrow 0}\,\limsup_{n\to\infty} E\big(\mathrm{e}^{-\mathrm{e}^{-\epsilon}\langle Y_n^S, f_{n,\delta}\rangle}\big) \le E\big(\mathrm{e}^{-\langle Y_\infty^S, f\rangle}\big). \tag{4.52}$$

This completes the proof of (4.34); the conclusion (4.35) then directly follows. □

4.4. Finishing touches

Let us first check that the arguments above already imply our main conclusion about the existence of the limit of processes $\{\eta_N^D\colon N\ge 1\}$:

Proof of Theorem 2.7 for $\lambda < 1/\sqrt{2}$. Pick $D\in\mathfrak{D}$ and assume without loss of generality that $\mathfrak{D}_0$ used above contained D. For any subsequence of N's for which the limit of the measures in question exists we then the representation (3.46) and the conclusions of Proposition 4.9 and 4.11 apply to all domains in $\mathfrak{D}_0$. We just have to show that the law of the limit measures is uniquely determined as that will prove the existence of the limit for all domains in $\mathfrak{D}_0$. (As our domain of interest was included there, the limit will then exist for all domains.)

By Exercise 4.8 it suffices to show that the law of $\langle Z_\lambda^D, f\rangle$ is determined for any continuous $f\colon D\times\mathbb{R}\to\mathbb{R}$ with compact support. Let D^n be the union of of all open dyadic squares of side 2^{-n} entirely contained in D. Letting n be so large that $\operatorname{supp}(f)\subseteq D^n\times[-n,n]$ and denoting $\widetilde{D}^n := D\smallsetminus\partial D^n$, Proposition 4.9 and Exercise 4.8(3) then imply

$$\langle Z_\lambda^D, f\rangle \overset{\text{law}}{=} \bigl\langle Z_\lambda^{D^n}, \mathrm{e}^{\alpha\lambda\Phi^{D,\widetilde{D}^n}} f\bigr\rangle \quad\text{with}\quad Z_\lambda^{D^n} \perp\!\!\!\perp \Phi^{D,\widetilde{D}^n}. \tag{4.53}$$

It follows that the law of the left-hand side is determined once the law of $Z_\lambda^{D^n}$ is determined. But Exercise 4.8(3) also shows that $Z_\lambda^{D^n}$ is the sum of independent realizations of Z_λ^S for S constituting the squares making up D^n. The laws of these Z_λ^S are determined by Proposition 4.11. $\square$

Concerning the proof of conformal invariance and full characterization by the LQG measure, we will need the following result:

Theorem 4.14 [Uniqueness of the GMC/LQG measure] *The law of the Gaussian Multiplicative Chaos measure $\mu_\infty^{D,\beta}$ does not depend on the choice of the orthonormal basis $\{\varphi_n\colon n\ge 1\}$ in $\mathsf{H}_0^1(D)$ that was used to define it.*

We will not prove this theorem in these notes as that would take us on a tangent that we do not wish to follow. We remark that the result has a rather neat proof due to Shamov from 2015 which was made possible by his ingenious characterization of the GMC measures using Cameron-Martin shifts. Earlier work of Kahane required uniform convergence of the covariances of the approximating fields.

Equipped with the uniqueness Theorem 4.14, let us now annotate the steps that identify Z_λ^D with the LQG-measure $\hat{c}\psi_\lambda^D(x)\mu_\infty^{D,\lambda\alpha}(\mathrm{d}x)$. For that pick any $D\in\mathfrak{D}$ and let $\{S_{k,i}\colon i=1,\dots,n(k)\}$ be the collection of open dyadic squares of side 2^{-k} that are entirely contained in D. Now observe:

Exercise 4.15 *Let H_k denote the subspace of all functions in $\mathsf{H}_0^1(D)$ that are harmonic in $S_{k,i}$ for each $i=1,\dots,n(k)$. Prove that*

$$\mathsf{H}_0^1(D) = \bigoplus_{k=0}^{\infty}\mathsf{H}_k \tag{4.54}$$

Next, for each $k\ge 1$, let $\{\widetilde{\varphi}_{k,j}\colon j\ge 1\}$ be an orthonormal basis in H_k with respect to the Dirichlet inner product. Denote $D^k := \bigcup_{j=1}^{n(k)} S_{k,j}$ with $D^0 := D$ and observe that $D^k\subset D^{k-1}$. Then show:

Exercise 4.16 *For $\{X_{k,j}\colon k,j\ge 1\}$ i.i.d. standard normals prove that, for each $k\ge 1$,*

$$\Phi^{D^{k-1},D^k} \overset{\text{law}}{=} \sum_{j\ge 1} X_{k,j}\widetilde{\varphi}_{k,j} \quad \text{on } D^k. \tag{4.55}$$

In particular, by the nesting property, for each $m \geq 1$,

$$\Phi^{D,D^m} \overset{\text{law}}{=} \sum_{k=1}^{m} \sum_{j \geq 1} X_{k,j} \widetilde{\varphi}_{k,j} \quad \text{on } D^m\,. \tag{4.56}$$

From here and Theorem 4.14 we now infer the following:

Exercise 4.17 *Prove that for any open dyadic square S*

$$Y_\infty^S(\mathrm{d}x) \overset{\text{law}}{=} \hat{c}\psi_\lambda^S(x)\mu_\infty^{S,\lambda\alpha}(\mathrm{d}x). \tag{4.57}$$

Similarly, show that for any $\widetilde{D} \subset D$ with $\mathrm{Leb}(D \smallsetminus \widetilde{D}) = 0$,

$$\mu_\infty^{D,\beta}(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\beta\Phi^{D,\widetilde{D}}(x) - \frac{1}{2}\beta^2 \mathrm{Var}[\Phi^{D,\widetilde{D}}(x)]}\, \mu_\infty^{\widetilde{D},\beta}(\mathrm{d}x)\,, \tag{4.58}$$

where $\Phi^{D,\widetilde{D}}$ and $\mu_\infty^{\widetilde{D},\beta}$ are regarded as independent.

The point here is that although the right-hand side (4.55) casts the binding field from D to D^m in the form akin to (2.39), the sum over j is infinite. One thus has to see that a suitable truncation to a finite sum will do as well. Once this exercise is solved, we just re-run the argument around (4.53) to get Theorem 2.17.

Once we identify the limit measure with the LQG-measure, the proof of conformal invariance is quite easy. A key point is to solve:

Exercise 4.18 [Conformal transform of GMC measure] *Let $f\colon D \mapsto f(D)$ be a conformal map between bounded and open domains in $\mathbb{C}$. Show that if $\{\varphi_n\colon n \geq 1\}$ is an orthonormal basis in $\mathsf{H}_0^1(D)$ with respect to the Dirichlet product, then $\{\varphi_n \circ f^{-1}\colon n \geq 1\}$ is an orthonormal basis in $\mathsf{H}_0^1(f(D))$. Prove that this implies*

$$\mu_\infty^{f(D),\beta} \circ f(\mathrm{d}x) \overset{\text{law}}{=} \big|f'(x)\big|^2 \mu_\infty^{D,\beta}(\mathrm{d}x). \tag{4.59}$$

To get the proof of Theorem 2.13, one just needs to observe:

Exercise 4.19 *For any conformal bijection $f\colon D \to f(D)$,*

$$\psi_\lambda^{f(D)}\big(f(x)\big) = \big|f'(x)\big|^{2\lambda^2}\, \psi_\lambda^D(x), \qquad x \in D. \tag{4.60}$$

4.5. Dealing with truncations

The above completes the proof of our results in the regime where second-moment calculations can be applied without truncations. To get the feeling what happens in the the complementary regime, $1/\sqrt{2} \leq \lambda < 1$, let us at least introduce the basic definitions and annotate the relevant steps.

Denote $\Lambda_r(x) := \{y \in \mathbb{Z}^d\colon |x-y|_\infty \leq r\}$. Given a discretized version D_N of a continuum domain D, for each $x \in D_N$ define

$$\Delta^k(x) := \begin{cases} \emptyset & \text{for } k = 0\,, \\ \Lambda_{\mathrm{e}^k}(x) & \text{for } k = 1, \dots, n(x) - 1\,, \\ D_N & \text{for } k = n(x)\,, \end{cases} \tag{4.61}$$

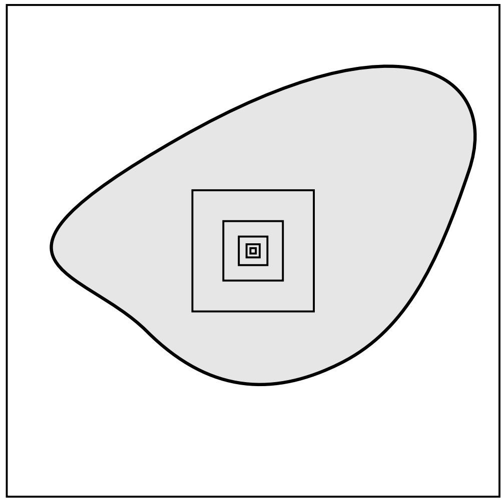

Figure 4.2: An illustration of the collection of sets $\Delta^k(x)$ above.

where $n(x) := \max\{n \geq 0 \colon \Lambda_{e^{n+1}}(x) \subseteq D_N\}$. See Fig. 4.2 for an illustration.
Recalling the definition of the field $\varphi^{D_N,\Delta^k}(x)$, we now define the truncation event

$$T_{N,M}(x) := \bigcap_{k=k_N}^{n(x)} \left\{ \left| \varphi^{D_N,\Delta^k}(x) - a_N \frac{n(x)-k}{n(x)} \right| \leq M\big[n(x)-k\big]^{3/4} \right\}, \tag{4.62}$$

where M is a parameter and $k_N := \frac{1}{8}\log(K_N) \approx \frac{1}{4}(1-\lambda^2)\log N$. Now we introduce the *truncated point measure*

$$\widehat{\eta}_N^{D,M} := \frac{1}{K_N} \sum_{x \in D_N} 1_{T_{N,M}(x)}\, \delta_{x/N} \otimes \delta_{h^{D_N}(x)-a_N} \tag{4.63}$$

The following monotonicity will be quite useful:

$$\eta_N^D \geq \eta_N^{D,M}, \qquad M \in (0,\infty). \tag{4.64}$$

Indeed, the tightness of $\{\widehat{\eta}_N^D \colon N \geq 1\}$ is thus inherited from $\{\eta_N^D \colon N \geq 1\}$ and, as $M \to \infty$, the limits points of the former increase to those of the latter. The requisite (now really ugly) 2nd moment calculations are then performed which yields the following conclusions for all $M < \infty$ and all $\lambda \in (0,1)$:

(1) Defining $\widehat{\Gamma}_N^{D,M}(b) := \{x \in D_N \colon h_x^{D_N} \geq a_N + b,\ T_{N,M}(x) \text{ occurs}\}$, we have

$$\sup_{N \geq 1} \frac{1}{K_N^2} E\big(|\widehat{\Gamma}_N^{D,M}(b)|^2\big) < \infty. \tag{4.65}$$

By a second-moment argument, the limits of $\{\widehat{\eta}_N^{D,M} \colon N \geq 1\}$ are non-trivial.

(2) The factorization property proved in Proposition 3.11 applies to limit points of $\{\widehat{\eta}_N^{D,M}\colon N \geq 1\}$ with Z_λ^D replaced by some $\widehat{Z}_\lambda^{D,M}$ instead.

The property (4.63) now implies that $M \mapsto \widehat{Z}_\lambda^{D,M}$ is pointwise increasing and so we may define

$$Z_\lambda^D(\cdot) := \lim_{M\to\infty} \widehat{Z}_\lambda^{D,M}(\cdot). \tag{4.66}$$

We then check that this measure has the properties in Exercise 4.8 as well as the Gibbs-Markov property from Proposition 4.9. However, although the limit in (4.66) exists in L^1 for all $\lambda(0,1)$, it does not in L^2 for $\lambda \geq 1/\sqrt{2}$ and so we have to keep using $\widehat{Z}_\lambda^{D,M}$ in whenever estimates involving second moments are needed.

This comes up only in one proof: "$\leq$" in (4.34). There we use $Z_\lambda^D(\cdot) \geq \widehat{Z}_\lambda^{D,M}(\cdot)$ and the fact that Z_λ^D satisfies the Gibbs-Markov property to dominate $Z_\lambda^D(\cdot)$ from below by measure

$$\widetilde{Z}_\lambda^S(\mathrm{d}x) := \sum_{i=1}^{4^n} \mathrm{e}^{\alpha\lambda\Phi^{S,\tilde{S}^n}}\, \widehat{Z}_\lambda^{S_{n,i},M}(\mathrm{d}x), \tag{4.67}$$

Then we perform the calculation after (4.24) with this measure instead of Z_λ^S modulo one caveat: In the proof of Lemma 4.13 we truncate to the event

$$\sup_{x\in\widetilde{S}_\delta^n} \Phi^{S,\widetilde{S}^n}(x) < 2\sqrt{g}\log(2^n) + c\sqrt{\log(2^n)}\,. \tag{4.68}$$

which has probability very close to one. On this event, writing again $L := 2^n$, the sum on the right-hand side of (4.45) is thus bounded by

$$c'\mathrm{e}^{2\sqrt{g}\alpha\lambda\log(L)+c\sqrt{\log(L)}}\,\mathrm{e}^{\frac12\alpha^2\lambda^2 g\log(L)}L^2L^{-2(2+2\lambda^2)} \tag{4.69}$$

Using the definition of α, this becomes $L^{-2(1-\lambda)^2+o(1)}$ which tends to zero as $n\to\infty$ for all $\lambda\in(0,1)$. The rest of the proof is then more or less the same.