Lecture 3

Intermediate level sets: factorization

The aim of this and the following lecture is to give a fairly detailed account of the proofs of Theorems 2.7–2.17 on the scaling limit of intermediate level sets. We will actually do this only in the regime where second-moment calculations work without the need for truncations; this requires restricting to $\lambda < 1/\sqrt{2}$. We will comment on what changes need to be made for the complementary set of λ's at the end of the next lecture.

3.1. Gibbs-Markov property of DGFF

A number of proofs in this lecture as well as later will use a special property of the DGFF that addresses the behavior of the field restrictions (via conditioning) on a subdomain. This property is the spatial analogue of the Markov property in one-parameter stochastic processes and it is a direct consequence of the fact that the law of the DGFF is a Gibbs measure (cf Definition 1.1). For this reason, we will attach the adjective Gibbs-Markov to this property, although the literature uses the term domain-Markov as well. The precise statement is as follows:

Lemma 3.1 [Gibbs-Markov property] *Let $U \subsetneq V \subsetneq \mathbb{Z}^2$ and denote*

$$\varphi_x^{V,U} := E\big(h_x^V \,\big|\, \sigma(h_z^V \colon z \in V \smallsetminus U)\big). \tag{3.1}$$

Then we have:

(1) A.e. sample of $x \mapsto \varphi_x^{V,U}$ is discrete harmonic on U with "boundary values" determined by $\varphi_x^{V,U} = h_x^V$ for each $x \in V \smallsetminus U$.

(2) The field $h^V - \varphi^{V,U}$ is independent of $\varphi^{V,U}$ and, in fact,

$$h^V - \varphi^{V,U} \overset{\text{law}}{=} h^U \tag{3.2}$$

Proof. Assume that V is finite for simplicity. Conditioning a multivariate Gaussian on part of the values preserves the multivariate Gaussian nature of the law.

Hence $\varphi^{V,U}$ and $h^V - \varphi^{V,U}$ are multivariate Gaussians that are, by properties of the conditional expectation, uncorrelated. It follows that $\varphi^{V,U} \perp\!\!\!\perp h^V - \varphi^{V,U}$.

Next let us prove that $\varphi^{V,U}$ has discrete-harmonic sample paths in U. To this end pick any $x \in U$ and note that the "smaller-always-wins" principle for nested conditional expectations yields

$$\varphi^{V,U}_x = E\Big(E\big(h^V_x \,\big|\, \sigma(h^V_{z'} \colon z' \neq x)\big) \,\Big|\, \sigma(h^V_z \colon z \in V \smallsetminus U)\Big). \tag{3.3}$$

In light of Definition 1.1 (and some routine limit arguments if V is infinite), the inner conditional expectation admits the explicit form

$$E\big(h^V_x \,\big|\, \sigma(h^V_{z'} \colon z' \neq x)\big) = \frac{\displaystyle\int_{\mathbb{R}} h_x \,\mathrm{e}^{-\frac18 \sum_{y\colon y\sim x}(h_y - h_x)^2} \mathrm{d}h_x}{\displaystyle\int_{\mathbb{R}} \mathrm{e}^{-\frac18 \sum_{y\colon y\sim x}(h_y - h_x)^2} \mathrm{d}h_x}, \tag{3.4}$$

where $y \sim x$ abbreviates $(x,y) \in E(\mathbb{Z}^2)$. Now

$$\begin{aligned}\frac14 \sum_{y\colon y\sim x}(h_y - h_x)^2 &= h_x^2 - 2\frac14 \sum_{y\colon y\sim x} h_y + \frac14 \sum_{y\colon y\sim x} h_y^2 \\ &= \Big(h_x - \frac14 \sum_{y\colon y\sim x} h_y\Big)^2 + \frac14 \sum_{y\colon y\sim x} h_y^2 - \Big(\frac14 \sum_{y\colon y\sim x} h_y\Big)^2.\end{aligned} \tag{3.5}$$

The last two terms factor from both the numerator and denominator on the right of (3.4). Shifting h_x by the average of the neighbors then gives

$$E\big(h^V_x \,\big|\, \sigma(h^V_{z'} \colon z' \neq x)\big) = \frac14 \sum_{y\colon y\sim x} h_y\,. \tag{3.6}$$

Using this in (3.3) shows that $\varphi^{V,U}$ has the mean-value property on U.

Finally, we need to show that $\widetilde{h}^U := h^V - \varphi^{V,U}$ has the law of h^U. The expectation of $\widetilde{h}^U$ is zero so we just need to verify that the covariances match. Here we note that, using the concept of the discrete harmonic measure H^U on U, we can write

$$\widetilde{h}^U_x = h^V_x - \sum_{z\in\partial U} H^U(x,z) h^V_z, \quad x \in U. \tag{3.7}$$

For any $x, y \in U$, this implies

$$\begin{aligned}\mathrm{Cov}(\widetilde{h}^U_x, \widetilde{h}^U_y) &= G^V(x,y) - \sum_{z\in\partial U} H^U(x,z) G^V(z,y) \\ &\quad - \sum_{z\in\partial U} H^U(y,z) G^V(z,x) + \sum_{z,\tilde z\in\partial U} H^U(x,z) H^U(y,\tilde z) G^V(z,\tilde z)\,.\end{aligned} \tag{3.8}$$

Now recall the representation (1.30) which casts $G^V(x,y)$ as $-\mathfrak{a}(x-y) + \phi(y)$ with ϕ harmonic on V. Plugging this in (3.8), the fact that

$$\sum_{z\in\partial U} H^U(x,z)\phi(z) = \phi(x) \tag{3.9}$$

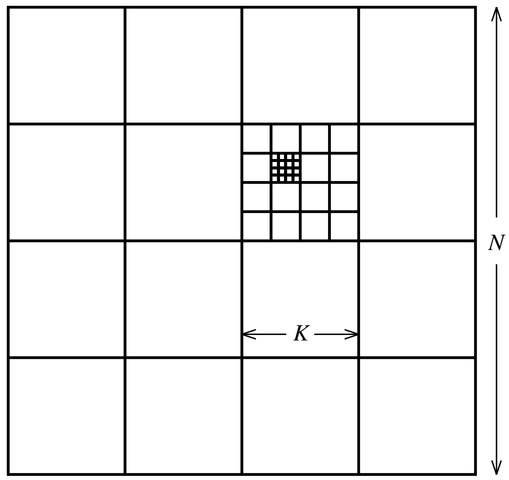

Figure 3.1: A typical setting for the application of the Gibbs-Markov property. The box $V_N = (0,N)^2 \cap \mathbb{Z}^2$ is partitioned into $(N/K)^2$ translates of boxes $V_K := (0,K)^2 \cap \mathbb{Z}^2$ of side K (assuming K divides N). This leaves a "line of sites" between any two adjacent translates of V_K. The DGFF on V_N is then partitioned as $h^{V_N} = h^{V_N^\circ} + \varphi^{V_N,V_N^\circ}$ with $h^{V_N^\circ} \perp\!\!\!\perp \varphi^{V_N,V_N^\circ}$, where V_N° is the union of the shown translates of V_K and φ^{V_N,V_N° has the law of the harmonic extension to V_N° of the values of h^{V_N} on $V_N \smallsetminus V_N^\circ$. As sketched, the translates of V_K can further be subdivided to produce a hierarchical description of the DGFF.

shows that all occurrences of ϕ in (3.8) cancel each other. As $x \mapsto \mathfrak{a}(z-x)$ is harmonic on U for any $z \in \partial U$, replacing $G^V(\cdot,\cdot)$ by $\mathfrak{a}(\cdot - \cdot)$ in the last two sums on the right of (3.8) makes these sums cancel each other as well.

We are thus left with the first two terms on the right of (3.8), where $G^V(\cdot,\cdot)$ is now substituted by $-\mathfrak{a}(\cdot - \cdot)$. The representation (1.30) then tells us that

$$\mathrm{Cov}(\widetilde{h}^U_x, \widetilde{h}^U_y) = G^U(x,y), \qquad x,y \in U. \tag{3.10}$$

Since both $\widetilde{h}^U$ and h^U vanish outside U, we have $\widetilde{h}^U \overset{\text{law}}{=} h^U$ as desired. □

Exercise 3.2 *Supply the missing (e.g., limiting) arguments to prove the Gibbs-Markov decomposition applies even to the situation when U and V are allowed to be infinite.*

We have seen that the monotonicity $V \mapsto G^V(x,y)$ allows for control of the variance of the DGFF in general domains by that in more regular ones. One of the important consequences of the Gibbs-Markov property is to give similar estimates for various probabilities involving a finite number of vertices. Here are some examples:

Exercise 3.3 *Suppose $U \subset V$ are non-empty and finite. Prove that for every $a \in \mathbb{R}$,*

$$P\big(h^U_x \geq a\big) \leq 2P\big(h^V_x \geq a\big), \qquad x \in U. \tag{3.11}$$

Similarly, for any relation $\mathcal{R} \subset \mathbb{Z}^d \times \mathbb{Z}^d$, show that also

$$P\big(\exists x,y \in U\colon (x,y) \in \mathcal{R},\, h_x^U, h_y^U \ge a\big) \\ \le 4P\big(\exists x,y \in V\colon (x,y) \in \mathcal{R},\, h_x^V, h_y^V \ge a\big)\,. \tag{3.12}$$

Similar ideas lead to:

Exercise 3.4 *Prove that for any $U \subset V$ non-empty and finite,*

$$E\big(\max_{x\in U} h_x^U\big) \le E\big(\max_{x\in V} h_x^V\big), \tag{3.13}$$

and so $U \mapsto E(\max_{x\in U} h_x^U)$ is non-decreasing with respect to set inclusion.

We will see that the monotonicity of expected maximum in the underlying domain has very deep consequences for the tightness of absolute maximum of DGFF.

A short way to write the Gibbs-Markov decomposition is as

$$h^V \overset{\text{law}}{=} h^U + \varphi^{V,U} \quad \text{where} \quad h^U \perp\!\!\!\perp \varphi^{V,U}. \tag{3.14}$$

A typical setting for the application of the Gibbs-Markov property is depicted in Fig. 3.1. There each of the small boxes (the translates of V_K) has its "private" independent copy of the DGFF. By (3.14), to get h^{V_N} these copies are "bound together" by an independent Gaussian field φ^{V_N,V_N° that, as far as its law is concerned, is just the harmonic extension of the values of h^{V_N} on the dividing lines that separate the small boxes from each other. For this reason we sometimes refer to φ^{V_N,V_N° as a *binding field*.

Iterations of the partitioning sketched in Fig. 3.1 lead to a hierarchical description of the DGFF on a box of side $N := 2^n$ as the sum (along root-to-leaf paths of length n) of a family of tree-indexed binding fields. If these binding fields could be regarded as constant on each of the "small" boxes, this would cast the DGFF as a Branching Random Walk. The fields are not constant, so this representation is only approximate. However, the connection of the DGFF to Branching Random Walk has still been extremely useful.

3.2. First moment of level-set size

Equipped with the Gibbs-Markov property, we are now ready to begin the proof of the scaling limit in Theorem 2.7. The key point is to estimate, as well as compute the asymptotic of, the first two moments of the level set

$$\Gamma_N^D(b) := \big\{x \in D_N\colon h_x^{D_N} \ge a_N + b\big\}. \tag{3.15}$$

Let us begin with the first-moment calculations. Assume that $\lambda \in (0,1)$, a domain $D \in \mathfrak{D}$ and an admissible sequence $\{D_N\colon N \ge 1\}$ of domains approximating D are fixed. Our first lemma is then:

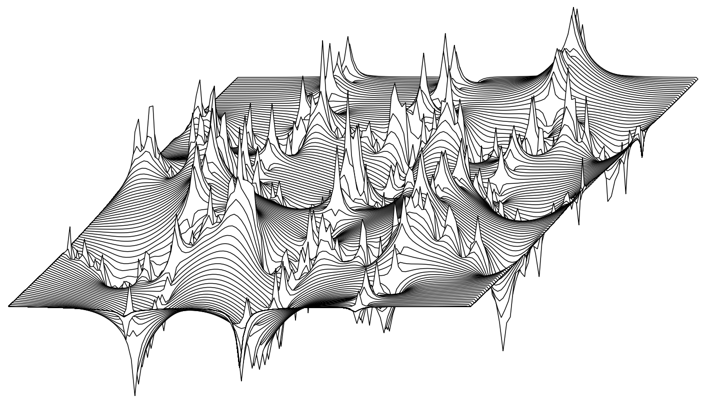

Figure 3.2: A sample of the binding field $\varphi^{V_{4N},V_{4N}^\circ}$ for the (first-level) partition depicted in Fig. 3.1. Here V_{4N}° is the union of 16 disjoint translates of V_N. Note that while the field is smooth inside the individual squares but becomes quite rough on the dividing lines of sites.

Lemma 3.5 [First moment upper bound] *For each $\delta \in (0,1)$ there is $c \in (0,\infty)$ such that for all $N \geq 1$, all $b \in \mathbb{R}$ with $|b| \leq \log N$ and all a_N with $\delta \log N \leq a_N \leq \delta^{-1}\log N$, and all $A \subset D_N$,*

$$E\big|\Gamma_N^D(b) \cap A\big| \leq cK_N \frac{|A|}{N^2}\,\mathrm{e}^{-\frac{a_N}{g\log N}b}\,. \tag{3.16}$$

Proof. Recalling the proof of upper bound in Theorem 2.1, the claim will follow by summing over $x \in D_N$ once we prove that, for some constant c,

$$P\big(h_x^{D_N} \geq a_N + b\big) \leq c\frac{1}{\sqrt{\log N}}\;\mathrm{e}^{-\frac{a_N^2}{2g\log N}}\;\mathrm{e}^{-\frac{a_N}{g\log N}b} \tag{3.17}$$

uniformly in $x \in D_N$ and in $b \in [-\log N, \log N]$. By (3.11), it suffices to prove this for $x := 0$ and D_N replaced $\widetilde{D}_N$ which is the box of side-length side length four times the ℓ^∞-diameter of D_N centered at any point of D_N. Theorem 1.17 ensures that the variance of $h_x^{\widetilde{D}_N}$ is within a constant $\tilde{c}$ of $g\log N$. Hence we get

$$P\big(h_x^{\widetilde{D}_N} \geq a_N + b\big) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{g\log N - \tilde{c}}}\int_b^\infty \mathrm{e}^{-\frac{1}{2}\frac{(a_N+s)^2}{g\log N+\tilde{c}}}\,\mathrm{d}s. \tag{3.18}$$

Bounding $(a_N+s)^2 \geq a_N^2 + 2a_N s$ and noting that the assumptions on a_N (and the inequality $(1+\eta)^{-1} \geq 1-\eta$ for $0<\eta<1$) imply

$$\frac{a_N^2}{g\log N + c} \geq \frac{a_N^2}{g\log N} - \frac{c}{g^2\delta}\,, \tag{3.19}$$

we get

$$\int_b^\infty \mathrm{e}^{-\frac{1}{2}\frac{(a_N+s)^2}{g\log N+\tilde{c}}}\,\mathrm{d}s \leq c'\mathrm{e}^{-\frac{a_N^2}{2g\log N}}\;\mathrm{e}^{-\frac{a_N}{g\log N+\tilde{c}}b} \tag{3.20}$$

for some constant $c' > 0$. As $a_N \leq \delta^{-1}\log N$ and $|b| \leq \log N$, the constant $\tilde{c}$ in the exponent can be dropped at the cost of another multiplicative (constant) term popping in the front. The claim follows. □

Note the following fact:

Exercise 3.6 *A sequence $\{\mu_n \colon n \geq 1\}$ of random Borel measures on a topological space $\mathscr{X}$ is tight with respect to the vague topology if and only if the sequence of random variables $\{\mu_N(K) \colon n \geq 1\}$ is tight for every compact $K \subset \mathscr{X}$.*

Lemma 3.5 then gives:

Corollary 3.7 [Tightness] *$\{\eta_N^D \colon N \geq 1\}$, regarded as measures on $\overline{D} \times (\mathbb{R} \cup \{+\infty\})$, is a tight sequence in the topology of vague convergence.*

Proof. Every compact sets in $\overline{D} \times (\mathbb{R} \cup \{+\infty\})$ is contained in $K_b := \overline{D} \times [b, \infty]$ for some $b \in \mathbb{R}$. The definition of η_N^D shows

$$\eta_N^D(K_b) = \frac{1}{K_N}\bigl|\Gamma_N^D(b)\bigr|. \tag{3.21}$$

Lemma 3.5 shows these have uniformly bounded expectations and so are tight as ordinary random variables. The claim follows by the above exercise. □

Tightness is usually associated with the phrase "mass is not escaping to infinity." However, for convergence of random measures in vague topology, tightness does not prevent convergence of total mass to zero. In order to rule that out, we will need a lower bound of the same order as the upper bound. This is achieved by:

Lemma 3.8 [First moment asymptotic] *Assume that a_N obeys* (2.27) *and let c_0 be as in* (1.33). *Then for all $b \in \mathbb{R}$ and all open $A \subseteq D$,*

$$E\bigl|\{x \in \Gamma_N^D(b) \colon x/N \in A\}\bigr| = \bigl(1 + o(1)\bigr)\frac{\mathrm{e}^{2c_0\lambda^2/g}}{\lambda\sqrt{8\pi}}\mathrm{e}^{-\alpha\lambda b}\Bigl[\int_A \psi_\lambda^D(x)\mathrm{d}x\Bigr]K_N. \tag{3.22}$$

where $o(1) \to 0$ as $N \to \infty$ uniformly on compact sets of b.

Proof. Thanks to the uniform control from Lemma 3.5, we may remove a small neighborhood of ∂D form A and thus assume that A has positive distance from D^{c}. We will proceed by extracting an asymptotic expression for $P(h_x^{D_N} \geq a_N + b)$ with x such that $x/N \in A$. For such x Theorem 1.17 gives

$$G^{D_N}(x, x) = g\log N + \theta_N(x), \tag{3.23}$$

where

$$\theta_N(x) := g\log N + c_0 + g\log r_D(x/N) + o(1)\,, \tag{3.24}$$

with $o(1) \to 0$ as $N \to \infty$ uniformly in $x \in D_N$ with $x/N \in A$. Using that $G^{D_N}(x, x)$ is the variance of $h_x^{D_N}$, we then get

$$P\bigl(h_x^{D_N} \geq a_N + b\bigr) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{g\log N + \theta_N(x)}}\int_b^\infty \mathrm{e}^{-\frac{1}{2}\frac{(a_N+s)^2}{g\log N + \theta_N(x)}}\,\mathrm{d}s\,. \tag{3.25}$$

The first occurrence of $\theta_N(x)$ does not affect the overall asymptotic as this quantity is bounded uniformly for all x under consideration. Expanding

$$(a_N+s)^2 = a_N^2 + 2a_N s + s^2 \tag{3.26}$$

and noting that (by decomposing the integration domain into $s \ll \log N$ and its complement) the s^2 term has negligible effect on the overall asymptotic of the integral, we find out

$$\int_b^\infty \mathrm{e}^{-\frac12 \frac{(a_N+s)^2}{g\log N+\theta_N(x)}}\,\mathrm{d}s = \bigl(1+o(1)\bigr)(\alpha\lambda)^{-1}\mathrm{e}^{-\frac12\frac{a_N^2}{g\log N+\theta_N(x)}}\,\mathrm{e}^{-\alpha\lambda b+o(1)} \tag{3.27}$$

We now use Taylor's Theorem (and the asymptotic of a_N) to get

$$\frac{a_N^2}{g\log N+\theta_N(x)} = \frac{a_N^2}{g\log N} - \frac{4\lambda^2}{g}\theta_N(x) + o(1) \tag{3.28}$$

with $o(1)\to 0$ again uniformly in all x under consideration. Invoking the definition of ψ_λ^D, this yields

$$P\bigl(h_x^{D_N} \ge a_N+b\bigr) = \bigl(1+o(1)\bigr)\frac{\mathrm{e}^{2c_0\lambda^2/g}}{\lambda\sqrt{8\pi}}\,\mathrm{e}^{-\alpha\lambda b}\psi_\lambda^D(x/N)\,\frac{K_N}{N^2}. \tag{3.29}$$

The result follows by summing this probability over x with $x/N \in A$ and using the continuity of ψ_λ^D to convert the resulting Riemann sum into an integral. $\square$

3.3. Second moment estimate

Our next task is to perform a rather tedious estimate on the second moment of the size of $\Gamma_N^D(b)$. It is here where we for simplicity limit the range of possible λ.

Lemma 3.9 [Second moment bound] *Suppose $\lambda \in (0, 1/\sqrt{2})$. For each $b_0 \in \mathbb{R}$ and each $D \in \mathfrak{D}$ there is $c_1 \in (0,\infty)$ such that for each $b \in [-b_0, b_0]$ and each $N \ge 1$,*

$$E\bigl(|\Gamma_N^D(b)|^2\bigr) \le c_1 K_N^2 \tag{3.30}$$

Proof. Assume $b := 0$ for simplicity (or absorb b into a_N). Writing

$$E\bigl(|\Gamma_N^D(0)|^2\bigr) = \sum_{x,y\in D_N} P\bigl(h^{D_N}(x) \ge a_N,\, h^{D_N}(y) \ge a_N\bigr). \tag{3.31}$$

we will need a good estimate on the probability on the right-hand side. Taking $\widetilde{D}_N$ to be a neighborhood of D_N of diameter twice the diameter of D_N, (3.12) shows

$$P\bigl(h^{D_N}(x) \ge a_N,\, h^{D_N}(y) \ge a_N\bigr) \le 4P\bigl(h^{\widetilde{D}_N}(x) \ge a_N,\, h^{\widetilde{D}_N}(y) \ge a_N\bigr)\,. \tag{3.32}$$

The Gibbs-Markov property in turn gives

$$h^{\widetilde{D}_N}(y) = \mathfrak{g}_x(y)h^{\widetilde{D}_N}(x) + \hat{h}^{\widetilde{D}_N\smallsetminus\{x\}}(y), \tag{3.33}$$

where

(1) $h^{\widetilde{D}_N}(x)$ and $\hat{h}^{\widetilde{D}_N \smallsetminus \{x\}}$ are independent,

(2) $\hat{h}^{\widetilde{D}_N \smallsetminus \{x\}}$ has the law of the DGFF in $\widetilde{D}_N \smallsetminus \{x\}$, and

(3) $\mathfrak{g}_x$ is harmonic in $\widetilde{D}_N \smallsetminus \{x\}$, vanishes outside $\widetilde{D}_N$ and obeys $\mathfrak{g}_x(x) = 1$.

Using this decomposition, the above probability is recast as

$$\begin{aligned} &P\big(h^{\widetilde{D}_N}(x) \ge a_N,\, h^{\widetilde{D}_N}(y) \ge a_N\big) \\ &\quad = \int_0^\infty P\Big(\hat{h}^{\widetilde{D}_N \smallsetminus \{x\}}(y) \ge a_N(1-\mathfrak{g}_x(y)) - s\mathfrak{g}_x(y)\Big) P\big(h^{\widetilde{D}_N}(x) - a_N \in \mathrm{d}s\big). \end{aligned} \tag{3.34}$$

Given $\delta > 0$ we can always bound the right-hand side by $P(h^{\widetilde{D}_N} \ge a_N)$ when $|x-y| \le \delta\sqrt{K_N}$. This permits us to assume that $|x-y| > \delta\sqrt{K_N}$ from now on. Since x, y lie "deep" inside $\widetilde{D}_N$ and $|x-y| > \delta\sqrt{K_N} = N^{1-\lambda^2+o(1)}$, we have

$$\mathfrak{g}_x(y) = \frac{G^{\widetilde{D}_N}(x,y)}{G^{\widetilde{D}_N}(x,x)} \le \frac{\log\frac{N}{|x-y|} + c}{\log N - c} \le 1 - (1-\lambda^2) + o(1) = \lambda^2 + o(1), \tag{3.35}$$

where $o(1) \to 0$ uniformly in $x, y \in D_N$. Assuming $s \in [0, a_N]$, from $\lambda < 1/\sqrt{2}$ we then have

$$a_N(1-\mathfrak{g}_x(y)) - s\mathfrak{g}_x(y) > \epsilon a_N \tag{3.36}$$

for some $\epsilon > 0$ as soon as N is large enough, uniformly in $x, y \in D_N$. The argument in Lemma 3.5 in conjunction with $\mathfrak{g}_x(y) \in [0,1]$ then show

$$\begin{aligned} &P\Big(\hat{h}^{\widetilde{D}_N \smallsetminus \{x\}}(y) \ge a_N(1-\mathfrak{g}_x(y)) - s\mathfrak{g}_x(y)\Big) \\ &\qquad \le \frac{c}{\sqrt{\log N}}\, \mathrm{e}^{-\frac{[a_N(1-\mathfrak{g}_x(y)) - s\mathfrak{g}_x(y)]^2}{2G(y,y)}} \le c\frac{K_N}{N^2}\, \mathrm{e}^{\mathfrak{g}_x(y)\frac{a_N^2}{g\log N} + \frac{a_N}{G(y,y)}\mathfrak{g}_x(y)s}, \end{aligned} \tag{3.37}$$

where we wrote G for $G^{\widetilde{D}_N \smallsetminus \{x\}}$ and used that $|G(y,y) - g\log N| \le c$ uniformly in $y \in D_N$. Then

$$P\big(h^{\widetilde{D}_N}(x) - a_N \in \mathrm{d}s\big) \le c\frac{K_N}{N^2}\, \mathrm{e}^{-\frac{a_N}{G(x,x)}s}\mathrm{d}s. \tag{3.38}$$

Since $G(x,x)/G(y,y) = 1 + o(1)$ and $\mathfrak{g}_x(y) \le \lambda^2 + o(1) < 1$, the integral in (3.34) over $s \in [0, a_N]$ yields a harmless multiplicative factor. Also, the middle inequality in (3.35) implies

$$\mathrm{e}^{\mathfrak{g}_x(y)\frac{a_N^2}{g\log N}} \le c\left(\frac{N}{|x-y|}\right)^{4\lambda^2+o(1)} \tag{3.39}$$

with $o(1) \to 0$ uniformly in $x, y \in D_N$ with $|x-y| > \delta\sqrt{K_N}$. From (3.32) we get

$$\begin{aligned} &P\big(h^{D_N}(x) \ge a_N,\, h^{D_N}(y) \ge a_N\big) \\ &\qquad \le P\big(h^{D_N}(x) \ge 2a_N\big) + c\left(\frac{K_N}{N^2}\right)^2 \left(\frac{N}{|x-y|}\right)^{4\lambda^2+o(1)} \end{aligned} \tag{3.40}$$

uniformly in $x, y \in D_N$ with$|x-y| > \delta\sqrt{K_N}$.

In order to finish the proof, we now write

$$E\big(|\Gamma_N^D(0)|^2\big) \le \sum_{\substack{x,y\in D_N\\ |x-y|\le\delta\sqrt{K_N}}} P\big(h^{D_N}(x) \ge a_N\big) + \sum_{\substack{x,y\in D_N\\ |x-y|>\delta\sqrt{K_N}}} P\big(h^{D_N}(x) \ge a_N,\, h^{D_N}(y) \ge a_N\big). \tag{3.41}$$

Summing over y and invoking Lemma 3.5 bounds the first term by a factor of order $(\delta K_N)^2$. The contribution of the first term on the right of (3.40) to the second sum is bounded via Lemma 3.5 as well:

$$P\big(h^{D_N}(x) \ge 2a_N\big) \le \frac{c}{\sqrt{\log N}}\mathrm{e}^{-2\frac{a_N^2}{g\log N}} = c\Big(\frac{K_N}{N^2}\Big)^2 \mathrm{e}^{-\frac{a_N^2}{g\log N}}\sqrt{\log N} \le c\delta\Big(\frac{K_N}{N^2}\Big)^2. \tag{3.42}$$

Plugging in also the second term on the right of (3.40), we thus get

$$E\big(|\Gamma_N^D(0)|^2\big) \le 2c\delta(K_N)^2 + c\Big(\frac{K_N}{N^2}\Big)^2 \sum_{\substack{x,y\in D_N\\ |x-y|>\delta\sqrt{K_N}}} \left(\frac{N}{|x-y|}\right)^{4\lambda^2+o(1)}. \tag{3.43}$$

Dominating the sum by $c(N^2)^2\int_{D\times D}|x-y|^{-4\lambda^2+o(1)}\mathrm{d}x\mathrm{d}y$, with the integral convergent due to $4\lambda^2 < 2$, also the second term on the right is of order $(K_N)^2$. □

As a corollary we now get:

Corollary 3.10 [Subsequential limits are non-trivial] *Suppose* $\lambda \in (0, 1/\sqrt{2})$. *Then every subsequential limit* η^D *of* $\{\eta_N^D\colon N \ge 1\}$ *obeys*

$$P\big(\eta^D(A\times[b,b']) > 0\big) > 0 \tag{3.44}$$

for any open and non-empty $A \subset D$ *and every* $b < b'$.

Proof. Abbreviate $X_N := \eta_N^D(A\times[b,b'])$. Then Lemma 3.8 implies

$$E(X_N) \underset{N\to\infty}{\longrightarrow} \hat{c}\Big[\int_A \psi_\lambda^D(x)\mathrm{d}x\Big]\big(\mathrm{e}^{-\lambda\alpha b} - \mathrm{e}^{-\lambda\alpha b'}\big) \tag{3.45}$$

where $\hat{c} := \mathrm{e}^{c_0\lambda^2/g}/(\lambda\sqrt{8\pi})$. This is positive and finite for any A and b, b' as above. On the other hand, Lemma 3.9 shows that $\sup_{N\ge1} E(X_N^2) < \infty$. The second-moment estimate then yields the claim. □

3.4. Second-moment asymptotic and factorization

At this point we know that the subsequential limits exist and are non-trivial (with positive probability). The final goal of this lecture is to prove:

Proposition 3.11 [Factorization] *Suppose $\lambda \in (0, 1/\sqrt{2})$. Then every subsequential limit η^D of $\{\eta_N^D \colon N \geq 1\}$ takes the form*

$$\eta^D(\mathrm{d}x\,\mathrm{d}h) = Z_\lambda^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\mathrm{d}h \tag{3.46}$$

where Z_λ^D is a random, a.s.-finite Borel measure on D with $P(Z_\lambda^D(D) > 0) > 0$.

As alluded to above, the proof relies on another hefty second-moment calculation. This is the content of:

Lemma 3.12 *For any $\lambda \in (0, 1/\sqrt{2})$, any open $A \subset D$, any $b \in \mathbb{R}$, and*

$$A_N := \{x \in \mathbb{Z}^2 \colon x/N \in A\} \tag{3.47}$$

we have

$$\lim_{N\to\infty} \frac{1}{K_N} E\Big|\big|\Gamma_N^D(0) \cap A_N\big| - \mathrm{e}^{\alpha\lambda b}\big|\Gamma_N^D(b) \cap A_N\big|\Big| = 0. \tag{3.48}$$

Proof (modulo a computation). We will instead prove

$$\lim_{N\to\infty} \frac{1}{K_N^2} E\left(\Big|\big|\Gamma_N^D(0) \cap A_N\big| - \mathrm{e}^{\alpha\lambda b}\big|\Gamma_N^D(b) \cap A_N\big|\Big|^2\right) = 0. \tag{3.49}$$

Writing

$$\big|\Gamma_N^D(\cdot) \cap A_N\big| = \sum_{x \in A_N} 1_{\{h_x^{D_N} \geq a_N + \cdot\}} \tag{3.50}$$

we get a sum of pairs of (signed) products of the various combinations of these indicators. The argument in the proof of Lemma 3.9 allows us to estimate the pairs where $|x - y| \leq \delta N$ by a quantity that vanishes as $N \to \infty$ and $\delta \downarrow 0$. It will thus suffice to show

$$\begin{aligned}\max_{\substack{x,y \in A_N \\ |x-y| > \delta N}} \bigg(& P\big(h_x^{D_N} \geq a_N,\, h_y^{D_N} \geq a_N\big) \\ & - \mathrm{e}^{\alpha\lambda b} P\big(h_x^{D_N} \geq a_N + b,\, h_y^{D_N} \geq a_N\big) \\ & \quad - \mathrm{e}^{\alpha\lambda b} P\big(h_x^{D_N} \geq a_N,\, h_y^{D_N} \geq a_N + b\big) \\ & \quad\quad + \mathrm{e}^{2\alpha\lambda b} P\big(h_x^{D_N} \geq a_N + b,\, h_y^{D_N} \geq a_N + b\big)\bigg) = o\Big(\frac{K_N^2}{N^4}\Big)\end{aligned} \tag{3.51}$$

as $N \to \infty$. A computation refining the argument in the proof of Lemma 3.9 (while aided by the fact that $|x - y| > \delta N$) now shows that, for any $b_1, b_2 \in \{0, b\}$,

$$\begin{aligned} & P\big(h^{D_N}(x) \geq a_N + b_1,\, h^{D_N}(y) \geq a_N + b_2\big) \\ & \qquad = \big(\mathrm{e}^{-\alpha\lambda(b_1+b_2)} + o(1)\big) P\big(h^{D_N}(x) \geq a_N,\, h^{D_N}(y) \geq a_N\big)\,. \end{aligned} \tag{3.52}$$

This then implies (3.51) and the whole claim. $\square$

Exercise 3.13 *Supply a detailed proof of* (3.52).

From Lemma 3.12 we get:

Corollary 3.14 *Suppose $\lambda \in (0, 1/\sqrt{2})$. Then subsequential limit η^D of $\{\eta^D_N \colon N \geq 1\}$ obeys the following: For any open $A \subset D$ and any $b \in \mathbb{R}$,*

$$\eta^D\big(A \times [b, \infty)\big) = \mathrm{e}^{-\alpha\lambda b}\eta^D\big(A \times [0, \infty)\big), \quad \text{a.s.} \tag{3.53}$$

Proof. In the notation of Lemma 3.12,

$$\eta^D_N\big(A \times [b, \infty)\big) = \frac{1}{K_N}\big|\Gamma^D_N(b) \cap A_N\big| \tag{3.54}$$

Taking a *joint* distributional limit of $\eta^D_N(A \times [b, \infty))$ and $\eta^D_N(A \times [b, \infty))$ along the given subsequence, Lemma 3.12 then readily gives the claim. □

Exercise 3.15 *Additional approximations are needed in order to make the conclusion of the previous proof fully justified for the assumed type of convergence of $\{\eta^D_N \colon N \geq 1\}$. Supply the missing details.*

We now give:

Proof of Proposition 3.11. For each Borel $A \subset D$ define

$$Z^D_\lambda(A) := (\alpha\lambda)\eta^D\big(A \times [0, \infty)\big) \tag{3.55}$$

Then Z^D_λ is an a.s.-finite (random) Borel measure on D. Corollary 3.14 and a simple limiting argument show that, for any $\mathcal{G}_\delta$-set $A \subset D$

$$\eta^D_N\big(A \times [b, \infty)\big) = (\alpha\lambda)^{-1} Z^D_\lambda(A)\mathrm{e}^{-\alpha\lambda b}, \quad \text{a.s.} \tag{3.56}$$

Letting $\mathcal{A}$ be the set of all finite unions of half-open dyadic boxes entirely contained in D, the fact that $\mathcal{A}$ is countable permits us to choose the null set in (3.56) such that the equality in (3.56) holds for all $A \in \mathcal{A}$ simultaneously, a.s. But both sides of (3.56) are Borel measures (in the first "variable") and since $\mathcal{A}$ is a generating class of Borel sets, equality holds simultaneously for all Borel $A \subset D$, a.s.

We now observe

$$(\alpha\lambda)^{-1} Z^D_\lambda(A)\mathrm{e}^{-\alpha\lambda b} = \int_{A \times [b, \infty)} Z^D_\lambda(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\mathrm{d}h \tag{3.57}$$

to check that η^D indeed has the desired form. □

We also record an important observation:

Corollary 3.16 *Assume $\lambda \in (0, 1/\sqrt{2})$ and denote*

$$\hat{c} := \frac{\mathrm{e}^{2c_0\lambda^2/g}}{\lambda\sqrt{8\pi}} \tag{3.58}$$

for c_0 as in (1.33). *Then Z^D_λ from* (3.46) *obeys*

$$E\big[Z^D_\lambda(A)\big] = \hat{c}\int_A \psi^D_\lambda(x)\mathrm{d}x \tag{3.59}$$

for each Borel $A \subseteq D$. Moreover, there is $c \in (0, \infty)$ such that for any open square $S \subset \mathbb{C}$,

$$E\big[Z^S_\lambda(A)^2\big] \leq c\,\mathrm{diam}(S)^{4+4\lambda^2}. \tag{3.60}$$

Proof. Thanks to uniform square integrability proved in Lemma 3.9, the convergence in probability is accompanied by convergence of the first moments. Then (3.59) follows from Lemma 3.8. To get also (3.60) we need a uniform version of the bound in Lemma 3.9. We will not perform the requisite calculation, just note that for a $c' \in (0,\infty)$ the following holds for all $D \in \mathfrak{D}$,

$$\limsup_{N\to\infty} \frac{1}{K_N^2}\, E\big(|\Gamma_N^D(0)|^2\big) \le c' \int_{D\times D} \left(\frac{[\operatorname{diam} D]^2}{|x-y|}\right)^{4\lambda^2} \mathrm{d}x\mathrm{d}y\,, \tag{3.61}$$

where $\operatorname{diam} D$ is the diameter of D in the Euclidean norm. We leave further details of the proof to the reader. □

This closes the first part of the proof of Theorem 2.7 which showed that every subsequential limit of the measures of interest factorizes into the desired product form.