Lecture 2

Maximum and intermediate values

In this lecture we will start discussing the main subject of interest in this course: geometric properties of the DGFF sample paths. After some introduction and pictures, we will focus attention on the behavior of the absolute maximum as well as the level sets at heights proportional to the absolute maximum. We will then state the main theorem on the scaling limit of such level sets linking the limit object to Gaussian multiplicative chaos and Liouville quantum gravity. The proofs of the main theorems are relegated to the forthcoming lectures.

2.1. Level set geometry

The existence of the scaling limit established in Theorem 1.26 indicates that the law of the DGFF is asymptotically scale invariant. Scale invariance of a random object usually entails one of the following two possibilities:

- either the object is trivial (e.g., degenerate, flat, non-random),
- or it is very interesting (e.g., chaotic or fractal, etc).

As is seen in Fig. 2.1, the DGFF definitely falls into the latter category.

Looking at Fig. 2.1 more closely, a natural first question is to understand the behavior of the (implicit) boundaries between warm and cold colors. As the field averages to zero, and should thus take both positive and negative values pretty much everywhere, this in particular amounts to looking at the contour lines *between* the regions where the field is positive and where it is negative. This has been done and corresponds to the beautiful work of Schramm and Sheffield, with later contributions due to Werner, Miller and others. It is thus known that (with proper formulation) these lines admit a scaling limit to a process of nested collections of loops called a *Conformal Loop Ensemble*. The individual contour lines are closely related to the *Schramm-Loewner process* SLE_κ with $\kappa = 4$.

Our interest in these lectures will be somewhat different as we will want to look at level sets at heights proportional to the scaling of the maximum. We call these

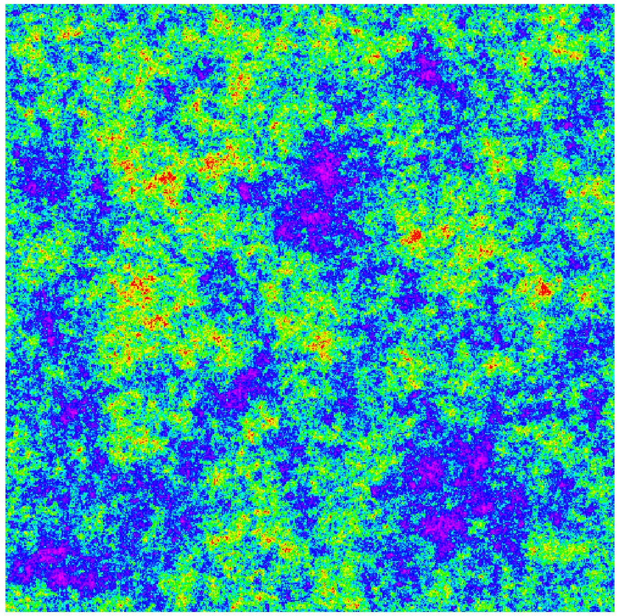

Figure 2.1: A sample of the DGFF on 300×300 square in $\mathbb{Z}^2$. The cold colors (violet and blue) indicate low values, the warm colors (yellow and red) indicate large values. The fractal nature of the sample is quite apparent.

intermediate level sets. Samples of such level sets are shown in Fig. 2.2.

The self-similar structure of the level sets in Fig. 2.2 is quite apparent. This motivates the following questions:

- Is there a way to take a scaling limit of the samples in Fig. 2.2?
- And if so, is there a way to characterize the limit object directly?

Our motivation for these arises directly from Donsker's Invariance Principle for random walks. There one first answers the second question by constructing a limit process called the Brownian Motion. Then one proves that, under diffusive scaling of space and time, all random walks with zero mean and second moments scale to that Brownian motion (or constant multiples thereof).

The goal of the lectures for the rest of the week is to answer these questions for intermediate level sets. We will do this only for one starting process (the DGFF above) so this cannot be thought as a full analogue of Donsker's Invariance Principle. Notwithstanding, the spirit of the result is quite similar.

2.2. Growth of absolute maximum

In order to set the scales for our future discussion, we first have to identify the growth-rate of the absolute maximum. Here an early result of Bolthausen, Deuschel

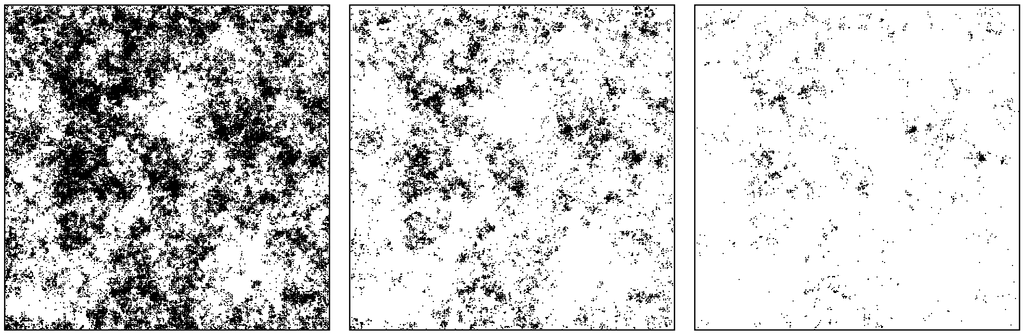

Figure 2.2: Plots of the points where the sample of the DGFF in Fig. 2.1 is at heights (as labeled left to right) above 0.1, 0.3 and 0.5-multiples of the absolute maximum, respectively. Higher level sets are too sparse to produce a visible effect.

and Giacomin from 2001 provided control of the leading-order asymptotic in square boxes. Their result reads:

Theorem 2.1 [Bolthausen, Deuschel, Giacomin 2001] *For* $V_N := (0,N)^2 \cap \mathbb{Z}^2$,

$$\max_{x\in V_N} h^{V_N}(x) = \bigl(2\sqrt{g} + o(1)\bigr)\log N \tag{2.1}$$

where $o(1) \to 0$ *in probability as* $N \to \infty$.

Proof of upper bound in (2.1). We start by noting the well-known tail estimate for centered normal random variables:

Exercise 2.2 *Prove that*

$$Z \overset{\text{law}}{=} \mathcal{N}(0,\sigma^2) \quad \Rightarrow \quad P(Z > a) \le \mathrm{e}^{-\frac{a^2}{2\sigma^2}}, \qquad a > 0. \tag{2.2}$$

We will want to use this for Z replaced by $h_x^{V_N}$ but for that we need need to bound the variance of $h_x^{V_N}$ uniformly in $x \in V_N$. Here we observe that, thanks to the monotonicity of $V \mapsto G^V(x,x)$ and translation invariance $G^{z+V}(z+x,z+y) = G^V(x,y)$, denoting $\widetilde{V}_N := (-N/2,N/2)^2 \cap \mathbb{Z}^2$, there is a $c \in \mathbb{R}$ such that

$$\max_{x\in V_N} \mathrm{Var}\bigl(h_x^{V_N}\bigr) \le \mathrm{Var}\bigl(h_0^{\widetilde{V}_{2N}}\bigr) \le g\log N + c, \tag{2.3}$$

where the last bound follows from the asymptotic in Theorem 1.17. Plugging this in (2.2), for any $\theta > 0$ we thus get

$$P\bigl(h_x^{V_N} > \theta\log N\bigr) \le \exp\Bigl\{-\frac{1}{2}\frac{\theta^2(\log N)^2}{g\log N + c}\Bigr\}. \tag{2.4}$$

Using that $(1+\lambda)^{-1} \geq 1-\lambda$ for $\lambda \in (0,1)$ we obtain

$$\frac{1}{g\log N + c} \geq \frac{1}{g\log N} - \frac{c}{(g\log N)^2} \tag{2.5}$$

as soon as N is sufficiently large. Then

$$\max_{x\in V_N} P\bigl(h_x^{V_N} > \theta\log N\bigr) \leq c' N^{-\frac{\theta^2}{2g}} \tag{2.6}$$

for $c' := \mathrm{e}^{\theta^2 c/(2g^2)}$ as soon as N is large enough. The union bound and the fact that $|V_N| \leq N^2$ then give

$$\begin{aligned} P\Bigl(\max_{x\in V_N} h_x^{V_N} > \theta\log N\Bigr) &\leq \sum_{x\in V_N} P\bigl(h_x^{V_N} > \theta\log N\bigr) \\ &\leq c'|V_N|N^{-\frac{\theta^2}{2g}} = c' N^{2-\frac{\theta^2}{2g}}. \end{aligned} \tag{2.7}$$

This tends to zero as $N\to\infty$ for any $\theta > 2\sqrt{g}$ thus proving "$\leq$" in (2.1). □

The proof of the complementary lower bound is considerably harder. The idea is to use the second-moment method but that requires working with a scale decomposition of the DGFF and computing the second moment under a suitable truncation on the various terms in this decomposition. We will not perform this calculation here as the result will follow as a corollary from Theorem 2.7 below.

Soon after the appearance of the above results, Daviaud was able to extend the control to the level sets of the form

$$\bigl\{x\in V_N\colon h_x^{V_N} \geq 2\sqrt{g}\,\lambda\log N\bigr\}, \tag{2.8}$$

where $\lambda \in (0,1)$. His result reads:

Theorem 2.3 [Daviaud 2004] *For any $\lambda \in (0,1)$,*

$$\#\bigl\{x\in V_N\colon h_x^{V_N} \geq 2\sqrt{g}\,\lambda\log N\bigr\} = N^{2(1-\lambda^2)+o(1)}, \tag{2.9}$$

where $o(1)\to 0$ in probability as $N\to\infty$.

Proof of "$\leq$" in (2.9). Let L_N denote the cardinality of the set in (2.8). Using the Markov inequality and the reasoning (2.6–2.7),

$$\begin{aligned} P\bigl(L_N \geq N^{2(1-\lambda^2)+\epsilon}\bigr) &\leq N^{-2(1-\lambda^2)-\epsilon} E(L_N) \\ &\leq c' N^{-2(1-\lambda^2)-\epsilon} N^{2-2\lambda^2} = c' N^{-\epsilon}. \end{aligned} \tag{2.10}$$

This tends to zero as $N\to\infty$ for any $\epsilon > 0$ thus proving "$\leq$" in (2.9). □

We will not give a full proof of the lower bound for all $\lambda \in (0,1)$ as that requires similar truncations as the corresponding bound for the maximum. However, these truncations are avoidable for λ small, so we will content ourselves with:

Proof of "$\geq$" in (2.9) *with positive probability for $\lambda < 1/\sqrt{2}$.* Define

$$Y_N := \sum_{x \in V_N^\epsilon} \mathrm{e}^{\beta h_x^{V_N}} \tag{2.11}$$

where $\beta > 0$ is a parameter to be adjusted later and $V_N^\epsilon := (\epsilon N, (1-\epsilon)N)^2 \cap \mathbb{Z}^2$ for some $\epsilon \in (0, 1/2)$ to be fixed for the rest of the calculation. This can be thought of as the normalizing factor (the partition function) for the Gibbs measure on V_N where state x gets energy $h_x^{V_N}$. Our first observation is:

Lemma 2.4 *For $\beta > 0$ such that $\beta^2 g < 2$ there is $c = c(\beta) > 0$ such that*

$$P\big(Y_N \geq c N^{2+\frac{1}{2}\beta^2 g}\big) \geq c \tag{2.12}$$

once N is sufficiently large.

Proof. We will prove this by invoking the *second moment method* whose driving force is the following inequality:

Exercise 2.5 *Let $Y \in L^2$ be a non-negative random variable. Prove that*

$$P\big(Y \geq \eta EY\big) \geq (1-\eta)^2 \frac{[E(Y)]^2}{E(Y^2)}, \quad \eta \in (0,1). \tag{2.13}$$

We begin by computing the first moment of Y_N. The fact that $E\mathrm{e}^X = \mathrm{e}^{EX + \frac{1}{2}\mathrm{Var}(X)}$ for any X normal yields

$$EY_N = \sum_{x \in V_N^\epsilon} \mathrm{e}^{\frac{1}{2}\beta^2 \mathrm{Var}(h_x^{V_N})}. \tag{2.14}$$

Writing $\widetilde{V}_N := (-N/2, N/2)^2 \cap \mathbb{Z}^2$, the monotonicity of $V \mapsto G^V(x,x)$ gives

$$\mathrm{Var}(h_0^{\widetilde{V}_{\epsilon N}}) \leq \mathrm{Var}(h_x^{V_N}) \leq \mathrm{Var}(h_0^{\widetilde{V}_N}) \tag{2.15}$$

Theorem 1.17 then implies

$$\sup_{N \geq 1} \max_{x \in V_N^\epsilon} \big|\mathrm{Var}(h_x^{V_N}) - g \log N\big| < \infty \tag{2.16}$$

As $|V_N^\epsilon|$ is of order N^2, using this in (2.14) we conclude that

$$cN^{2+\frac{1}{2}\beta^2 g} \leq EY_N \leq c^{-1} N^{2+\frac{1}{2}\beta^2 g} \tag{2.17}$$

holds for some constant $c \in (0,1)$ and all $N \geq 1$.

Next we will compute the second moment of Y_N. Using the notation for the Green function, we have

$$E(Y_N^2) = \sum_{x,y \in V_N^\epsilon} \mathrm{e}^{\frac{1}{2}\beta^2 [G^{V_N}(x,x) + G^{V_N}(y,y) + 2G^{V_N}(x,y)]}. \tag{2.18}$$

Invoking (2.16) and (1.44) we thus get

$$E(Y_N^2) \le c' N^{\beta^2 g} \sum_{x,y \in V_N^\epsilon} \left(\frac{N}{|x-y| \vee 1} \right)^{\beta^2 g}. \tag{2.19}$$

For $\beta^2 g < 2$ the sum is dominated by pairs x and y with $|x-y|$ of order N. The sum is thus of order N^4 and so we conclude

$$E(Y_N^2) \le c'' N^{\beta^2 g + 4} \tag{2.20}$$

for some constant $c'' > 0$. By (2.17), this bound is proportional to $[EY_N]^2$ so using this in (2.13) (with, e.g., $\eta := 1/2$) readily yields the claim. $\square$

Next we will need to observe that the main contribution to Y_N comes from the set of points where the field is roughly equal $\beta g \log N$:

Lemma 2.6 *For any $\delta > 0$,*

$$P\left(\sum_{x \in V_N^\epsilon} 1_{\{|h_x^{V_N} - \beta g \log N| > (\log N)^{1/2+\delta}\}} \, \mathrm{e}^{\beta h_x^{V_N}} \ge \delta N^{2 + \frac12 \beta^2 g} \right) \underset{N\to\infty}{\longrightarrow} 0. \tag{2.21}$$

Proof. By (2.16), we may instead prove this for $\beta g \log N$ replaced by $\beta \mathrm{Var}(h_x^{V_N})$. Using the Markov inequality, the probability is then bounded by

$$\frac{1}{\delta N^{2+\frac12 \beta^2 g}} \sum_{x \in V_N^\epsilon} E\Bigl(1_{\{|h_x^{V_N} - \beta \mathrm{Var}(h_x^{V_N})| > (\log N)^{1/2+\delta}\}} \, \mathrm{e}^{\beta h_x^{V_N}} \Bigr) \tag{2.22}$$

Changing variables inside the (single-variable) Gaussian integral gives

$$\begin{aligned} E\Bigl(1_{\{|h_x^{V_N} - \beta \mathrm{Var}(h_x^{V_N})| > (\log N)^{1/2+\delta}\}} \, \mathrm{e}^{\beta h_x^{V_N}} \Bigr) \qquad & \\ = \mathrm{e}^{-\frac12 \beta^2 \mathrm{Var}(h_x^{V_N})} P\bigl(|h_x^{V_N}| > (\log N)^{1/2+\delta} \bigr) & \\ \le c N^{-\frac12 \beta^2 g} \mathrm{e}^{-c''(\log N)^{2\delta}} & \end{aligned} \tag{2.23}$$

for some $c, c' > 0$, where we used again (2.16). This bounds the probability in the statement by a constant times $\delta^{-1} \mathrm{e}^{-c''(\log N)^{2\delta}}$, which tends to zero as $N \to \infty$. $\square$

Combining the results of the two lemmas we readily infer

$$P\left(\sum_{x \in V_N^\epsilon} 1_{\{h_x^{V_N} \ge \beta g \log N - (\log N)^{1/2+\delta}\}} \ge \frac{c}{2} N^{2 + \frac12 \beta^2 g - \beta^2 g} \mathrm{e}^{-\beta (\log N)^{1/2+\delta}} \right) \ge \frac{c}{2} \tag{2.24}$$

as soon as N is sufficiently large. Setting β so that $\beta g \log N - (\log N)^{1/2+\delta} = 2\sqrt{g} \lambda \log N$ gives $2 - \frac12 \beta^2 g = 2(1-\lambda^2) + O(\log N)^{-1/2+\delta})$ and so, since $V_N^\epsilon \subset V_N$,

$$P\bigl(L_N \ge N^{2(1-\lambda^2) - c'(\log N)^{1/2+\delta}} \bigr) \ge \frac{c}{2} \tag{2.25}$$

holds for some constant $c' \in \mathbb{R}$ once N is large enough. This is "$\geq$" in (2.9) with $o(1) \to 0$ with a uniformly positive probability. The claim applies only to β such that $\beta^2 g < 2$, which means that it covers only $\lambda < 1/\sqrt{2}$. $\square$

Having the lower bound with positive probability is actually sufficient to complete the proof of (2.9) as stated. The key additional tool needed for this is the Gibbs-Markov decomposition of the DGFF which will be discussed in the next lecture. (The application to the above proof will be given as an exercise.)

It is actually remarkable that the first-moment calculation alone is able to nail the correct leading order of the maximum as well as the asymptotic size of the level set (2.8). As that calculation did not involve correlations between the DGFF at different vertices, the same estimate would apply to i.i.d. Gaussians with the same growth of the variances. This (and many subsequent derivations) may lead one to think that the extreme values behave somehow like those of i.i.d. Gaussians. However, this is very far from truth, as seen in Fig. 2.3.

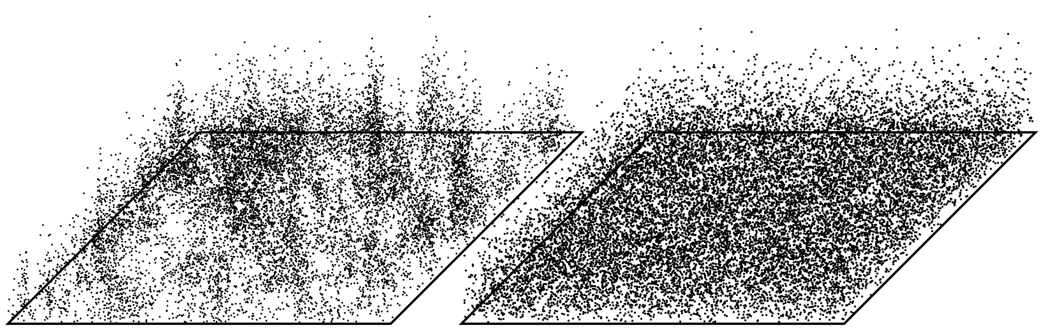

Figure 2.3: Left: A sample of the level set (2.8) on a square of side $N := 300$ with $\lambda := 0.2$. Right: A corresponding sample for i.i.d. normals with mean zero and variance $g \log N$. Although these two samples live on the same "vertical scale", their local structure is very different.

Related to this is the fact that the factor $1 - \lambda^2$ in the exponent is ubiquitous in this subject area. Indeed, it appears in various forms in the study of thick points of Brownian motion and, as was just noted, i.i.d. Gaussians with variance $g \log N$. A paper by Chatterjee, Dembo and Ding (arXiv:1310.5175) gives (generous) conditions under which such a factor should be expected.

2.3. Intermediate level sets

The main objective in this part of the course is to show that the intermediate level set (2.8) admit a non-trivial *scaling limit* whose law can be explicitly characterized. A key starting point is proper formulation of what it means to take a scaling limit. Indeed, scaling the box down to a unit size, the set (2.8) becomes increasingly dense everywhere so taking its limit using, e.g., the topology of Hausdorff convergence does not seem useful. The new idea here is to encode the set into the point measure on $D \times \mathbb{R}$ of the form

$$\sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h_x^{D_N} - a_N}, \tag{2.26}$$

where a_N is a scale sequence such that

$$\frac{a_N}{\log N} \underset{N\to\infty}{\longrightarrow} 2\sqrt{g}\,\lambda \tag{2.27}$$

for some $\lambda \in (0,1)$. A sample from (2.26) can be identified with the picture on the left of Fig. 2.3. Theorem 2.3 indicates that the measure in (2.26) will have unbounded mass even if restricted to bounded intervals in the second variable, and so suitable normalization is required. We will show that this can be done (somewhat surprisingly) by a *deterministic* sequence of the form

$$K_N := \frac{N^2}{\sqrt{\log N}}\, \mathrm{e}^{-\frac{a_N^2}{2g\log N}} \tag{2.28}$$

As is directly checked, (2.27) implies $K_N = N^{2(1-\lambda^2)+o(1)}$ so the normalization is consistent with with Theorem 2.3. Our main result, proved jointly with O. Louidor in a recent posting (arXiv:1612.01424), is then:

Theorem 2.7 [Scaling limit of intermediate level sets] *For each $\lambda \in (0,1)$ and each $D \in \mathfrak{D}$ there is an a.s.-finite random Borel measure Z_λ^D on D such that for any a_N satisfying* (2.27) *and any admissible sequence $\{D_N : N \geq 1\}$ of lattice approximations of D, the normalized point measure*

$$\eta_N^D := \frac{1}{K_N} \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h^{D_N}(x) - a_N} \tag{2.29}$$

obeys

$$\eta_N^D \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} Z_\lambda^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\mathrm{d}h, \tag{2.30}$$

where $\alpha := 2/\sqrt{g}$. Moreover, $Z_\lambda^D(A) > 0$ for every non-empty open $A \subset D$ a.s.

A remark is perhaps in order on what it means that random measures converge in law. The space of Radon measures on $D \times \mathbb{R}$ (of which η_N^D is an example) is naturally endowed with topology of vague convergence. Then $\eta_N^D \to \eta^D$ in law if for every $f : D \times \mathbb{R} \to \mathbb{R}$ which is continuous and compactly supported, the integrals of f with respect to η_N^D converge in law to the integral of f with respect to η^D. A subtlety of the above theorem is that the convergence actually happens on a larger space, namely, $\overline{D} \times (\mathbb{R} \cup \{+\infty\})$. This means that we can take the above statement even for functions which take non-trivial values on ∂D in the x-variable and/or at $+\infty$ in the h-variable. This then readily implies:

Corollary 2.8 *For the setting of Theorem 2.7,*

$$\frac{1}{K_N}\#\{x \in D_N : h^{D_N}(x) \geq a_N\} \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} (\alpha\lambda)^{-1}\, Z_\lambda^D(D). \tag{2.31}$$

Proof. Apply Theorem 2.7 to η_N^D integrated against $f(x,y) := 1_{[0,\infty)}(h)$. $\square$

Exercise 2.9 *Apply suitable monotone limits to check that the convergence in* (2.30) *— which restrict* a priori *only integrals of these measures with respect to compactly supported continuous functions — can be applied to functions of the form*

$$f(x,h) := 1_A(x)1_{[a,b]}(h) \tag{2.32}$$

for $A \subset D$ either open or closed and $a < b$.

We remark that Corollary 2.8 extends, quite considerably, Theorem 2.3 originally proved by Daviaud in 2004. Trying to get more feeling for what Theorem 2.7 says about the positions of the points in the level set, we also state:

Corollary 2.10 *Let X_N be a point from $\{x/N\colon x \in D_N,\, h_x^{D_N} \ge a_N\}$ chosen uniformly at random. Then*

$$\frac{1}{N}X_N \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \widehat{X} \quad \textit{with} \quad \text{law}(\widehat{X}) = \frac{Z_\lambda^D(\cdot)}{Z_\lambda^D(D)}. \tag{2.33}$$

Proof. We easily check that, for any $f\colon \overline{D} \to \mathbb{R}$ continuous (and thus bounded),

$$E\big[f(X_N/N)\big] = \frac{\langle \eta_N^D, f \otimes 1_{[0,\infty)}\rangle}{\langle \eta_N^D, 1_{[0,\infty)}\rangle}, \tag{2.34}$$

where $(f \otimes 1_{[0,\infty)})(x,h) := f(x)1_{[0,\infty)}(h)$ and the brackets denote the integral of the function with respect to the measure. Applying Theorem 2.7, we get

$$\frac{\langle \eta_N^D, f1_{[0,\infty)}\rangle}{\langle \eta_N^D, 1_{[0,\infty)}\rangle} \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \frac{\int_D f(x) Z_\lambda^D(\mathrm{d}x)}{Z_\lambda^D(D)}. \tag{2.35}$$

This is what is stated above. $\square$

Exercise 2.11 *The statement* (2.35) *harbors a technical caveat: we are taking the distributional limit of a ratio of two random variables, each of which converges in law. Fill the details needed to justify the conclusion.*

The spatial part of the right-hand side of (2.30) thus tells us about the "intensity" of the sets in the pictures in Fig. 2.3. Concerning the values of the field, we may be tempted to say that these are Gumbel "distributed" with decay exponent $\alpha\lambda$. This is not justified by the statement per say as the measure on the right of (2.30) is not a probability (it is not even finite). Still, one can perhaps relate this to the corresponding problem for i.i.d. Gaussians with variance $g \log N$; cf Fig. 2.3 again. As a straightforward exercise in extreme-value statistics, we in fact pose:

Exercise 2.12 *Prove the same type of convergence, with the same K_N and with Z_λ^D replaced by (a constant times) the Lebesgue measure on D, for the measure η_N^D associated with standard i.i.d. Gaussians with covariance $g \log N$.*

We rush to add that (as we will explain later) Z_λ^D is a.s. singular with respect to the Lebesgue measure. This vindicates, one more time, Fig. 2.3.

2.4. Link to Liouville Quantum Gravity

As we will see, the random measures $\{Z^D_\lambda\colon D\in\mathfrak{D}\}$ (or, rather their laws) are very closely related. We will later give a list of properties that characterize these laws uniquely. From these properties one can derive the following transformation rule under the conformal maps between the underlying domains:

Theorem 2.13 [Conformal convariance] *Let $\lambda\in(0,1)$. Under any conformal bijection $f\colon D\to f(D)$ between the admissible domains $D,f(D)\in\mathfrak{D}$, the laws of the above measures transform as*

$$Z^{f(D)}_\lambda\circ f(\mathrm{d}x)\overset{\text{law}}{=}|f'(x)|^{2+2\lambda^2}\,Z^D_\lambda(\mathrm{d}x). \tag{2.36}$$

Recall that the simply connected domains are all included in $\mathfrak{D}$ and that $r_D(x)$ denotes the conformal radius of D from x. The following is now a simple consequence of the above theorem:

Exercise 2.14 *Show that in the class of bounded, simply-connected $D\subset\mathbb{C}$, the law of*

$$\frac{1}{r_D(x)^{2+2\lambda^2}}\,Z^D_\lambda(\mathrm{d}x) \tag{2.37}$$

is invariant under conformal maps. Prove that this measure is infinite a.s.

As a consequence, the law of Z^D_λ for any bounded and simply connected D can thus be reconstructed from the law on, say, the open unit disc. However, we can even give an independent construction of the law of Z^D_λ using the ideas from Kahane's theory of multiplicative chaos.

Let $\mathsf{H}^1_0(D)$ denote the closure of the set of smooth, functions with compact support in D in the topology induced by the Dirichlet inner product

$$\langle f,g\rangle_\nabla:=\frac14\int_D\nabla f(x)\cdot\nabla g(x)\,\mathrm{d}x\,, \tag{2.38}$$

where ∇f is now the ordinary (continuum) gradient. For $\{X_n\colon n\ge1\}$ i.i.d. standard normals and $\{f_n\colon n\ge1\}$ an orthonormal basis in $\mathsf{H}^1_0(D)$, let

$$\varphi_n(x):=\sum_{k=1}^n X_kf_k(x). \tag{2.39}$$

These are to be thought of as regularizations of the CGFF. Indeed, we have:

Exercise 2.15 *For any smooth $f\colon D\to\mathbb{R}$, let $\varphi_n(f):=\int_D f(x)\varphi_n(x)\mathrm{d}x$. Show that $\varphi_n(f)$ converges, as $n\to\infty$, in L^2 to a CGFF in the sense of Definition 1.28.*

For each $\beta\in[0,\infty)$, define the random measure

$$\mu^{D,\beta}_n(\mathrm{d}x):=1_D(x)\mathrm{e}^{\beta\varphi_n(x)-\frac{\beta^2}{2}E[\varphi_n(x)^2]}\,\mathrm{d}x. \tag{2.40}$$

The following observation goes back to Kahane in 1985:

Lemma 2.16 [Gaussian Multiplicative Chaos] *There exists a random, a.s. finite (albeit possibly trivial) Borel measure $\mu_\infty^{D,\beta}$ on D such that for each measurable $A \subset D$*

$$\mu_n^{D,\beta}(A) \underset{n\to\infty}{\longrightarrow} \mu_\infty^{D,\beta}(A), \quad \text{a.s.} \tag{2.41}$$

Proof. For each $n \in \mathbb{N}$ define

$$M_n := \mu_n^{D,\beta}(A) \quad \text{and} \quad \mathcal{F}_n := \sigma(X_1, \dots, X_n). \tag{2.42}$$

We claim that $\{M_n : n \geq 1\}$ is a martingale with respect to $\{\mathcal{F}_n : n \geq 1\}$. Using the regularity of the underlying measure space (to apply Fubini-Tonelli)

$$E(M_{n+1}|\mathcal{F}_n) = E\big(\mu_{n+1}^{D,\beta}(A) \,\big|\, \mathcal{F}_n\big) = \int_D \mathrm{d}x\, E\big(\mathrm{e}^{\beta\varphi_{n+1}(x)-\frac{\beta^2}{2}E[\varphi_{n+1}(x)^2]} \,\big|\, \mathcal{F}_n\big)\,. \tag{2.43}$$

The additive structure of φ_n now gives

$$\begin{aligned} &E\big(\mathrm{e}^{\beta\varphi_{n+1}(x)-\frac{\beta^2}{2}E[\varphi_{n+1}(x)^2]} \,\big|\, \mathcal{F}_n\big) \\ &\qquad = \mathrm{e}^{\beta\varphi_n(x)-\frac{\beta^2}{2}E[\varphi_n(x)^2]} E\big(\mathrm{e}^{\beta f_{n+1}(x)X_{n+1}-\frac{1}{2}\beta^2 f_{n+1}(x)^2 E[X_{n+1}^2]}\big) \\ &\qquad = \mathrm{e}^{\beta\varphi_n(x)-\frac{\beta^2}{2}E[\varphi_n(x)^2]}. \end{aligned} \tag{2.44}$$

Using this in (2.43), the right-hand side then wraps back into $\mu_n^{D,\beta}(A) = M_n$ and so $\{M_n : n \geq 1\}$ is a martingale as claimed.

The martingale $\{M_n : n \geq 1\}$ is non-negative and so the Martingale Convergence Theorem yields $M_n \to M_\infty$ a.s. In order to identify the limit in terms of a random measure, we have to rerun the above argument as follows: For any bounded measurable $f : D \to \mathbb{R}$ define

$$\phi_n(f) := \int f \,\mathrm{d}\mu_n^{D,\beta}\,. \tag{2.45}$$

Then the same argument as above shows that $\phi_n(f)$ is a bounded martingale and so $\phi_n(f) \to \phi_\infty(f)$. Specializing to continuous f, the immediate bound

$$\big|\phi_n(f)\big| \leq \mu_n^{D,\beta}(D)\, \|f\|_{C(D)} \tag{2.46}$$

along with the above a.s. convergence $M_n \to M_\infty$ yields

$$\big|\phi_\infty(f)\big| \leq M_\infty \|f\|_{C(D)}. \tag{2.47}$$

Restricting to a countable dense subclass of f to manage proliferation of null sets, $f \mapsto \phi_\infty(f)$ extends to a continuous linear functional on $C(D)$ a.s. whose value for any f still agrees with $\phi_\infty(f)$ (constructed by the above limit) a.s. The Riesz Representation Theorem then casts $\phi_\infty(f)$ as

$$\phi_\infty(f) = \int f \,\mathrm{d}\mu_\infty^{D,\beta} \tag{2.48}$$

for some (random) Borel measure $\mu_\infty^{D,\beta}$. By straightforward approximation arguments, we then get (2.41) (with the null set depending on A) as well. $\square$

As it turns out, the law of the measure $\mu_\infty^{D,\beta}$ is independent of the choice of the underlying basis in $\mathsf{H}_0^1(D)$. (This has been proved gradually starting with somewhat restrictive Kahane's theory and culminating in a recent paper by Shamov). Moreover, it is also known that for each $\beta \in (0, \beta_c)$, where

$$\beta_c := \alpha = 2/\sqrt{g} \tag{2.49}$$

we have $\mu_\infty^{D,\beta}(D) > 0$ a.s. (This will independently follow from our results below.) The measure $\mu_\infty^{D,\beta}$ is called the *Gaussian multiplicative chaos* associated with the continuum Gaussian Free Field. We now claim:

Theorem 2.17 **[Z_λ^D-measures as LQG measure]** *Assume the setting of Theorem 2.7 with $\lambda \in (0,1)$ and denote*

$$\psi_\lambda^D(x) := \exp\Big\{ 2\lambda^2 \int_{\partial D} \Pi^D(x, \mathrm{d}z) \log|x - z| \Big\}. \tag{2.50}$$

Then there is $\hat{c} \in (0, \infty)$ such that for all $D \in \mathfrak{D}$,

$$Z_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} \hat{c}\psi_\lambda^D(x)\, \mu_\infty^{D,\lambda\alpha}(\mathrm{d}x). \tag{2.51}$$

Note that ψ_λ^D is just the $2\lambda^2$-th power of the conformal radius $r_D(x)$. The measure on the right of (2.51) (without the constant c) is called the *Liouville Quantum Gravity* (LQG) measure in D for parameter $\beta := \lambda\alpha$. This object is currently being heavily studied in connection with random conformally-invariant geometry.

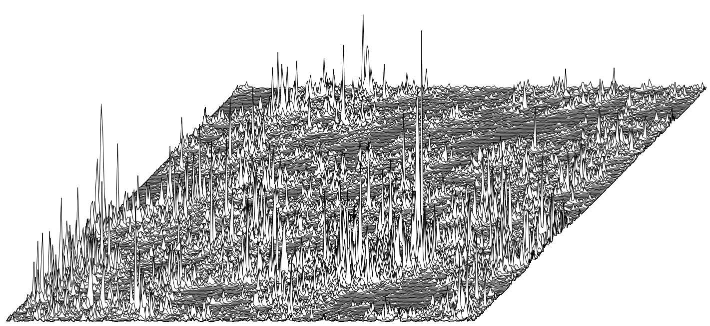

Figure 2.4: A sample of the LQG measure $\psi_\lambda^D(x)\mu_\infty^{D,\lambda\alpha}(\mathrm{d}x)$ for D a unit square and $\lambda = 0.3$. The high points indicate places of high local intensity.

We will not discuss the LQG measures much in these lectures, although (as we have seen above) they will keep popping up in our various theorems. The proofs of the above results will be given in the forthcoming two lectures.