Lecture 12

Random walk in DGFF landscape

In this lecture we turn our attention to a different problem than discussed so far: a random walk in a random environment. The connection with the main theme of these lectures is through the specific choice of the random walk dynamics which we interpret as motion of a random particle in an electric field which is itself a sample of the DGFF. We first state the results, obtained in a recent joint work with Jian Ding and Subhajit Goswami, on the behavior of this random walk. Then we proceed to develop the key method of the proof, which is based on control of effective resistivity in the associated random conductance model.

12.1. Statement of main results

We wish to consider a random walk that somehow describes motion of a charged particle in a rapidly varying electrical field. We will fit this into the framework of the theory of random walks in random environment (RWRE) as follows: The walk will be confined to the hypercubic lattice $\mathbb{Z}^d$. The electrical field will be given as a configuration $\{h_x\colon x\in\mathbb{Z}^d\}$ with h_x denoting the *electrostatic potential* at x. Fixing a parameter $\beta>0$ corresponding to the *inverse temperature*, the walk is then a discrete-time Markov chain with state space $\mathbb{Z}^d$ and transition probabilities

$$\mathsf{P}_h(x,y) := \frac{\mathrm{e}^{\beta(h_y-h_x)}}{\displaystyle\sum_{z\colon (x,z)\in E(\mathbb{Z}^d)} \mathrm{e}^{\beta(h_z-h_x)}}\, 1_{(x,y)\in E(\mathbb{Z}^d)} \tag{12.1}$$

The walk thus tends to move in the direction of increasing electrostatic potential with β modulating the overall strength of this effect. It is not surprise that, at least for fields with pronounced local maxima, this can lead to trapping.

The above Markov chain is defined for any sample of h. However, as usual in RWRE theory, we will require that

$$\text{law of } \{\mathsf{P}_h(x,\cdot)\colon x\in\mathbb{Z}^d\} \text{ is stationary under shifts of } \mathbb{Z}^d. \tag{12.2}$$

As $\mathsf{P}_h(x,\cdot)$ depends only on the differences of the field, it suffices to impose

$$\text{law of } \{h_x - h_y \colon (x,y) \in E(\mathbb{Z}^d)\} \text{ is stationary under shifts of } \mathbb{Z}^d. \tag{12.3}$$

A number of natural examples can be considered, with i.i.d. random fields or, in fact, any stationary random field $\{h_x \colon x \in \mathbb{Z}^d\}$ obviously satisfying (12.3). However, our desire is to work with fields that exhibit logarithmic correlations of which the two-dimensional DGFF is an example.

Our motivation for focussing on log-correlated fields comes from the 2004 papers of Carpentier and Le Doussal and Castillo and Le Doussal, who discovered, on the basis of physics arguments, that such environments exhibit the following effects:

(1) trapping makes the walk behave subdiffusively with the diffusive exponent ν, defined via $|X_n| = n^{\nu+o(1)}$, depending non-trivially on β, and

(2) $\beta \mapsto \nu(\beta)$ undergoes a phase transition (i.e., a change in analytic dependence) as β varies through a critical point β_c.

The log-correlated class is in fact deemed critical for the above phase transition to occur. The purpose of these lectures it to demonstrate that (1-2) indeed happen in at least one example in spatial dimension $d = 2$.

Since the DGFF on $\mathbb{Z}^2$ does not exist, we will henceforth take

$$h := \text{ DGFF in } \mathbb{Z}^2 \smallsetminus \{0\}. \tag{12.4}$$

This does fall into the class of systems introduced above; indeed, we have:

Exercise 12.1 *Show that h obeys* (12.3).

Let $X = \{X_n \colon n \geq 0\}$ denote a sample path of the Markov chain. We will write P_h^x for the law of X with $P_h^x(X_0 = x) = 1$, use E_h^x to denote expectation with respect to P_h^x and write $\mathbb{P}$ to denote the law of the DGFF on $\mathbb{Z}^2 \smallsetminus \{0\}$. Our first result about X, proved in a joint 2016 paper with J. Ding and S. Goswami, is then:

Theorem 12.2 [Heat-kernel decay] *For each $\beta > 0$ and each $\delta > 0$,*

$$\mathbb{P}\left(\frac{1}{T}\mathrm{e}^{-(\log T)^{1/2+\delta}} \leq P_h^0(X_{2T} = 0) \leq \frac{1}{T}\mathrm{e}^{(\log T)^{1/2+\delta}}\right) \underset{T\to\infty}{\longrightarrow} 1. \tag{12.5}$$

Note that there is not dependence of the statement on β. (Indeed, it applies even to $\beta = 0$ when X is just the simple symmetric random walk on $\mathbb{Z}^2$.) Hence, as far as the leading order of the return probabilities (a.k.a. *heat kernel*) is concerned, the walk behaves just as the simple random walk. Note, however, that the $\mathrm{e}^{\pm(\log T)^{1/2+\delta}}$ terms are unfortunately too large to determine whether X is recurrent or transient.

Although the propensity of the walk to move towards larger values of the field does not seem to affect the (leading order) heat kernel decay, the effect on the path properties is quite detectable. For each set $A \subset \mathbb{Z}^2$, define

$$\tau_A := \inf\{n \geq 0 \colon X_n \in A\}. \tag{12.6}$$

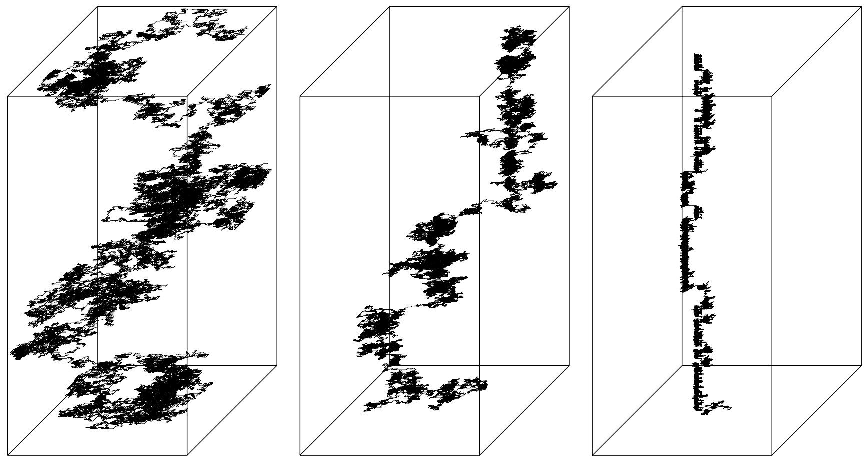

Figure 12.1: Runs of 100000 steps of the random walk with transition probabilities (12.1) and β equal to 0.2, 0.6 and 1.2 multiples of β_c. Time runs upwards along the vertical axis. Trapping effects are quite apparent.

Denote

$$B(N) := [-N, N]^2 \cap \mathbb{Z}^2. \tag{12.7}$$

Then we have:

Theorem 12.3 [Subdiffusive expected exit time] *For each $\beta > 0$ and each $\delta > 0$,*

$$\mathbb{P}\left(N^{\psi(\beta)} e^{-(\log N)^{1/2+\delta}} \le E_h^0(\tau_{B(N)^c}) \le N^{\psi(\beta)} e^{(\log N)^{1/2+\delta}} \right) \underset{N\to\infty}{\longrightarrow} 1, \tag{12.8}$$

where, for $\beta_c := \sqrt{\pi/2}$,

$$\psi(\beta) := \begin{cases} 2 + 2(\beta/\beta_c)^2, & \text{if } \beta \le \beta_c, \\ 4\beta/\beta_c, & \text{if } \beta \ge \beta_c. \end{cases} \tag{12.9}$$

Note that the functional of $\psi(\beta)$ dependence on β takes quite a familiar form: For $\lambda := \beta/\beta_c$ we have $\psi(\beta) = 2 + 2\lambda^2$, which is the scaling exponent associated with the intermediate level set at height λ-multiple of the absolute maximum. Also note that, for $\beta > 0$, we have $\psi(\beta) > 0$. The walk thus takes considerably longer (in expectation) to exit a box than the simple random walk. This can be interpreted as a version of *subdiffusive behavior*.

A standard definition of subdiffusive behavior is via the spatial spread of the walk at large times. Here we can report only a one-way bound:

Corollary 12.4 [Subdiffusive lower bound] *For each $\beta > 0$ and each $\delta > 0$,*

$$P_h^0\left(|X_T| \ge T^{1/\psi(\beta)} e^{-(\log N)^{1/2+\delta}}\right) \underset{N\to\infty}{\longrightarrow} 1, \quad \text{in } \mathbb{P}\text{-probability}. \tag{12.10}$$

Unfortunately, the more relevant upper bound is elusive at this point although we believe that our methods can be boosted to include a matching (leading-order) upper bound as well.

The method of proof is based on the following simple rewrite of the transition probabilities from (12.1):

$$\mathsf{P}_h(x,y) := \frac{\mathrm{e}^{\beta(h_y+h_x)}}{\pi_h(x)}\,1_{(x,y)\in E(\mathbb{Z}^d)}\,, \tag{12.11}$$

where

$$\pi_h(x) := \sum_{z\colon (x,z)\in E(\mathbb{Z}^d)} \mathrm{e}^{\beta(h_z+h_x)}. \tag{12.12}$$

This, as we will explain in the next section, phrases the problem as a *random walk among random conductances*, with the conductance of edge $(x,y)\in E(\mathbb{Z}^d)$ given by

$$c(x,y) := \mathrm{e}^{\beta(h_y+h_x)}. \tag{12.13}$$

As is readily checked, X is a reversible Markov chain with stationary measure π_h. In $d\ge 3$ we can take h to be the DGFF in all of $\mathbb{Z}^d$ in which case the conductances are stationary. However, for $d=2$ and $h :=$ DGFF on $\mathbb{Z}^2\smallsetminus\{0\}$, making the Markov chain reversible carries the price of losing stationarity. (Recall that a similar situation occurs for Sinai's RWRE on $\mathbb{Z}$.) The benefit of stationarity is that makes the Markov chain amenable to analysis via methods of *electrostatic theory*.

Interpreting the underlying graph as an electric network with edge (x,y) having resistance $r(x,y) := 1/c(x,y)$, the key notion to consider is then the effective resistance $R_{\text{eff}}(0,B(N)^{\text{c}})$ from 0 to $B(N)^{\text{c}}$. We will define this quantity precisely in the next section; let us just say that this is the value of voltage difference one needs to put between 0 and $B(N)^{\text{c}}$ to have unit current pass through the network. For the effective resistance we then get:

Theorem 12.5 [Effective resistance growth] *For each $\beta>0$,*

$$\limsup_{N\to\infty}\,\frac{\log R_{\text{eff}}(0,B(N)^{\text{c}})}{(\log N)^{1/2}(\log\log N)^{1/2}}<\infty,\qquad \mathbb{P}\text{-a.s.} \tag{12.14}$$

and, for each $\delta>0$, also

$$\liminf_{N\to\infty}\,\frac{\log R_{\text{eff}}(0,B(N)^{\text{c}})}{(\log N)^{1/2}/(\log\log N)^{1+\delta}}>0,\qquad \mathbb{P}\text{-a.s.} \tag{12.15}$$

We can write both conclusions into one expression as

$$R_{\text{eff}}(0,B(N)^{\text{c}}) = \mathrm{e}^{(\log N)^{1/2+o(1)}},\quad N\to\infty. \tag{12.16}$$

In particular, $R_{\text{eff}}(0,B(N)^{\text{c}})\to\infty$ as $N\to\infty$ and so from the standard criteria of recurrence and transience of Markov chains (to be discussed in the next section as well) we get:

Corollary 12.6 *For $\mathbb{P}$-a.e. realization of h, the Markov chain X is recurrent.*

12.2. A crash course on electrostatic theory

Consider an unoriented finite, connected graph $\mathfrak{G} = (\mathsf{V}, \mathsf{E})$. An assignment of *resistance* $r_e \in (0, \infty)$ to each $e \in \mathsf{E}$ then makes $\mathfrak{G}$ an electric network. An alternative description uses *conductances* $\{c_e : e \in \mathsf{E}\}$ where

$$c_e := \frac{1}{r_e}. \tag{12.17}$$

We will exchangeably write $r(x,y)$ for r_e when $e = (x,y)$, and similarly for $c(x,y)$. Note that these are symmetric quantities, $r(x,y) = r(y,x)$ and $c(x,y) = c(y,x)$ whenever $(x,y) \in \mathsf{E}$.

Next we define some key notions of the theory. For any two distinct $u, v \in \mathsf{V}$, let

$$\mathcal{F}(u,v) := \bigl\{ f \text{ function } \mathsf{V} \to \mathbb{R} \colon \, f(u) = 1,\, f(v) = 0 \bigr\}. \tag{12.18}$$

We interpret such f as an assignment of a *voltage* to vertices of V; $f \in \mathcal{F}(u,v)$ then has unit voltage difference between u and v. For any $f \colon \mathsf{V} \to \mathbb{R}$ define its Dirichlet energy by

$$\mathcal{E}(f) := \sum_{(x,y) \in \mathsf{E}} c(x,y) \bigl[f(y) - f(x) \bigr]^2, \tag{12.19}$$

where each edge is counted only once.

Definition 12.7 [Effective conductance] *The infimum*

$$C_{\text{eff}}(u,v) := \inf\bigl\{ \mathcal{E}(f) \colon f \in \mathcal{F}(u,v) \bigr\}. \tag{12.20}$$

is the effective conductance from u to v.

Note that $C_{\text{eff}}(u,v) > 0$ since $\mathfrak{G}$ is assumed connected and conductances are assumed to be strictly positive.

Next we define the notion of (electric) current as follows:

Definition 12.8 [Current] *Let $\vec{\mathsf{E}}$ denote the set of oriented edges in $\mathfrak{G}$, with both orientations present. A* current from u to v *is an assignment i_e of a real number to each $e \in \vec{\mathsf{E}}$ such that, writing $i(x,y)$ for i_e with $e = (x,y)$,*

$$i(x,y) = -i(y,x), \quad (x,y) \in \vec{\mathsf{E}} \tag{12.21}$$

and

$$\sum_{y \colon (x,y) \in \vec{\mathsf{E}}} i(x,y) = 0, \quad x \in \mathsf{V} \smallsetminus \{u,v\}. \tag{12.22}$$

The first condition expresses the natural condition that current flowing along (x,y) is the opposite of the current flowing along (y,x). The second condition then forces that the current is conserved at all points but u and v. Next we observe:

Lemma 12.9 [Value of current] *For each current i from u to v,*

$$\sum_{y \colon (u,x) \in \vec{\mathsf{E}}} i(u,x) = \sum_{y \colon (x,v) \in \vec{\mathsf{E}}} i(x,v) \tag{12.23}$$

Proof. Conditions (12.21–12.22) imply

$$0 = \sum_{(x,y)\in\vec{\mathsf{E}}} i(x,y) = \sum_{x\in\mathsf{V}} \sum_{y\colon (x,y)\in\vec{\mathsf{E}}} i(x,y) = \sum_{y\colon (u,y)\in\vec{\mathsf{E}}} i(u,y) + \sum_{y\colon (v,y)\in\vec{\mathsf{E}}} i(v,y). \quad (12.24)$$

Employing (12.21), we then get (12.23). □

A natural interpretation of (12.23) is that the current incoming to the network at u equals the outgoing current at v. (Note that this may be false in infinite networks.) We call the common value in (12.23) the *value of current* i, with the notation $\operatorname{val}(i)$. It is natural to single out the currents with unit value into

$$\mathcal{I}(u,v) := \bigl\{i\colon \text{ current from } u \text{ to } v \text{ with } \operatorname{val}(i) = 1\bigr\}. \quad (12.25)$$

For each current i, its Dirichlet energy is then given by

$$\widetilde{\mathcal{E}}(i) := \sum_{e\in E} r_e (i_e)^2, \quad (12.26)$$

where, again, each edge enters only once into the sum.

Definition 12.10 [Effective resistance] *The infimum*

$$R_{\text{eff}}(u,v) := \inf\bigl\{\widetilde{\mathcal{E}}(i)\colon i \in \mathcal{I}(u,v)\bigr\} \quad (12.27)$$

is the effective resistance from u *to* v*.*

Note that $R_{\text{eff}}(u,v) < \infty$ since the resistances are strictly positive and $\mathcal{I}(u,v) \neq \emptyset$ due to the assumed connectivity of $\mathfrak{G}$.

It is quite clear that the effective resistance and effective conductance must be closely related. For instance, by (12.17) they are clearly reciprocals of each other in the network with two vertices and one edge. We observe:

Lemma 12.11 *For any two distinct* $u, v \in \mathsf{V}$,

$$\mathcal{E}(f)\widetilde{\mathcal{E}}(i) \ge 1, \quad f \in \mathcal{F}(u,v),\, i \in \mathcal{I}(u,v). \quad (12.28)$$

In particular, $R_{\text{eff}}(u,v)C_{\text{eff}}(u,v) \ge 1$*.*

Proof. Let $f \in \mathcal{F}(u,v)$ and $i \in \mathcal{I}(u,v)$. By a symmetrization argument and the definition of unit current,

$$\begin{aligned}\sum_{(x,y)\in\mathsf{E}} i(x,y)\bigl[f(x) - f(y)\bigr] &= \frac{1}{2}\sum_{x\in\mathsf{V}} \sum_{y\colon (x,y)\in\mathsf{E}} i(x,y)\bigl[f(x) - f(y)\bigr] \\ &= \sum_{x\in\mathsf{V}} f(x) \sum_{y\colon (x,y)\in\mathsf{E}} i(x,y) = f(u) - f(v) = 1. \quad (12.29)\end{aligned}$$

On the other hand, (12.17) and the Cauchy-Schwarz inequality yield

$$\begin{aligned}&\sum_{(x,y)\in\mathsf{E}} i(x,y)\bigl[f(x) - f(y)\bigr] \\ &\quad = \sum_{(x,y)\in\mathsf{E}} \sqrt{r(x,y)}\, i(x,y)\, \sqrt{c(x,y)}\, \bigl[f(x) - f(y)\bigr] \\ &\quad \le \widetilde{\mathcal{E}}(i)^{1/2}\mathcal{E}(f)^{1/2}. \quad (12.30)\end{aligned}$$

This gives (12.28). The second part follows by optimizing over f and i. □

We now claim:

Theorem 12.12 [Electrostatic duality] *For any distinct $u, v \in \mathsf{V}$,*

$$C_{\text{eff}}(u,v) = \frac{1}{R_{\text{eff}}(u,v)} \tag{12.31}$$

Proof. Since $\mathcal{I}(u,v)$ can be identified with a closed convex subset of $\mathbb{R}^{\mathsf{E}}$ and $i \mapsto \widetilde{\mathcal{E}}(i)$ with a strictly convex function on $\mathbb{R}^{\mathsf{E}}$ with compact level sets, there is a unique minimizer i of (12.27). We claim that i obeys the *Kirchhoff cycle law*: for each $n \geq 1$ and each $x_0, x_1, \dots, x_n = x_0 \in \mathsf{V}$

$$\sum_{k=1}^{n} r(x_k, x_{k+1}) i(x_k, x_{k+1}) = 0. \tag{12.32}$$

This is seen by considering the current j defined by $j(x_k, x_{k+1}) = -j(x_{k+1}, x_k) = 1$ for $k = 1, \dots, n$ and $j(x,y) = 0$ on all edges not belonging to the cycle $(x_0, \dots, x_n)$. Then $i + aj \in \mathcal{I}(u,v)$ for any $a \in \mathbb{R}$ and so, since i is the minimizer,

$$\widetilde{\mathcal{E}}(i + aj) = \widetilde{\mathcal{E}}(i) + a \sum_{k=1}^{n} r(x_k, x_{k+1}) i(x_k, x_{k+1}) + a^2 \widetilde{\mathcal{E}}(j) \geq \widetilde{\mathcal{E}}(i). \tag{12.33}$$

Taking $a \downarrow 0$ then shows "$\geq$" in (12.32) and taking $a \uparrow 0$ then proves equality.

The fact that $e \mapsto r_e i(e)$ obeys the cycle law implies that it is a gradient of a function. Specifically, we claim that there is $f \colon \mathsf{V} \to \mathbb{R}$ such that $f(v) = 0$ and

$$f(y) - f(x) = r(x,y) i(x,y), \quad (x,y) \in \mathsf{E}. \tag{12.34}$$

To see this, consider any path $x_0 = v, x_1, \dots, x_n = x$ with $(x_k, x_{k+1}) \in \mathsf{E}$ and let $f(x)$ be the sum of $r_e i(e)$ for edges along this path. The cycle condition then ensures that the value of $f(x)$ is independent of the path chosen. Hence we get also (12.34). A key point is to determine the value $f(u)$. Here we note that (12.34) equals $\widetilde{\mathcal{E}}(i)$ with the quantity on the left of (12.29) and so

$$\widetilde{\mathcal{E}}(i) = f(u) - f(v) = f(u). \tag{12.35}$$

The function $\tilde{f}(x) := f(x)/R_{\text{eff}}(u,v)$ thus belongs to $\mathcal{F}(u,v)$ and since, as is directly checked, $\mathcal{E}(f) = \widetilde{\mathcal{E}}(i) = R_{\text{eff}}(u,v)$, we get

$$C_{\text{eff}}(u,v) \leq \mathcal{E}(\tilde{f}) = \frac{1}{R_{\text{eff}}(u,v)^2} \mathcal{E}(f) = \frac{1}{R_{\text{eff}}(u,v)}. \tag{12.36}$$

This gives $C_{\text{eff}}(u,v) R_{\text{eff}}(u,v) \leq 1$, complementing the inequality from Lemma 12.11. The claim follows. □

There is a natural extension of the effective resistance/conductance to arguments which are themselves sets. For any pair of disjoint sets $A, B \subset \mathsf{V}$, we thus define $R_{\text{eff}}(A,B)$ to be the effective resistance $R_{\text{eff}}(\langle A \rangle, \langle B \rangle)$ in the network where all edges between the vertices in A as well as between the vertices in B, have been dropped and the vertices in A then merged into a vertex $\langle A \rangle$ and those in B merged into $\langle B \rangle$. In engineering language, this amounts to *shorting* those vertices.

12.3. Markov chain connections & network reduction

Every electric network is naturally associated with a Markov chain on V whose transition probabilities are given by

$$\mathsf{P}(x,y) := \frac{c(x,y)}{\pi(x)} 1_{(x,y)\in\mathsf{E}} \quad \text{where} \quad \pi(x) := \sum_{y\colon (x,y)\in\mathsf{E}} c(x,y). \tag{12.37}$$

The symmetry $c(x,y) = c(y,x)$ the translates into the condition

$$\pi(x)\mathsf{P}(x,y) = \pi(y)\mathsf{P}(y,x) \tag{12.38}$$

thus making π the reversible measure. Writing P^x for the law of the Markov chain started at x, we then have:

Proposition 12.13 [Connection to Markov chain] *The variational problem* (12.20) *has a unique minimizer f which is given by*

$$f(x) = P^x(\tau_u < \tau_v) \tag{12.39}$$

where $\tau_z := \inf\{n \geq 0\colon X_n = z\}$.

Proof. Define an operator $\mathcal{L}$ on $\ell^2(\mathsf{V})$ by

$$\mathcal{L}f(x) := \sum_{y\colon (x,y)\in\mathsf{E}} c(x,y)\big[f(y) - f(x)\big]. \tag{12.40}$$

(This is an analogue of the discrete Laplacian we encountered earlier in these lectures.) As is easy to check, the minimizer of (12.20) obeys $\mathcal{L}f(x) = 0$ for all $x \neq u,v$, with boundary values $f(y) = 1$ and $f(v) = 0$. As $x \mapsto P^x(\tau_u < \tau_v)$ obeys exactly the same set of conditions, the claim follows by noting that the solution to the (Dirichlet) problem is unique by, say, the Maximum Principle. □

Corollary 12.14 *Letting* $\hat{\tau}_x := \inf\{n \geq 1\colon X_n = x\}$,

$$\frac{1}{R_{\text{eff}}(u,v)} = \pi(u)P^u(\hat{\tau}_u > \tau_v). \tag{12.41}$$

Proof. Let f be the minimizer of (12.20). In light of $\mathcal{L}f(x) = 0$ for all $x \neq u,v$, symmetrization arguments and $f \in \mathcal{F}(u,v)$ show

$$\mathcal{E}(f) = -\sum_{x\in\mathsf{V}} f(x)\mathcal{L}f(x) = -f(u)\mathcal{L}f(u) - f(v)\mathcal{L}f(v) = -\mathcal{L}f(u). \tag{12.42}$$

Now (12.39) implies

$$\begin{aligned} -\mathcal{L}f(u) &= \pi(u) - \sum_{x\colon (u,x)\in\mathsf{E}} c(u,x)P^x(\tau_x < \tau_v) \\ &= \pi(u)\Big[1 - \sum_{x\colon (u,x)\in\mathsf{E}} \mathsf{P}(u,x)P^x(\tau_x < \tau_v)\Big] = \pi(u)P^u(\hat{\tau}_u > \tau_v)\,, \end{aligned} \tag{12.43}$$

where we used the Markov property and the fact that $u \neq v$ implies $\hat{\tau}_u \neq \tau_v$. The claim now follows from the Electrostatic Duality. $\square$

The above provides an electrostatic criterion for recurrence/transience of a Markov chains X on an infinite connected electric network with $r_e \in (0,\infty)$ for each edge in the underlying graph. Let $B(x,r)$ denote the ball in the graph-theoretical metric of radius r centered at x. Note that:

Exercise 12.15 *Denote by $C_{\text{eff}}(x, B(x,r)^{\text{c}})$, resp., $R_{\text{eff}}(x, B(x,r)^{\text{c}})$ the effective conductance, resp., resistance in the network where $B(x,r)^{\text{c}}$ has been collapsed into a single vertex. Prove, by employing a shorting argument, that $r \mapsto C_{\text{eff}}(x, B(x,r)^{\text{c}})$ is non-increasing.*

This and the Electrostatic Duality permit us to define

$$R_{\text{eff}}(x,\infty) := \lim_{r\to\infty} R_{\text{eff}}\big(x, B(x,r)^{\text{c}}\big) \tag{12.44}$$

We then have:

Corollary 12.16 [Characterization of recurrence/transience]

$$R_{\text{eff}}(x,\infty) = \infty \quad \Leftrightarrow \quad X \text{ is recurrent.} \tag{12.45}$$

Proof. Corollary 12.14 shows that $P^x(\hat{\tau}_x > \tau_{B(x,r)^{\text{c}}})$ is proportional to $C_{\text{eff}}(u, B(x,r)^{\text{c}})$. Since $\tau_{B(x,r)^{\text{c}}} \geq r$, $P^x(\hat{\tau}_x = \infty)$ is proportional to $R_{\text{eff}}(x,\infty)^{-1}$. $\square$

The advantage of translating properties of Markov chains into electric network language is that we can now manipulate networks using operations that do not always have a natural counterpart, or type of monotonicity, in the context of Markov chains. We will refer to these using the term *network reduction*. An example of such reduction is the subject of:

Exercise 12.17 [Network reduction] *Let $V' \subset V$ and, for $f\colon V' \to \mathbb{R}$, define*

$$\mathcal{E}'(f) := \inf\big\{\mathcal{E}(g)\colon g(x) = f(x)\ \forall x \in V'\big\}. \tag{12.46}$$

Prove that $\mathcal{E}'(f)$ is still a Dirichlet energy of the form

$$\mathcal{E}'(f) = \frac{1}{2}\sum_{x,y\in V'} c'(x,y)\big[f(y) - f(x)\big]^2 \tag{12.47}$$

where

$$c'(x,y) = \pi(x)P^x\big(X_{\hat{\tau}_{V'}} = y\big) \tag{12.48}$$

with $\hat{\tau}_A := \inf\{n \geq 1\colon X_n \in A\}$.

A simple example is the situation when V' has only two vertices. Indeed, for any $u \neq v$, we have

$$V' = \{u,v\} \quad \Rightarrow \quad c'(u,v) = C_{\text{eff}}(u,v). \tag{12.49}$$

Other examples are the subjects of:

Exercise 12.18 [Series law] *Suppose $\mathfrak{G}$ contains a string of vertices $x_1, \dots, x_{n+1}$ such that $(x_i, x_{i+1}) \in \mathsf{E}$ for each $i = 1, \dots, n$ and such that, for $i = 2, \dots, n$, the vertex x_i has no other neighbors than x_{i-1} and x_{i+1}. Prove that in the reduced network with $V' := V \smallsetminus \{x_2, \dots, x_n\}$ the string is replaced by an edge (x_0, x_n) with resistance*

$$r'(x_0, x_n) := \sum_{i=1}^{n} r(x_i, x_{i+1}) \tag{12.50}$$

Exercise 12.19 [Parallel law] *Suppose $\mathfrak{G}$ contains n edges $e_1, \dots, e_n$ between vertices x and y of conductances $c(e_i)$, respectively. Prove that we can replace these by a single edge e with conductance*

$$c'(e) := \sum_{i=1}^{n} c(e_i). \tag{12.51}$$

We note that the effective resistances/conductances between different pairs of vertices do not convey direct information concerning the network itself. Notwithstanding, in some situation the identification is possible:

Exercise 12.20 [Star-triangle transformation] *Suppose $\mathfrak{G}$ is a network on three nodes $\{1, 2, 3\}$, and for each i, j let c_{ij} denote the conductance of the edge (i, j). Let R_{ij} denote the effective resistance between node i and node j. Then,*

$$\frac{c_{12}}{c_{12} + c_{13}} = \frac{R_{13} + R_{23} - R_{12}}{2R_{23}}. \tag{12.52}$$

[Hint: Prove equivalence of the "triangle" network to the that looking like a "star" which has an additional vertex $\{0\}$ and only edges $(0, i)$, $i = 1, 2, 3$.]

12.4. Path-cut representations

The network reduction ideas naturally lead to estimates on effective conductance or resistance using geometric objects such as paths and cuts. Here a path P from u to v is a sequence of edges $e_1, \dots, e_n$, which we think of as oriented for this purpose, with e_1 having initial point u and e_n terminal endpoint v, and the initial point of e_{i+1} equal to terminal point of e_i for each $i = 1, \dots, n-1$. We will often identify P with the set of these edges, making $e \in P$ meaningful.

Lemma 12.21 *Suppose $\mathcal{P}$ is a finite set of edge disjoint paths (i.e., $\forall P, P' \in \mathcal{P}$ with $P \neq P'$ implying $P \cap P' = \emptyset$) from u to v. Then*

$$R_{\text{eff}}(u, v) \le \Big[\sum_{P \in \mathcal{P}} \frac{1}{\sum_{e \in P} r_e}\Big]^{-1} \tag{12.53}$$

Proof. The idea is to route a suitable amount of current along each path to define a unit current from u to v. Let $\overline{R}$ denote the quantity on the right of (12.53) and, for each $P \in \mathcal{P}$, denote $i_P := \overline{R} / \sum_{e \in P} r_e$. Note that then $\sum_P i_P = 1$ so letting, for each $e \in \mathsf{E}$,

$$i(e) := \sum_{P \in \mathcal{P} \colon e \in P} i_P \tag{12.54}$$

defines $i \in \mathcal{I}(u,v)$. A calculation shows $\widetilde{\mathcal{E}}(i) = \overline{R}$ and so $R_{\text{eff}}(u,v) \le \overline{R}$. □

A natural question to ask is whether the above bound can possibly be sharp. It turns out that what stand in the way of this is the edge-disjointness requirement. This is overcome in:

Proposition 12.22 [Path representation of effective resistance] *Let $\mathfrak{P}_{u,v}$ denote the set of all multisets edges whose graph union contains a path from u to v. Then*

$$R_{\text{eff}}(u,v) = \inf_{\mathcal{P} \in \mathfrak{P}_{u,v}} \inf_{\{r_{e,P} \colon e \in \mathsf{E}, P \in \mathcal{P}\} \in \mathfrak{R}_{\mathcal{P}}} \Big[\sum_{P \in \mathcal{P}} \frac{1}{\sum_{e \in P} r_{e,P}} \Big]^{-1}, \tag{12.55}$$

where $\mathfrak{R}_{\mathcal{P}}$ is the set of all assignments $\{r_{e,P} : e \in \mathsf{E}, P \in \mathcal{P}\} \in \mathbb{R}_+^{\mathsf{E} \times \mathcal{P}}$ such that

$$\sum_{P \in \mathcal{P}} \frac{1}{r_{e,P}} \le \frac{1}{r_e}, \quad e \in E(\mathcal{G}). \tag{12.56}$$

The infima in (12.56) *are (jointly) achieved.*

Proof. Pick $\mathcal{P}$ and $\{r_{e,P} \colon P \in \mathcal{P}\}$ satisfying (12.56). Now split each edge e into a collection of edges $\{e_P \colon P \in \mathcal{P}\}$ and assign resistance $r_{e,P}$ to e_P. If strict inequality holds in (12.56), introduce a dummy copy $\tilde{e}$ of e and assign conductance $c_{\tilde{e}} := 1/r_e - \sum_{P \in \mathcal{P}} 1/r_{e,P}$ to $\tilde{e}$. The Parallel Law shows that this is an equivalent network but now with $\mathcal{P}$ being (naturally interpreted as) mutually edge disjoint. Lemma 12.21 shows that "$\le$" holds in (12.55).

To get equality in (12.55), let $i_\star \in \mathcal{I}(u,v)$ be such that $\widetilde{\mathcal{E}}(i_\star) = R_{\text{eff}}(u,v)$. We will now run an algorithm that identifies currents i_k from u and v (not necessarily of unit value) obeying the Kirchhoff cycle law and paths P_k. First solve:

Exercise 12.23 *Suppose $e \mapsto i(e)$ is a current from u to v obeying the Kirchhoff cycle law* (12.32) *and $\text{val}(i) > 0$. Show that there is a path P from u to v such that $i(e) > 0$ for each $e \in P$.*

The algorithm is then defined as follows. INITIATE by $i_0 := i_\star$. If $i_{k-1} = 0$ then STOP, else use Exercise 12.23 to find a path P_k from u to v where $i_k(e) > 0$ for each $e \in P_k$. Then set $\alpha_k := \min_{e \in P_k} i_{k-1}(e)$, let

$$i_k(e) := i_{k-1}(e) - \alpha_k 1_{\{e \in P_k\}} \tag{12.57}$$

and, noting that i_k obeys the Kirchhoff cycle law, REPEAT.

As $\{e \in E \colon i_k(e) \neq 0\}$ is strictly decreasing in k, the algorithm will terminate after a finite number of steps. This means

$$\sum_{k \colon e \in P_k} \alpha_k = i_\star(e) \quad \text{and so} \quad \sum_k \alpha_k = \text{val}(i_\star) = 1. \tag{12.58}$$

Set $r_{e,P_k} := i_\star(e) r_e / \alpha_k$. Hence we get

$$R_{\text{eff}}(u,v) = \sum_{e \in \mathsf{E}} r_e i_\star(e)^2 = \sum_{e \in \mathsf{E}} \sum_{k \colon e \in P_k} r_{e,P_k} \alpha_k^2 = \sum_k \alpha_k^2 \sum_{e \in P_k} r_{e,P_k}. \tag{12.59}$$

Minimizing the right-hand side subject to the constraint $\sum_k \alpha_k = 1$ shows that $R_{\text{eff}}(u,v)$ is no less than the quantity on the right of (12.55) (without the infima). As (12.58) shows $\sum_k 1/r_{e,P_k} = 1/r_e$ for each $e \in \mathsf{E}$, the claim follows. $\square$

A similar statement applies to the effective conductance. Here one needs a notion of a *cut*, or *cut-set* form u to v which is a set of edges in E such that every path from u to v must us an edge in this set. We then have:

Lemma 12.24 [Nash-Williams criterion] *For any collection Π of edge-disjoint cut-sets from u to v,*

$$C_{\text{eff}}(u,v) \le \Big[\sum_{\pi\in\Pi} \frac{1}{\sum_{e\in\pi} c_e}\Big]^{-1} \tag{12.60}$$

Proof. Let $i \in \mathcal{I}(u,v)$. The proof will be based on:

Exercise 12.25 *For any cut-set π from u to v, $\sum_{e\in\pi} i(e) = 1$.*

Indeed, once this is settled, the Cauchy-Schwarz inequality tells us

$$1 = \Big[\sum_{e\in\pi} i(e)\Big]^2 \le \Big[\sum_{e\in\pi} r_e i(e)^2\Big]\Big[\sum_{e\in\pi} c_e\Big]. \tag{12.61}$$

The assumed edge-disjointness of the cut-sets in Π then yields

$$\mathcal{E}(i) \ge \sum_{\pi\in\Pi}\sum_{e\in\pi} r_e i(e)^2 \ge \sum_{\pi\in\Pi} \frac{1}{\sum_{e\in\pi} c_e}. \tag{12.62}$$

As this holds for all $i \in \mathcal{I}(u,v)$, the claim follows by the Electrostatic Duality. $\square$

The above lemma is easy to prove when the cutsets are *nested* meaning that they can be ordered in a sequence $\pi_1,\dots,\pi_n$ such that π_i separates π_{i-1} (as well as u) from π_{i+1} (as well as v). However, as the above proof shows, this geometric restriction is not needed (and, in fact, would be inconvenient to carry around).

We call (12.60) "the Nash-Williams criterion" because it is useful in proving recurrence of an infinite network. Indeed, to prove recurrence it suffices to construct a disjoint family of cut-sets whose total resistances add up to infinity. As far as computation of $C_{\text{eff}}(u,v)$ is concerned, (12.60) is generally not sharp, but that so predominantly due to the requirement of edge-disjointness. Indeed, we have the following analogue of Proposition 12.22:

Proposition 12.26 [Cut-set representation of effective conductance] *Let $\mathfrak{S}_{u,v}$ denote the set of all finite collections of cutsets between u and v. Then*

$$C_{\text{eff}}(u,v) = \inf_{\Pi\in\mathfrak{S}_{u,v}} \inf_{\{c_{e,\pi}\colon e\in\mathsf{E},\pi\in\Pi\}\in\mathfrak{C}_\Pi} \Big[\sum_{\pi\in\Pi} \frac{1}{\sum_{e\in\pi} c_{e,\pi}}\Big]^{-1}, \tag{12.63}$$

where $\mathfrak{C}_\Pi$ is the set of all assignments $\{c_{e,\pi}\colon e\in\mathsf{E},\pi\in\Pi\} \in \mathbb{R}_+^{\mathsf{E}\times\Pi}$ such that

$$\sum_{\pi\in\Pi} \frac{1}{c_{e,\pi}} \le \frac{1}{c_e}, \quad e\in\mathsf{E}. \tag{12.64}$$

The infima in (12.63) *are (jointly) achieved.*