Lecture 11

Local structure of extremal points

In this lecture we augment the prior description of the point processes associated with extremal local maxima by adding information about the local behavior of the field in a neighborhood thereof. A number of interesting corollaries are stated at the end concerning the cluster-process structure of the extremal level sets, the Poisson-Dirichlet limit of the Gibbs measure associated with the DGFF, the Liouville Quantum Gravity in the so called glassy phase and freezing phenomenon.

11.1. Cluster at absolute maximum

Our interest in this lecture is on the local behavior of the field near its large values. We will refer to these values vaguely as *cluster*. A natural starting point, and pretty much all that will technically be required, is the situation near the absolute maximum. We will proceed by conditioning on the location of the maximum; the translation invariance of the DGFF permits us to shift this location to the origin. The desired conclusion is then the content of:

Theorem 11.1 [Cluster law] *Let $D \in \mathfrak{D}$ with $0 \in D$ and let $\{D_N \colon N \geq 1\}$ be an admissible sequence of approximating domains. Then for each $t \in \mathbb{R}$ and each $f \in C_c(\mathbb{R}^{\mathbb{Z}^2})$ depending only on a finite number of coordinates,*

$$E\big(f(h_0^{D_N} - h^{D_N}) \,\big|\, h_0^{D_N} = m_N + t,\, h^{D_N} \leq h_0^{D_N}\big) \underset{N\to\infty}{\longrightarrow} E_\nu(f) \tag{11.1}$$

where ν is a measure defined from $\phi :=$ DGFF on $\mathbb{Z}^2 \smallsetminus \{0\}$ via

$$\nu(\cdot) := \lim_{r\to\infty} P\bigg(\phi + \frac{2}{\sqrt{g}}\mathfrak{a} \in \cdot \,\bigg|\, \phi_x + \frac{2}{\sqrt{g}}\mathfrak{a}(x) \geq 0 \colon |x| \leq r\bigg) \tag{11.2}$$

with $\mathfrak{a}$ denoting the potential kernel on $\mathbb{Z}^2$.

The existence of the limit in (11.2) is part of the statement of the theorem. However, this can be seen already from:

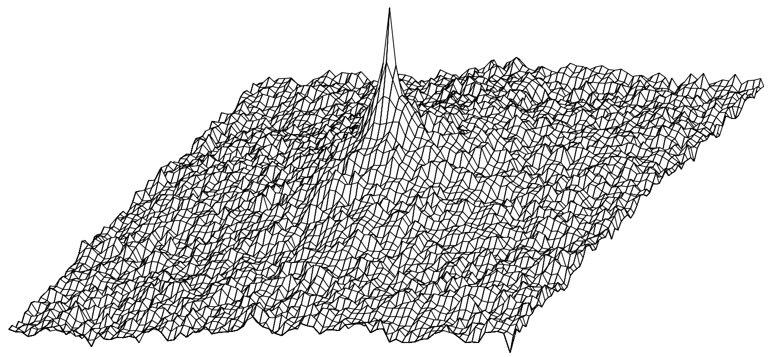

Figure 11.1: A sample of the configuration of the DGFF in the vicinity of its (large) absolute maximum.

Exercise 11.2 *Let ν_r be the conditional measure on the right of* (11.2)*. Prove that $r \mapsto \nu_r$ is stochastically increasing. [Hint: This is true for any strong-FKG measure.]*

This means that $r \mapsto \nu_r(A)$ is increasing on increasing events and so the limit in (11.2) exists in the sense of convergence of finite-dimensional distribution functions. The problem is that ν_r is a measure on a non-compact space and the interpretation of the limit as a distribution thus requires a proof of tightness. This additional ingredient will be supplied by our proof of Theorem 11.1.

The fact that the limit takes the form in (11.2) can be understood on the basis of a simple heuristic calculation. Indeed, conditioning the field on $h_0^{D_N} = m_N + t$ shifts the mean of $h_0^{D_N} - h_x^{D_N}$ by the quantity with $N \to \infty$ asymptotic

$$(m_N + t)\big(1 - \mathfrak{g}^{D_N}(x)\big) \underset{N\to\infty}{\longrightarrow} \frac{2}{\sqrt{g}}\mathfrak{a}(x). \tag{11.3}$$

A variance computation then identifies the asymptotic law of $h_0^{D_N} - h_x^{D_N}$ as $\frac{2}{\sqrt{g}}\mathfrak{a}(x)$ plus the pinned DGFF; see Fig. 11.1. The conditioning on 0 being the maximum then forces the additional conditioning on positivity in (11.2). This would more or less prove the result directly, except for the following caveat:

Theorem 11.3 *There exists $c_\star \in (0,\infty)$ such that*

$$P\bigg(\phi_x + \frac{2}{\sqrt{g}}\mathfrak{a}(x) \ge 0\colon |x| \le r\bigg) = \frac{c_\star}{\sqrt{\log r}}\big(1 + o(1)\big), \quad r \to \infty. \tag{11.4}$$

The conditioning in (11.2) is thus increasingly singular and so it is hard to imagine that one could control the limit solely by manipulations with weak convergence.

We remark that the proof of Theorem 11.3 along with the asymptotic (1.33) for the potential kernel tell us that $\max_{|x|\le r}\phi_x$ will grow to the leading order as $r \mapsto 2\sqrt{g}\log r$. An interesting question is to determine the precise subleading order (which we expect to be $\log\log r$-order). This would be a version of the Law of the Iterated Logarithm for the pinned DGFF.

11.2. Random walk based estimates

The proof of Theorem 1.11 will be based on the concentric decomposition of the DGFF developed in Sections 8.2–8.4. The main difference is that, as these sections were devoted to the proof of the tightness of the lower tail of the maximum, we were not allowed to assume that in estimates there. With the tightness now settled in Lemma 8.21, Lemma 8.11 can be rephrased as:

Lemma 11.4 *There is $a > 0$ such that each $k = 1, \dots, n$ and each $t \geq 0$,*

$$P\biggl(\Bigl|\max_{x\in\Delta^k\smallsetminus\Delta^k}\bigl[\chi_{k-1}(x)+\chi_k(x)+h'_k(x)\bigr]-m_{2^k}\Bigr|\geq t\biggr)\leq \mathrm{e}^{-at}. \tag{11.5}$$

This allows for control of the deviations of the field h^{D_N} from $-S_k$ in both directions which upgrades Lemma 8.16 into the form:

Lemma 11.5 [Reduction to random walk event] *Assume h^{D_N} is realized as the sum on the right of* (8.30). *There is a numerical constant $C > 0$ such that uniformly in the above setting, the following holds for each $k = 0, \dots, n$ and each $t \in \mathbb{R}$:*

$$\begin{aligned}\{S_{n+1}=0\}\cap\bigl\{S_k\geq R_K(k)+|t|\bigr\}&\\ \subseteq\{h_0^{D_N}=0\}\cap\bigl\{h^{D_N}&\leq(m_N+t)(1-\mathfrak{g}^{D_N})\text{ on }\Delta^k\smallsetminus\Delta^{k-1}\bigr\}\\ &\subseteq\{S_{n+1}=0\}\cap\bigl\{S_k\geq -R_K(k)-|t|\bigr\}.\end{aligned} \tag{11.6}$$

where K is the control variable from Definition 8.15 and

$$R_k(\ell):=C[1+\Theta_k(\ell)]. \tag{11.7}$$

We will now use the random walk $\{S_0, \dots, S_n\}$ to control all important aspects of the conditional expectation in the statement of Theorem 11.1.

First note that the event $\bigcap_{k=0}^n\{S_K\geq -R_K(k)-|t|\}$ encases all of the events of interest and so we can use it as the basis for estimates of various undesirable scenarios. (This is necessary because the relevant events will have probability tending to zero proportionally to $1/n$.) In particular, we upgrade Lemma 8.19 to the form:

Lemma 11.6 *There are $c_1, c_2 > 0$ such that for all $n \geq 1$ and all $k = 1, \dots, \lfloor n/2\rfloor$,*

$$P\biggl(\{K>k\}\cap\bigcap_{\ell=0}^n\{S_\ell\geq -R_k(\ell)-|t|\}\,\bigg|\,S_{n+1}=0\biggr)\leq c_1\frac{1+t^2}{n}\mathrm{e}^{-c_2(\log k)^2} \tag{11.8}$$

Since the target decay is order-$1/n$, we see that in the forthcoming derivations we can assume $\{K \leq k\}$ for k sufficiently large but independent of n. Lemma 8.18 then takes the form:

Lemma 11.7 [Entropic repulsion] *For each $t \in \mathbb{R}$ there is $c > 0$ such that for all $n \geq 1$ and all $k = 1, \dots, \lfloor n/2 \rfloor$*

$$P\biggl(\{S_k, S_{n-k} \geq k^{1/6}\} \cap \bigcap_{\ell=k+1}^{n-k-1} \{S_\ell \geq R_k(\ell) + |t|\} \\ \biggm| \bigcap_{\ell=0}^{n} \{S_K \geq -R_k(\ell) - |t|\} \cap \{S_{n+1} = 0\} \biggr) \geq 1 - ck^{-\frac{1}{16}} \tag{11.9}$$

Consider now the expectation in the statement of Theorem 11.1. We first invoke Lemma 8.3 to shift the conditioning event to $h_0^{D_N} = 0$ at the cost of adding the term $(m_N + t)\mathfrak{g}^{D^N}$ to all occurrences of the field. Denoting

$$m_N(t,x) := (m_N + t)(1 - \mathfrak{g}^{D_N}(x)) \tag{11.10}$$

the expectation can be written as the ratio

$$\frac{E\Bigl(f\bigl(m_N(t,x) - h^{D_N}\bigr) 1_{\{h^{D_N} \leq m_N(t,\cdot)\}} \Bigm| h_0^{D_N} = 0 \Bigr)}{E\Bigl(1_{\{h^{D_N} \leq m_N(t,\cdot)\}} \Bigm| h_0^{D_N} = 0 \Bigr)} \tag{11.11}$$

Both the numerator and the denominator have the same structure, so we will just focus on the numerator. We claim:

Proposition 11.8 *For each $\epsilon > 0$ and each $t_0 > 0$ there is $k_0 \geq 1$ such that for all k with $k_0 \leq k \leq n^{1/6}$ and all $t \in [-t_0, t_0]$,*

$$\Biggl| E\Bigl(f\bigl(m_N(t,x) - h^{D_N}\bigr) 1_{\{h^{D_N} \leq m_N(t,\cdot)\}} \Bigm| h_0^{D_N} = 0 \Bigr) \\ - E\biggl(f\bigl(\tfrac{2}{\sqrt{g}}\mathfrak{a} + \phi_k\bigr) 1_{\{\phi_k + \frac{2}{\sqrt{g}}\mathfrak{a} \geq 0 \text{ in } \Delta^k\}} 1_{\{S_k, S_{n-k} \in [k^{1/6}, k^2]\}} \Bigl(\prod_{\ell=k}^{n-k} 1_{\{S_\ell \geq 0\}\}} \Bigr) \\ \times 1_{\{h^{D_N} \leq m_N(t,\cdot) \text{ in } D_N \smallsetminus \Delta^{n-k}\}} \biggm| h_0^{D_N} = 0 \biggr) \Biggr| \leq \frac{\epsilon}{n}, \tag{11.12}$$

where

$$\phi_k(x) := h_0^{\Delta^k} - h_x^{\Delta^k}. \tag{11.13}$$

Proof (sketch). Invoking the sets underlying the concentric decomposition, we write the "hard" event in the expectation as the intersection of an "inner", "middle" and "outer" event,

$$1_{\{h^{D_N} \leq m_N(t,\cdot)\}} = 1_{\{h^{D_N} \leq m_N(t,\cdot) \text{ in } \Delta^k\}} \\ \times 1_{\{h^{D_N} \leq m_N(t,\cdot) \text{ in } \Delta^{n-k} \smallsetminus \Delta^k\}} 1_{\{h^{D_N} \leq m_N(t,\cdot) \text{ in } D_N \smallsetminus \Delta^{n-k}\}}. \tag{11.14}$$

Plugging this in the expectation and invoking Lemma 11.6 to insert $\{K \leq k\}$ into the expectation, the bounds in (11.6) permit us to replace the "middle" event

$$\{h^{D_N} \leq m_N(t,\cdot) \text{ in } \Delta^{n-k} \smallsetminus \Delta^k\} \tag{11.15}$$

by the event

$$\bigcap_{\ell=k}^{n-k}\{S_\ell \geq \pm(R_k(\ell)+|t|)\} \tag{11.16}$$

with the sign depending on we aim to get upper or lower bounds. Lemma 11.7 then tells us that the difference between and these upper and lower bounds is negligible, and so we may further replace $\{S_\ell \geq \pm(R_k(\ell)+|t|)\}$ by $\{S_\ell \geq 0\}$.

The restriction to $S_k, S_{n-k} \geq k^{1/6}$ then comes via Lemma 11.7 and the bounds $S_k, S_{n-k} \leq k^2$ arise from the restriction to $\{K \leq k\}$ and the fact that $R_k(k) \leq k$ for k large. We can also use continuity of f to replace $m_N(t,\cdot)$ in the argument of f by its limit value (11.3). Finally, noting that, conditional on $h_0^{D_N}$ we have

$$h_x^{D_N} = -\phi_k(x) + \sum_{\ell>k}\big[b_\ell(x)\varphi_\ell(0)+\chi_\ell\big], \tag{11.17}$$

we use the entropy repulsion arguments to replace the "inner" event

$$\{h^{D_N} \leq m_N(t,\cdot) \text{ in } \Delta^k\} \tag{11.18}$$

by

$$\{\phi_k + \tfrac{2}{\sqrt{g}}\mathfrak{a} \geq 0 \text{ in } \Delta^k\}\,. \tag{11.19}$$

This requires showing that the entropic repulsion creates enough of a gap to neglect the sum on the right of (11.17) as well as the difference between $m_N(t,\cdot)$ and $\frac{2}{\sqrt{g}}\mathfrak{a}$ without much cost in overall expectation. Since the quantity under expectation remains concentrated on $\bigcap_{\ell=0}^{n}\{S_\ell \geq -R_k(\ell)-|t|\}$, we can use Lemma 11.7 to drop the restriction to $\{K \leq k\}$ and get the desired result. $\square$

A key point to observe now is that, conditionally on S_k and S_{n-k} and $S_{n+1}=0$, the "inner" field ϕ_k, the random variables $\{S_\ell\colon \ell=k,\dots,n-k\}$, and the "outer" field $\{h_x^{D_N}\colon x \in D_N \smallsetminus \Delta^{n-k}\}$ are independent. (This is the reason why we strove to get ϕ_k into the "inner" event. The restriction to $S_{n+1}=0$ allows us to label the random walk "backwards" in the "outer" part of the domain.) This allows us to replace the product of indicators of $\{S_\ell \geq 0\}$ by its conditional expectation given S_k and S_{n-k}. We then invoke:

Lemma 11.9 *For each $t_0>0$ there is $c>0$ such that for all $1 \leq k \leq n^{1/6}$,*

$$\left| P\Big(\bigcap_{\ell=k}^{n-k}\{S_\ell \geq 0\}\,\Big|\,\sigma(S_k,S_{n-k})\Big) - \frac{2}{g\log 2}\frac{S_kS_{n-k}}{n}\right| \leq c\frac{k^4}{n}\frac{S_kS_{n-k}}{n} \tag{11.20}$$

holds everywhere on $\{S_k, S_{n-k} \in [k^{1/6},k^2]\}$.

Proof (idea). We will only explain the form of the leading term leaving the error to a reference to the aforementioned 2016 joint paper with O. Louidor. Calling $x := S_k$ and $y := S_{n-k}$, the probability is lower bounded by

$$P\Big(B_t \geq 0\colon t \in [t_k,t_{n-k}]\,\Big|\,\sigma(B_{t_k},B_{t_{n-k}})\Big)\,, \tag{11.21}$$

where we used the fact that the random walk has Gaussian steps to embed it into the Brownian motion $\{B_t \colon t \geq 0\}$ via

$$S_k := B_{t_k} \quad \text{where} \quad t_k := \sum_{\ell=0}^{k-1} \mathrm{Var}\bigl(\varphi_\ell(0)\bigr). \tag{11.22}$$

Note that, in light of Lemma 8.8, we know that

$$t_k = \bigl(g \log 2 + o(1)\bigr)k. \tag{11.23}$$

Next we observe:

Exercise 11.10 *For B a standard Brownian motion, prove that for any $x, y > 0$ and $t > 0$,*

$$P^x\bigl(B_s \geq 0 \colon 0 \leq s \leq t \,\big|\, B_t = y\bigr) = 1 - \exp\bigl\{-2\tfrac{xy}{t}\bigr\}. \tag{11.24}$$

For $xy \ll t$, the expression on the right of (11.24) is asymptotic to $2\frac{xy}{t}$. This shows

$$\begin{aligned} P\Bigl(\bigcap_{\ell=k}^{n-k} \{S_\ell \geq 0\} \,\Big|\, \sigma(S_k, S_{n-k})\Bigr) &\gtrsim \frac{2S_k S_{n-k}}{t_{n-k} - t_k} \\ &= \frac{2}{g \log 2} \frac{S_k S_{n-k}}{n} \bigl(1 + o(1)\bigr) \end{aligned} \tag{11.25}$$

whenever $k^4 \ll n$.

To get a similar upper bound, one writes the Brownian motion on interval $[t_\ell, t_{\ell+1}]$ as a linear curve connecting S_ℓ to $S_{\ell+1}$ plus a Brownian bridge. Then we observe that the entropic repulsion pushes the walk far away from the positivity constraint so that these Brownian bridges do not affect the resulting probability much. $\square$

Define the quantities

$$\Xi^{\mathrm{in}}_\ell(f) := E\Bigl(f\bigl(\phi_k + \tfrac{2}{\sqrt{g}}\mathfrak{a}\bigr) 1_{\{\phi_k + \frac{2}{\sqrt{g}}\mathfrak{a} \geq 0 \text{ in } \Delta^k\}} 1_{\{S_k \in [k^{1/6}, k^2]\}} S_k\Bigr) \tag{11.26}$$

and

$$\Xi^{\mathrm{out}}_{N,\ell}(t) := E\biggl(1_{\{S_{n-k} \in [k^{1/6}, k^2]\}} S_{n-k}\, 1_{\{h^{D_N} \leq m_N(t,\cdot) \text{ in } D_N \smallsetminus \Delta^{n-k}\}} \,\bigg|\, h^{D_N}_0 = 0\biggr) \tag{11.27}$$

The stated independence along with the asymptotic for the conditional probability in Lemma 11.9 then shows

$$E\Bigl(f\bigl(m_N(t,x) - h^{D_N}\bigr) 1_{\{h^{D_N} \leq m_N(t,\cdot)\}} \,\Big|\, h^{D_N}_0 = 0\Bigr) = \frac{2}{n} \frac{\Xi^{\mathrm{in}}_k(f) \Xi^{\mathrm{out}}_{N,k}(t)}{g \log 2} \bigl(1 + o(1)\bigr) \tag{11.28}$$

with $o(1) \to 0$ in the limits as $N \to \infty$ followed by $k \to \infty$. Using this in the ratio (11.11), the quantity $\Xi^{\mathrm{out}}_{N,\ell}(t)$ *cancels* and, since the right-hand side depends on N only through the $o(1)$ terms that tend to zero, we get:

Corollary 11.11 *For any f as above,*

$$\lim_{N\to\infty} E\Bigl(f\bigl(h^{D_N}_0 - h^{D_N}\bigr) \,\Big|\, h^{D_N}_0 = m_N + t,\, h^{D_N} \leq h^{D_N}_0\Bigr) = \lim_{k\to\infty} \frac{\Xi^{\mathrm{in}}_k(f)}{\Xi^{\mathrm{in}}_k(1)} \tag{11.29}$$

where both limits exist.

11.3. Full process convergence

Thanks to the representation of the pinned DGFF in Exercise 8.13, the above derivation applies, albeit in somewhat simpler terms, also the limit of the probabilities in (11.2). The difference is that here the random walk is not constrained to $S_{n+1} = 0$. This affects the asymptotics of the relevant probability as follows:

Lemma 11.12 *For any $f \in C_c(\mathbb{R}^{\mathbb{Z}^2})$ depending only on a finite number of coordinates,*

$$E\Big(f\big(\phi + \tfrac{2}{\sqrt{g}}\mathfrak{a}\big)1_{\{\phi+\frac{2}{\sqrt{g}}\mathfrak{a}\geq 0 \text{ in } \Delta^r\}}\Big) = \frac{1}{\sqrt{\log 2}}\frac{\Xi_k^{\text{in}}(f)}{\sqrt{r}}\big(1+o(1)\big)\,, \tag{11.30}$$

where $o(1) \to 0$ as $r \to \infty$ followed by $k \to \infty$.

Note that the asymptotic $1/\sqrt{r}$ is exactly that of a random walk of r steps to stay positive. Indeed, the reader will readily check:

Exercise 11.13 *Let $\{B_t \colon t \geq 0\}$ be the standard Brownian motion with P^x denoting the law started from $B_0 = x$. Prove that for all $x > 0$ and all $t > 0$,*

$$\sqrt{\frac{2}{\pi}}\,\frac{x}{\sqrt{t}}\Big(1 - \frac{x^2}{2t}\Big) \leq P^x\big(B_s \geq 0 \colon 0 \leq s \leq t\big) \leq \sqrt{\frac{2}{\pi}}\,\frac{x}{\sqrt{t}}\,. \tag{11.31}$$

We will not give further details concerning the proof of Lemma 11.12 as that amounts to repetitions that the reader may not find illuminating. Rather we move on to:

Proof of Theorem 11.1. From the previous lemma we have

$$\lim_{r\to\infty} P\bigg(\phi + \frac{2}{\sqrt{g}}\mathfrak{a} \in \cdot \,\bigg|\, \phi_x + \frac{2}{\sqrt{g}}\mathfrak{a}(x) \geq 0 \colon |x| \leq r\bigg) = \lim_{k\to\infty} \frac{\Xi_k^{\text{in}}(f)}{\Xi_k^{\text{in}}(1)}. \tag{11.32}$$

Jointly with Corollary 11.11, this proves equality of the limits in the statement.

To see that ν concentrates on $\mathbb{R}^{\mathbb{Z}^2}$ we observe that all derivations above were uniform in f varying throughout any fixed equicontinuous and bounded family of functions of given number of variables. Taking $f \uparrow 1$ along such a family can then be interchanged with the $k \to \infty$ limit in (11.32). This implies $\nu(\mathbb{R}^{\mathbb{Z}^2}) = 1$. $\square$

We will now show how this can be built into the proof of Theorem 9.3. Consider the three coordinate process $\eta^D_{N,r}$ defined in (9.6) and let $f \in C_c(D \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^d})$ depend only on a finite number of coordinates of the "cluster" variable. The idea is to consider the Laplace transform of $\langle \eta^D_{N,r}, f\rangle$, condition on the location of, and value of the field at, the relevant local maxima and wrap the result into a Laplace transform of a function of just the first two variables only. To this we then apply the already proved limit result.

The implementation of this will require working with a slight modification of our original process. Given $x \in D_N$ and a sample of h^{D_N}, define the field

$$\Phi^{M,x}(z) := \sum_{y \in D_N \cap \partial\Lambda_M(x)} H^{(D_N \cap \Lambda_M(x)) \smallsetminus \{x\}}(z,y)\, h^{D_N}(y). \tag{11.33}$$

This is the harmonic extension of the values of h^{D_N} distance M away from x while pretending that the value at x is zero. Note that $\Phi^{r,x}(x) = 0$. Then we set

$$\widehat{\eta}^D_{N,M} := \sum_{x\in D_N} 1_{\{x\in\widehat{\Theta}_{N,M}\}}\delta_{x/N}\otimes\delta_{h_x^{D_N}-m_N}\otimes\delta_{\{h_x^{D_N}-h_{x+z}^{D_N}+\Phi^{M,x}_{x+z}\colon z\in\mathbb{Z}^2\}}, \tag{11.34}$$

where

$$\widehat{\Theta}_{N,M} := \Bigl\{x\in D_N\colon h_x^{D_N} = \max_{y\in\Lambda_{3M}(x)}(h_y^{D_N}-\Phi^{r,x}_y)\Bigr\} \tag{11.35}$$

We now observe that the two processes are very close to each other:

Lemma 11.14 *For any f as above,*

$$\lim_{r\to\infty}\limsup_{N\to\infty}\Bigl|\,E\bigl(\mathrm{e}^{\langle\eta^D_{N,r_N},f\rangle}\bigr)-E\bigl(\mathrm{e}^{\langle\widehat{\eta}^D_{N,N/r},f\rangle}\bigr)\Bigr| = 0. \tag{11.36}$$

Proof (idea). We need to show two things: First, the field $\Phi^{r,x}$ is very small near x. This follows from the fact that it has harmonic paths and is equal to zero at x with the boundary values more or less averaging away. Second, we need to show that the points $\widehat{\Theta}_{N,N/r}$ where $h_x^{D_N} = m_N + O(1)$ coincide with the r_N-local extrema of h^{D_N} of that height with high probability. This again boils down to smallness of $\Phi^{N/r,x}$ in $\Lambda_r(x)$ along with the fact that the local maximum will always be strict, and the fact that points at height $m_N + O(1)$ are either closer than r or farther than N/r with high probability (cf Theorem 9.2). □

We are now ready to give:

Proof of Theorem 9.3. Suppose r is so large that f does not depend on cluster variables outside $\Lambda_r(0)$. We begin by invoking the inclusion-exclusion formula to get

$$\mathrm{e}^{\langle\eta^D_{N,N/r},f\rangle} = \sum_{A\subset D_N}\prod_{x\in A}\Bigl[1_{\{x\in\widehat{\Theta}_{N,N/r}\}}\bigl(\mathrm{e}^{-f(x/N,h_x^{D_N}-m_N,\dots)}-1\bigr)\Bigr] \tag{11.37}$$

where the dots denote the cluster variables $\{h_x^{D_N}-h_{x+z}^{D_N}+\Phi^{r,x}_{x+z}\colon z\in\mathbb{Z}^2\}$. The key point is that the product of the indicators is non-zero only if any pair of distinct points in A is at least distance $3N/r$ away. Assuming $2r < N/r$, for any A possibly contributing to (11.37), any two distinct sets from $\{\Lambda_{N/r}(x)\colon x\in A\}$ are well separated. This means that we can take conditional expectation given

$$\mathcal{F}_{A,r} := \sigma\Bigl(\Bigl\{h_y^{D_N}\colon y\in A\cup\bigcap_{x\in A}\Lambda_{N/r}(x)^{\mathrm{c}}\Bigr\}\Bigr) \tag{11.38}$$

and use the Gibbsian property of the DGFF to get

$$\begin{aligned}E\Bigl(\prod_{x\in A}\Bigl[1_{\{x\in\widehat{\Theta}_{N,N/r}\}}\bigl(\mathrm{e}^{-f(x/N,h_x^{D_N}-m_N,\dots)}-1\bigr)\Bigr]\,\Big|\,\mathcal{F}_{A,r}\Bigr)\\ =\prod_{x\in A}E\Bigl(1_{\{x\in\widehat{\Theta}_{N,N/r}\}}\bigl(\mathrm{e}^{-f(x/N,h_x^{D_N}-m_N,\dots)}-1\bigr)\,\Big|\,\mathcal{F}_{A,r}\Bigr)\end{aligned} \tag{11.39}$$

Now note that on $\{x \in \widehat{\Theta}_{N,N/r}\}$ the value of the field at x dominates all values in $\Lambda_{N/r}(x)$ and, once that is arranged, the values of h^{D_N} outside $\Lambda_{N/r}(x)$ are only restricted by the value of $h^{D_N}_x$. Also note that, for $x \in A$ and any event $B \in \mathbb{R}^{\Lambda_r(x)}$,

$$P\big(h^{D_N} - \Phi^{N/r,x} \in B \big| \mathcal{F}_{A,x}\big) = P\big(h^{\Lambda_{N/r}(x)} \in B \,\big|\, h_0^{\Lambda_{N/r}(x)} = t\big)\Big|_{t:=h_0^{D_N}} \tag{11.40}$$

Since $f(x/N, h^{D_N}_x - m_N, \dots)$ depends only on the coordinates in $\Lambda_r(x)$, we thus get

$$\begin{aligned} E\Big(1_{\{x\in\widehat{\Theta}_{N,r}\}}\big(\mathrm{e}^{-f(x/N,h_x^{D_N}-m_N,\dots)}-1\big)\,\Big|\,\mathcal{F}_{A,r}\Big) \\ = E\Big(1_{\{x\in\widehat{\Theta}_{N,r}\}}\big(\mathrm{e}^{-f_{N,r}(x/N,h_x^{D_N}-m_N)}-1\big)\,\Big|\,\mathcal{F}_{A,r}\Big) \end{aligned} \tag{11.41}$$

where, abbreviating $\Lambda_{N/r} := \Lambda_{N/r}(0)$,

$$\mathrm{e}^{-f_{N,r}(x,t)} := E\Big(\mathrm{e}^{-f(x,t,h_0^{\Lambda_{N/r}}-h^{\Lambda_{N/r}})}\,\Big|\,h^{\Lambda_{N/r}} \le h_0^{\Lambda_{N/r}},\, h_0^{\Lambda_{N/r}} = m_N + t\Big) \tag{11.42}$$

Wrapping the inclusion-exclusion formula back together, we thus get

$$E\big(\mathrm{e}^{\langle \eta^D_{N,N/r}, f\rangle}\big) = E\big(\mathrm{e}^{\langle \eta^D_{N,N/r}, f_{N,r}\rangle}\big) \tag{11.43}$$

Since t is restricted to a compact set by the support restriction on f, Theorem 11.1 shows that the function $f_{N,r}$ is uniformly approximated by f_ν defined

$$f_\nu(x,t) := -\log\big[E_\nu\,\mathrm{e}^{-f(x,t,\phi)}\big]\,. \tag{11.44}$$

The tightness of the processes $\{\eta^D_{N,r_N} \colon N \ge 1\}$ and routine approximations based on Theorem 9.2 then show

$$E\big(\mathrm{e}^{\langle \eta^D_{N,N/r}, f_{N,r}\rangle}\big) = E\big(\mathrm{e}^{\langle \eta^D_{N,r_N}, f_\nu\rangle}\big) + o(1) \tag{11.45}$$

with $o(1) \to 0$ as $N \to \infty$, where (we note) we again replaced N/r by r_N in the second occurrence of the point process.

As $f_\nu \in C_c(D \times \mathbb{R})$, the convergence of the two-coordinate process now yields

$$E\big(\mathrm{e}^{\langle \eta^D_{N,r_N}, f_\nu\rangle}\big) \underset{N\to\infty}{\longrightarrow} E\Big(\exp\Big\{-\int_{D\times\mathbb{R}} Z^D(\mathrm{d}x)\otimes\mathrm{e}^{-\alpha h}\mathrm{d}h\,\big(1-\mathrm{e}^{-f_\nu(x,h)}\big)\Big\}\Big) \tag{11.46}$$

Now observe

$$\begin{aligned} \int_{D\times\mathbb{R}} Z^D(\mathrm{d}x)\otimes\mathrm{e}^{-\alpha h}\mathrm{d}h\,\big(1-\mathrm{e}^{-f_\nu(x,h)}\big) \\ = \int_{D\times\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^2}} Z^D(\mathrm{d}x)\otimes\mathrm{e}^{-\alpha h}\mathrm{d}h\otimes\nu(\mathrm{d}\phi)\big(1-\mathrm{e}^{-f(x,h,\phi)}\big) \end{aligned} \tag{11.47}$$

to write the limit as the Laplace transform of $\mathrm{PPP}(Z^D(\mathrm{d}x)\otimes\mathrm{e}^{-\alpha h}\mathrm{d}h\otimes\nu(\mathrm{d}\phi))$. This holds for a generating class of functions f and so the claim follows. $\square$

Lemma 11.12 then also pretty much the asymptotic (11.4):

Proof of Theorem 11.3, mail idea. Routine (by now) upper and lower bounds using random walk $\{S_k\colon k\ge 1\}$ show that the probability in the statement is of order $1/\sqrt{r}$. Lemma 11.12 then shows

$$c_1 < \Xi_k^{\text{in}}(1) < c_2, \quad k \ge 1, \tag{11.48}$$

for some constants $c_1, c_2 \in (0,\infty)$. The statement of Lemma 11.12 then permits to take $r\to\infty$ independently of $k\to\infty$ (albeit in this order) which means that

$$\Xi_\infty^{\text{in}}(1) := \lim_{k\to\infty} \Xi_k^{\text{in}}(1) \tag{11.49}$$

exists, is positive and finite. Since the $r\log 2$ is, to the leading order, the logarithm of the diameter of Δ^r, the claim follows with $c_\star := \Xi_\infty^{\text{in}}(1)$. □

11.4. Some corollaries

Having established the limit of the structured point measure, we can go back to the "ordinary" extreme value process and extract its limit form as well:

Corollary 11.15 [Cluster process] *Under the above assumptions,*

$$\sum_{x\in D_N} \delta_{x/N}\otimes\delta_{h_x^{D_N}-m_N} \ \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \ \sum_{i\in\mathbb{N}}\sum_{z\in\mathbb{Z}^2} \delta_{(x_i,\,h_i-\phi_z^{(i)})}. \tag{11.50}$$

where

- $\{(x_i,h_i)\colon i\in\mathbb{N}\}$ *are points in a sample from* $\text{PPP}(Z^D(\mathrm{d}x)\otimes \mathrm{e}^{-\alpha h}\mathrm{d}h)$*, and*
- $\{\phi^{(i)}\colon i\in\mathbb{N}\}$ *are i.i.d. samples from* ν*.*

The measure on the right is locally finite on $D\times\mathbb{R}$ *a.s.*

Note the limit process on the right of (11.50) takes the form of a *cluster process*. This term generally refers to a collection of random points obtained by first sampling points in a Poisson point process and then associating with each point a cluster of points. The clusters are independent from each other although the points within each cluster can be heavily dependent.

Another observation that is derived from the above limit law concerns the Gibbs measure on D_N associated with the DGFF on D_N as follows:

$$\mu_{\beta,N}^D(\{x\}) := \frac{1}{\mathfrak{Z}_N(\beta)}\,\mathrm{e}^{\beta h_x^{D_N}} \quad\text{where}\quad \mathfrak{Z}_N(\beta) := \sum_{x\in D_N} \mathrm{e}^{\beta h_x^{D_N}}. \tag{11.51}$$

In order to study the scaling limit of this object, we associate the value $\mu_{\beta,N}^D(\{x\})$ with a point mass at x/N. From the convergence of properly normalized measure

$$\sum_{x\in D_N} \mathrm{e}^{\beta h_x^{D_N}}\delta_{x/N} \tag{11.52}$$

to the Liouville Quantum Gravity for $\beta < \beta_c := \alpha$ it is known that

$$\sum_{z\in D_N} \mu^D_{\beta,N}\big(\{z\}\big)\delta_{z/N}(\mathrm{d}x) \;\underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}}\; \frac{Z^D_\lambda(\mathrm{d}x)}{Z^D_\lambda(D)} \tag{11.53}$$

where $\lambda := \beta/\beta_c$ and where Z^D_λ is the measure we saw in the discussion of the intermediate level sets (for $\lambda < 1$). The result extends (although the proof details of this are scarce) to the case $\beta = \beta_c$, where we get the $\widehat{Z}^D(\mathrm{d}x)$ instead. The supercritical case $\beta > \beta_c$ has been open for quite a while. It was finally settled in:

Corollary 11.16 [Poisson-Dirichlet limit for the Gibbs measure] *Let* $\mathrm{PD}(s)$ *denote the Poisson-Dirichlet law with parameter* $s \in (0,1)$. *For all* $\beta > \beta_c := \alpha$,

$$\sum_{z\in D_N} \mu^D_{\beta,N}\big(\{z\}\big)\delta_{z/N}(\mathrm{d}x) \;\underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}}\; \sum_{i\in\mathbb{N}} p_i\delta_{X_i}, \tag{11.54}$$

where $\{X_i\}$ *are (conditionally on* Z^D*) i.i.d. with common law* $\widehat{Z}^D$, *while* $\{p_i\}\overset{\text{law}}{=}\mathrm{PD}(\beta_c/\beta)$ *is independent of* Z^D *and thus also* $\{X_i\}$.

We recall that $\mathrm{PD}(s)$ is a law on decreasing sequences of non-negative numbers with total sum equal to one obtained by taking a sample from the Poisson process on $[0,\infty)$ with intensity $x^{-1-s}\mathrm{d}x$, ordering the points and normalizing them. The above corollary actually follows from our description of the supercritical Liouville Quantum Gravity measure. Given a (Borel) probability measure Q on $\mathbb{C}$ and a parameter $s > 0$, define the point measure $\Sigma_{s,Q}$ by

$$\Sigma_{s,Q}(\mathrm{d}x) := \sum_{i\in\mathbb{N}} q_i\,\delta_{X_i}, \tag{11.55}$$

where $\{q_i\}$ enumerates the sample points of a Poisson process on $[0,\infty)$ with intensity $x^{-1-s}\mathrm{d}x$ and $\{X_i\}$ are independent samples from Q, independent of the $\{q_i\}$.

Theorem 11.17 [Liouville measure in the glassy phase] *Let* Z^D *and* ν *be as in Theorem 9.3. For each* $\beta > \beta_c := \alpha$ *there is* $c(\beta) \in (0,\infty)$ *such that*

$$\sum_{z\in D_N} \mathrm{e}^{\beta(h_z-m_N)}\delta_{z/N}(\mathrm{d}x) \;\underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}}\; c(\beta)\,Z^D(D)^{\beta/\beta_c}\;\Sigma_{\beta_c/\beta,\widehat{Z}^D}(\mathrm{d}x), \tag{11.56}$$

where Z^D *is sampled first and* $\Sigma_{\beta_c/\beta,\widehat{Z}^D}$ *is defined conditionally on* Z^D. *Moreover,*

$$c(\beta) = \beta^{-\beta/\beta_c}\big[E_\nu\big(Y^\beta(\phi)^{\beta_c/\beta}\big)\big]^{\beta/\beta_c} \quad \text{with} \quad Y^\beta(\phi) := \sum_{x\in\mathbb{Z}^2} \mathrm{e}^{-\beta\phi_x}. \tag{11.57}$$

In particular, $E_\nu(Y^\beta(\phi)^{\beta_c/\beta}) < \infty$ *for each* $\beta > \beta_c$.

Note that the limit laws in (11.54) and (11.56) are purely atomic, in contrast to the limits of the subcritical measures (11.52) which are singular with respect to the Lebesgue measure but non-atomic.

The reader might wonder how is it possible that the rather complicated structure of the cluster law ν only manifests itself though the expectation of the quantity $Y^\beta(\phi)$. This can, more or less, be traced to the following property of the Gumbel law:

Exercise 11.18 *Let $\{h_i : i \in \mathbb{N}\}$ be samples from* $\mathrm{PPP}(\mathrm{e}^{-\alpha h}\mathrm{d}h)$ *and let $\{X_i : i \in \mathbb{N}\}$ be independent, i.i.d. random variables with $c := E\mathrm{e}^{\alpha X_1} < \infty$. Prove that*

$$\big\{h_i + X_i : i \in \mathbb{N}\big\} \overset{\text{law}}{=} \mathrm{PPP}\big(c\mathrm{e}^{-\alpha h}\mathrm{d}h\big). \tag{11.58}$$

The mechanism behind the above corollary is that the contribution of the cluster associated with a given large local maximum then projects into the shift of the local maximum by an independent random variable. Exercise 11.18 shows that this is in law equivalent to a deterministic shift by $\alpha^{-1}\log c$.

Our final corollary concerns the behavior of the function

$$G_{N,\beta}(t) := E\bigg(\exp\Big\{-\mathrm{e}^{-\beta t}\sum_{x\in D_N}\mathrm{e}^{\beta h_x^{D_N}}\Big\}\bigg), \tag{11.59}$$

which, we observe, is a reparametrization of the Laplace transform of the normalizing constant $\mathfrak{Z}_N(\beta)$ from (11.51). In their work on Branching Brownian Motion, Derrida and Spohn and later Fyodorov and Bouchaud observed that, in a suitable limit, an analogous quantity ceases to depend on β once β crosses a critical threshold. They referred to this as *freezing*. Our control above is able to establish the same phenomenon for the quantity arising from DGFF:

Corollary 11.19 [Freezing] *For each $\beta > \beta_\mathrm{c} := \alpha$ there is $\tilde{c}(\beta) \in \mathbb{R}$ such that*

$$G_{N,\beta}\big(\,t + m_N + \tilde{c}(\beta)\big) \underset{N\to\infty}{\longrightarrow} E\big(\mathrm{e}^{-Z^D(D)\,\mathrm{e}^{-\alpha t}}\big). \tag{11.60}$$

The constant $\tilde{c}(\beta)$ depends only on the law of ν and that only via the expectation in (11.57).

We refer to the aforementioned 2016 joint paper with O. Louidor for further consequences of the above limit theorem and additional details.