Lecture 10

Nailing the intensity measure

In this lecture we first observe that the intensity measure associated with a subsequential limit of the extremal point process is in one-to-one correspondence with the limit distribution of the DGFF maximum. A theorem of Bramson, Ding and Zeitouni, which we state without proof, ensures that the limit law of the maximum exists and so all subsequential limits converge to the same object. Next we state properties that link the intensity measure in various domains; e.g., restriction to a subdomain (Gibbs-Markov) and conformal maps between domains. These give rise to a set of conditions that identify the intensity measure uniquely. We also sketch a proof that these conditions are satisfied by, and the intensity measure thus agrees with, a version the critical Liouville Quantum Gravity.

10.1. Limit law for DGFF maximum

On the way to the proof of Theorem 9.3 we have so far shown that any *subsequential* limit point of the measures $\{\eta^D_{N,r_N}\colon N\ge 1\}$, restricted to the first two coordinates, is a Poisson Point Process with intensity $Z^D(\mathrm{d}x)\otimes \mathrm{e}^{-\alpha h}\mathrm{d}h$. Our next task to to prove the existence of the limit. Here we observe the fact used in the last proof of the previous lecture:

Lemma 10.1 *Suppose $N_k\to\infty$ is such that $\eta^D_{N_k,r_{N_k}}\overset{\text{law}}{\longrightarrow}\mathrm{PPP}(Z^D(\mathrm{d}x)\otimes \mathrm{e}^{-\alpha h}\mathrm{d}h)$. Then for each $t\in\mathbb{R}$,*

$$P\Bigl(\max_{x\in D_{N_k}} h_x^{D_{N_k}} < m_{N_k}+t\Bigr)\underset{k\to\infty}{\longrightarrow} E\Bigl(\mathrm{e}^{-Z^D(D)\alpha^{-1}\mathrm{e}^{-\alpha t}}\Bigr)\tag{10.1}$$

Proof. We have

$$\bigl\{\max_{x\in D_N} h_x^{D_N} < m_N+t\bigr\}=\Bigl\{\eta^D_{N,r_N}\bigl(D\times[t,\infty)\bigr)=0\Bigr\}\tag{10.2}$$

Suitable approximation arguments then show that, along the given subsequence, the probability of the latter event converges to that of $P(\eta^D(D\times[t,\infty))=0)$. Noting that the probability that PPP(M) has no points in set A is $\mathrm{e}^{-M(A)}$, the claim follows by a straightforward calculation. $\square$

A key observation to make is that, as t varies through $\mathbb{R}$, the parameter $\alpha^{-1}\mathrm{e}^{-\alpha t}$ varies through $(0,\infty)$. The limit distribution of the maximum thus determines the Laplace transform of the random variable $Z^D(D)$. Hence we get:

Corollary 10.2 *If* $\max_{x\in D_N} h_x^{D_N} - m_N$ *converges in distribution, then the law of* $Z^D(D)$ *is the same for every subsequential limit of* $\{\eta^D_{N,r_N}\colon N\ge 1\}$.

The premise to this corollary has been supplied by:

Theorem 10.3 [Bramson, Ding and Zeitouni 2013] *Recall* $V_N := (0,N)^2\cap\mathbb{Z}^2$. *Then, as* $N\to\infty$, *the centered maximum* $\max_{x\in V_N} h_x^{V_N} - m_N$ *of DGFF in* V_N *converges in law to a non-degenerate random variable.*

We do not attempt to give the proof of this result as that would take us on a tour involving comparisons with Modified Branching Random Walk. Notwithstanding, let us at least highlight some of the main ideas.

The proof aims to show that any two subsequential limits of the centered maximum will give the same result. This is achieved by considering a Gibbs-Markov decomposition of the DGFF on V_{KN}, with both K and N large, into independent copies of the DGFF on K^2 translates $\{V_N^{(i)}\colon i=1,\dots,K^2\}$ of V_N and the binding field, and checking what having the field on V_{KN} being close to m_{NK} means for the the corresponding translate of V_N.

Explicitly, denoting $V_{KN}^\circ := \bigcup_{i=1}^{K^2} V_N^{(i)}$, we realize the DGFF on V_{KN} as

$$h^{V_{KN}} = h^{V_{KN}^\circ} + \varphi^{V_{KN},V_{KN}^\circ} \quad\text{with}\quad h^{V_{KN}^\circ} \perp\!\!\!\perp \varphi^{V_{KN},V_{KN}^\circ} \tag{10.3}$$

We can in fact always ignore the small neighborhood of the "dividing lines" between the boxes $\{V_N^{(i)}\colon i=1,\dots,K^2\}$ since we have:

Lemma 10.4 *There is* $c>0$ *such that for any* $N\ge 1$ *and any* $A\subset D_N$,

$$P\bigl(\max_{x\in A} h_x^{D_N} = \max_{x\in D_N} h_x^{D_N}\bigr) \le c\frac{|A|}{N^2}. \tag{10.4}$$

As a corollary, we get:

Exercise 10.5 *Use Lemma 10.4 to prove that, for every subsequential limit* η^D *of processes of interest, the associated* Z^D *measure obeys*

$$A\subset\overline{D}\ \text{Borel},\ \mathrm{Leb}(A)=0 \quad\Rightarrow\quad Z^D(A)=0\quad\text{a.s.} \tag{10.5}$$

Next let x_i denote the vertex at the center (resolving ties arbitrarily) of $V_N^{(i)}$. Then

$$\mathrm{Cov}\Bigl(\varphi_{x_i}^{V_{KN},V_{KN}^\circ},\varphi_{x_j}^{V_{KN},V_{KN}^\circ}\Bigr) = \mathrm{Cov}\bigl(h_{x_i}^{V_{KN}},h_{x_j}^{V_{KN}}\bigr) - \mathrm{Cov}\bigl(h_{x_i}^{V_N^{(i)}},h_{x_j}^{V_N^{(j)}}\bigr) \tag{10.6}$$

and so, checking the cases $i = j$ and $i \neq j$ separately, we get

$$\mathrm{Cov}\Big(\varphi_{x_i}^{V_{KN},V_{KN}^\circ},\varphi_{x_j}^{V_{KN},V_{KN}^\circ}\Big) = g\log\Big(\frac{KN}{|x_i-x_j|\vee N}\Big)+O(1). \tag{10.7}$$

This means that $\{\varphi_{x_i}^{V_{KN},V_{KN}^\circ}\colon i=1,\dots,K^2\}$ behaves very much like the DGFF in V_K. This might suggest that we could replace the values of $\varphi^{V_{KN},V_{KN}^\circ}$ on $V_N^{(i)}$ by the value at x_i but this is true only to the leading order as the field

$$x\mapsto \varphi_x^{V_{KN},V_{KN}^\circ}-\varphi_{x_i}^{V_{KN},V_{KN}^\circ},\quad x\in V_N^{(i)}, \tag{10.8}$$

retains variance of order unity (after all, it scales to a non-trivial Gaussian field with smooth, but non-constant sample paths).

To see how scales pass from level NK to level N, solve:

Exercise 10.6 *Prove that for $c>0$ small enough,*

$$\sup_{N\ge1}P\Big(\max_{i=1,\dots,K^2}\varphi_{x_i}^{V_{KN},V_{KN}^\circ}>2\sqrt{g}\log K-c\log\log K\Big)\underset{K\to\infty}{\longrightarrow}0 \tag{10.9}$$

If we ignore the variations of the field in (10.8), and assuming $K\le N$, this means that for $x\in V_N^{(i)}$, the assumption $h_x^{V_{KN}}=m_{KN}+O(1)$ implies

$$h_x^{V_N^{(i)}}\ge m_{KN}-2\sqrt{g}\log K+c\log\log K\ge m_N+c\log\log K+O(1), \tag{10.10}$$

i.e., $h^{V_N^{(i)}}$ takes an *unusually high* value at x. We thus only have to tie the maxima at different scales under the situation when these are conditioned to be already quite large. Since the variations caused by the field (10.8) remains relevant in this regime as well, we have to control *both* the value and the position of the maximum. The proof is thus reduced to proving:

Proposition 10.7 *There is a constant $c_\star>0$ such that*

$$P\big(\max_{x\in V_N}h_x^{V_N}\ge m_N+t\big)=\big[c_\star+o(1)\big]t\mathrm{e}^{-\alpha t} \tag{10.11}$$

where $o(1)\to0$ in the limit $N\to\infty$ followed by $t\to\infty$. Furthermore, there is a bounded and continuous function $\psi\colon(0,1)^2\to[0,\infty)$ such that for all $A\subset(0,1)^2$ open,

$$P\Big(N^{-1}\operatorname*{argmax}_{V_N}h^{V_N}\in A\,\Big|\,\max_{x\in V_N}h_x^{V_N}\ge m_N+t\Big)=o(1)+\int_A\psi(x)\mathrm{d}x\,, \tag{10.12}$$

where $o(1)\to0$ in the limit $N\to\infty$ followed by $t\to\infty$.

This is still done by way of the above decomposition of V_{KN} but now for the event that the maximum of $h^{V_{KN}}$ is much in excess of m_{KN}. We refer the reader to the original paper for further details.

Let us now comment on how the above implies the uniqueness of the subsequential limit of the processes $\{\eta_{N,r_N}^D\colon N\ge1\}$. First off, all what we stated above was for square boxes V_N but it can be extended to any admissible sequence D_N of lattice approximations of $D \in \mathfrak{D}$. Defining, for $A \subset D$ with a non-empty interior,

$$h_{A,\star}^{D_N} := \max_{\substack{x \in D_N \\ x/N \in A_i}} h_x^{D_N}. \tag{10.13}$$

the key is then to note that the techniques underlying the proof of Theorem 10.3 also give:

Lemma 10.8 *Let $A_1, \dots, A_k \subset D$ be disjoint open sets. Then the joint law of*

$$\{h_{A_i,\star}^{D_N} - m_N \colon i = 1, \dots, k\} \tag{10.14}$$

admits a non-degenerate weak limit as $N \to \infty$.

A proof of this lemma can be found in the 2013 joint paper of the lecturer with O. Louidor. Next we note:

Lemma 10.9 *For any subsequential limit $\eta^D = \text{PPP}(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h)$ of processes $\{\eta_{N,r_N}^D \colon N \geq 1\}$, any disjoint open sets $A_1, \dots, A_k \subset D$ and any $t_1, \dots, t_k \in \mathbb{R}$,*

$$P\bigl(h_{A_i,\star}^{D_N} < m_N + t_i \colon i = 1, \dots, k\bigr) \underset{N\to\infty}{\longrightarrow} E\bigl(\mathrm{e}^{-\sum_{i=1}^{K} Z^D(A_k)\alpha^{-1}\mathrm{e}^{-\alpha t_i}}\bigr). \tag{10.15}$$

Proof. We can write the left-hand side as the probability of $\langle \eta_{N,r_N}^D, f\rangle = 0$, where $f := \sum_{i=1}^k 1_{A_i} \otimes 1_{[t_i,\infty)}$. The right-hand side is checked to be the probability that $\langle \eta^D, f\rangle = 0$ and so the claim follows by approximating f by bounded continuous functions with compact support in $\overline{D} \times (\mathbb{R} \cup \{\infty\})$. □

This permits us to give:

Proof of Theorem 9.3, first two coordinates. Lemmas 10.8 and 10.9 imply that the joint law of $(Z^D(A_1), \dots, Z^D(A_k))$ is the same for every subsequential limit η^D of our processes of interest. This means that we know the law of $\langle Z^D, f\rangle$ for any f of the form $f = \sum_{i=1}^k a_i 1_{A_i}$ with A_i open disjoint. Noting that every bounded and continuous f can be approximated by a function of the form $\sum_{i=1}^k a_i 1_{\{a_{i-1} \leq f < a_i\}}$ with $\text{Leb}(f = a_i) = 0$ for every $i = 1, \dots, k$, by the result of Exercise 10.5 we get uniqueness of the law of $\langle Z^D, f\rangle$ for any $f \in C_{\mathrm{c}}(D \times \mathbb{R})$. By the Riesz Representation Theorem, this identifies the law of Z^D uniquely. We must have $Z^D(D) \in (0, \infty)$ a.s. because the maximum is tight at scale $m_N + O(1)$. □

10.2. Properties of Z^D-measures

As was the case of intermediate level sets, once the convergence issue has been settled, the next natural question is: What is Z^D? And: Can its law be independently characterized? The structure of the limit process offers a number of interpretations. First we ask the reader to solve:

Exercise 10.10 *Let* $\{(x_i, h_i)\colon i \in \mathbb{Z}\}$ *be points in a sample of* $\mathrm{PPP}(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h)$. *Show that*

$$\big\{h_i - \alpha^{-1} \log Z^D(D)\colon i \in \mathbb{N}\big\} \tag{10.16}$$

has Gumbel law with intensity $\mathrm{e}^{-\alpha h}\mathrm{d}h$.

This shows that (the scaled logarithm of) the total mass acts as a random shift for the values of the local maxima of the DGFF. (In particular, the *gap* between the i-th and $i+1$-st local maximum are distributed as the gaps in the Gumbel law.) Another, perhaps more interesting, aspect to highlight is:

Lemma 10.11 *Let* X_N *denote the position where* $h^{D_N}_{X_N} = \max_{x \in D_N} h^{D_N}_x$. *Then for any* $A \subset D$ *open with* $\mathrm{Leb}(\partial A) = 0$ *and any* $t \in \mathbb{R}$,

$$P\Big(\tfrac{1}{N}X_N \in A,\, \max_{x \in D_N} h^{D_N}_x < m_N + t\Big) \underset{N\to\infty}{\longrightarrow} E\big(\widehat{Z}^D(A)\mathrm{e}^{-\alpha^{-1}\mathrm{e}^{-\alpha t}Z^D(D)}\big). \tag{10.17}$$

where

$$\widehat{Z}^D(A) := \frac{Z^D(A)}{Z^D(D)}. \tag{10.18}$$

Proof. Lemma 10.8 along with a continuity argument based on $\mathrm{Leb}(\partial A) = 0$ implying $Z^D(\partial A) = 0$ show

$$\big(h^{D_N}_{A,\star} - m_N,\, h^{D_N}_{A^{\mathrm{c}},\star} - m_N\big) \overset{\text{law}}{\longrightarrow} (h^\star_A, h^\star_{A^{\mathrm{c}}}) \tag{10.19}$$

The continuity of the law of the DGFF yields

$$\begin{aligned} P\Big(\tfrac{1}{N}X_N \in A,\, \max_{x \in D_N} h^{D_N}_x < m_N + t\Big) &= P\big(h^{D_N}_{A^{\mathrm{c}},\star} < h^{D_N}_{A,\star},\, h^{D_N}_{A,\star} - m_N < t\big) \\ &= P\big(h^{D_N}_{A^{\mathrm{c}},\star} \le h^{D_N}_{A,\star},\, h^{D_N}_{A,\star} - m_N \le t\big). \end{aligned} \tag{10.20}$$

Since the first line is the measure of an open set while the second line is the measure of the closure thereof, the standard facts about the convergence in law imply

$$P\Big(\tfrac{1}{N}X_N \in A,\, \max_{x \in D_N} h^{D_N}_x < m_N + t\Big) \underset{N\to\infty}{\longrightarrow} P\big(h^\star_{A^{\mathrm{c}}} < h^\star_A,\, h^\star_A < t\big). \tag{10.21}$$

Now we invoke:

Exercise 10.12 *Prove that*

$$h^\star_A \overset{\text{law}}{=} \inf\big\{t \in \mathbb{R}\colon \eta^D(A \times [t,\infty)) = 0\big\} \tag{10.22}$$

and that this in fact applies jointly to $(h^\star_A, h^\star_{A^{\mathrm{c}}})$.

This means that we can now rewrite the probability on the right of (10.21) in terms of maximal points in the sample of η^D. For $\eta = \mathrm{PPP}(M \otimes \mathrm{e}^{\alpha h}\mathrm{d}h)$ with a fixed M, the probability density of the maximal point in A is $\mathrm{e}^{-\alpha h}M(A)\mathrm{e}^{-\alpha^{-1}\mathrm{e}^{-\alpha h}M(A)}$ while the probability that no point in A^{c} will be above h is $\mathrm{e}^{-\alpha^{-1}\mathrm{e}^{-\alpha h}M(A^{\mathrm{c}})}$. Hence,

$$P\big(h^\star_{A^{\mathrm{c}}} < h^\star_A,\, h^\star_A < t\big) = \int_{-\infty}^{t} \mathrm{e}^{-\alpha h}\, E\Big(Z^D(A)\mathrm{e}^{-\alpha^{-1}\mathrm{e}^{-\alpha h}Z^D(D)}\Big)\mathrm{d}h \tag{10.23}$$

The result now follows by integrating. □

Hereby we get:

Corollary 10.13 *The measure $A \mapsto E(\widehat{Z}^D(A))$ is the $N \to \infty$ weak limit of the marginal law of the (a.s.-unique) maximizer of h^{D_N} scaled by N.*

Although the laws of $\{Z^D\colon D \in \mathfrak{D}\}$ are defined individually, just as for the measures arising from the intermediate level sets, they are very much interrelated. We will now articulate properties that underpin these relations.

Theorem 10.14 [Properties of Z^D-measures] *The family $\{Z^D\colon D \in \mathfrak{D}\}$ of intensities obeys the following properties:*

(1) $Z^D(A) = 0$ *a.s. for any* $A \subset \overline{D}$ *with* $\text{Leb}(A) = 0$,

(2) *for any* $a \in \mathbb{C}$ *and any* $b > 0$,

$$Z^{a+bD}(a + b\mathrm{d}x) \overset{\text{law}}{=} b^4 Z^D(\mathrm{d}x), \tag{10.24}$$

(3) *if* $D \cap \widetilde{D} = \emptyset$, *then*

$$Z^{D\cup\widetilde{D}}(\mathrm{d}x) \overset{\text{law}}{=} Z^D(\mathrm{d}x) + Z^{\widetilde{D}}(\mathrm{d}x), \quad Z^D \perp\!\!\!\perp Z^{\widetilde{D}}, \tag{10.25}$$

(4) *if* $\widetilde{D} \subseteq D$ *and* $\text{Leb}(D \smallsetminus \widetilde{D}) = 0$, *then for* $\Phi^{D,\widetilde{D}} = \mathcal{N}(0, C^{D,\widetilde{D}})$ *and* $\alpha := 2/\sqrt{g}$,

$$Z^D(\mathrm{d}x) \overset{\text{law}}{=} Z^{\widetilde{D}}(\mathrm{d}x)\, \mathrm{e}^{\alpha\Phi^{D,\widetilde{D}}(x)}, \quad Z^{\widetilde{D}} \perp\!\!\!\perp \Phi^{D,\widetilde{D}}, \tag{10.26}$$

(5) *there is* $\hat{c} \in (0,\infty)$ *such that for all open* $A \subseteq D$,

$$\lim_{\lambda\downarrow 0} \frac{E(Z^D(A)\mathrm{e}^{-\lambda Z^D(D)})}{\log(1/\lambda)} = \hat{c}\int_A \psi^D(x)\,\mathrm{d}x, \tag{10.27}$$

where $\psi^D(x) := r_D(x)^2$.

Before we get on to the proof of this theorem, let us make a few remarks. First off, with the exception of (5), appropriate versions of these properties are shared by the whole family of measures $\{Z^D_\lambda\colon D \in \mathfrak{D}\}$ introduced early in this course. The condition (5) is an exception as it shows that, unlike the measures Z^D_λ, we have $EZ^D(A) = \infty$ for any non-empty open $A \subset D$. This is what stands in the way of proving uniqueness of the law of Z^D by the argument underlying Proposition 4.11.

Proof of Theorem 10.14, (1) and (3). (1) was already stated as Exercise 10.5 while (3) is a simple consequence of the fact that the DGFF on domains separated by at least two lattice steps are independent. (This is one placed where the first condition (1.23) of admissible approximations enters.) □

Next we will address the statement of the *Gibbs-Markov property* in (4):

Proof of Theorem 10.14(4). For a while we can follow the proof of Proposition 4.9 on the Gibbs-Markov property for the measures arising from the intermediate level sets. Let $\widetilde{D}$ and D as in the statement, let $f \in \mathbb{C}_c(\widetilde{D} \times \mathbb{R})$ with $f \geq 0$ and recall the notation

$$f_\Phi(x,h) := f\bigl(x,\, h + \Phi^{D,\widetilde{D}}(x)\bigr)\,. \tag{10.28}$$

The argument leading up to (4.24) then ensures

$$\langle \eta^D, f\rangle \overset{\text{law}}{=} \langle \eta^{\widetilde{D}}, f_\Phi\rangle, \quad \Phi^{D,\widetilde{D}} \perp\!\!\!\perp \eta^{\widetilde{D}}, \tag{10.29}$$

where η^D and $\eta^{\widetilde{D}}$ are the limit processes in the respective domains. The Poisson nature of the limit process then gives, via a routine change of variables,

$$\begin{aligned} E\bigl(\mathrm{e}^{-\langle \eta^{\widetilde{D}}, f_\Phi\rangle}\bigr) &= E\Bigl(\exp\Bigl\{-\int Z^{\widetilde{D}}(\mathrm{d}x)\mathrm{e}^{-\alpha h}\mathrm{d}h\,\bigl(1-\mathrm{e}^{-f(x,h+\Phi^{D,\widetilde{D}}(x))}\bigr)\Bigr\}\Bigr) \\ &= E\Bigl(\exp\Bigl\{-\int Z^{\widetilde{D}}(\mathrm{d}x)\mathrm{e}^{\alpha\Phi^{D,\widetilde{D}}(x)}\mathrm{e}^{-\alpha h}\mathrm{d}h\,\bigl(1-\mathrm{e}^{-f(x,h)}\bigr)\Bigr\}\Bigr) \end{aligned} \tag{10.30}$$

Comparing this with the expression one would get for the Laplace transform of $\langle \eta^D, f\rangle$, and using that the above f's are sufficient to determine the intensity measure, the claim follows. $\square$

From here we get:

Exercise 10.15 *Prove that the measure Z^D charges every non-empty open subset of D with probability one. In particular,* $\operatorname{supp} Z^D = \overline{D}$ *a.s.*

The Gibbs-Markov property yields another useful fact:

Exercise 10.16 *Let $D \in \mathfrak{D}$ and assume that $\{D^n : n \geq 1\} \in \mathfrak{D}$ are such that $D^n \uparrow D$ with $C^{D,D^n}(x,y) \to 0$ locally uniformly on D. Then*

$$Z^{D^n}(\mathrm{d}x) \xrightarrow[n\to\infty]{\text{law}} Z^D(\mathrm{d}x). \tag{10.31}$$

Proof of Theorem 10.14(2). The previous exercise allows us to assume that both a and b are rational. As aN and bN will then be integer for an infinite number of N, the existence of the limit permits us to in fact assume that $a \in \mathbb{Z}^2$ and $b \in \mathbb{N}$. The invariance of the law of Z^D under integer-valued shifts is a trivial consequence of the similar invariance of the DGFF. Concerning the behavior under scaling, here we note that if $\{D_N : N \geq 1\}$ approximates D, then $\{D_{bN} : N \geq 1\}$ approximates bD. The only item to worry about is the centering of the field which changes by

$$m_{bN} - m_N = 2\sqrt{g}\,\log(b) + o(1). \tag{10.32}$$

Following this change through the limit procedure yields

$$Z^{bD}(b\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha 2\sqrt{g}\,\log(b)} Z^D(\mathrm{d}x). \tag{10.33}$$

The claim follows by noting that $\alpha 2\sqrt{g} = 4$. $\square$

Proof of Theorem 10.14(5), sketch. The starting point is an extension of Proposition 10.7 to all $D \in \mathfrak{D}$. This is technically achieved by considering two domains, $\widetilde{D} \subset D$ with $\widetilde{D}$ a square, picking their approximating domains $\{D_N\}$ and $\{\widetilde{D}_N\}$ respectively, and coupling the fields between these via the Gibbs-Markov property,

$$h^{D_N} = h^{\widetilde{D}_N} + \varphi^{D_N,\widetilde{D}_N}, \quad h^{\widetilde{D}_N} \perp\!\!\!\perp \varphi^{D_N,\widetilde{D}_N} \tag{10.34}$$

We then claim without proof the following intuitive fact: *Conditional on the maximum of the DGFF in D_N to be large, the position of the maximizer of h^{D_N} will with high probability coincide with the position of the maximizer of $h^{\widetilde{D}_N}$.* Denoting the limit law of the scale maximizer/centered maximum for domains D_N by $(X_\star, h_\star)$ and letting $(\widetilde{X}_\star, \widetilde{h}_\star)$ be the corresponding object of $\widetilde{D}_N$, we thus get

$$P\big(X_\star \in A,\, h_\star > t\big) = \big(1 + o(1)\big) P\Big(\widetilde{X}_\star \in A,\, h_\star + \Phi^{D,\widetilde{D}}(\widetilde{X}_\star) > t\Big), \tag{10.35}$$

where $N \to \infty$ could be taken in light of Lemma 10.11 and where $o(1) \to 0$ in the limit $t \to \infty$. Since $\widetilde{D}$ is a square, Proposition 10.7 and a judicious shift of t by $\Phi^{D,\widetilde{D}}(X_\star)$ then show

$$t^{-1} \mathrm{e}^{\alpha t} P\Big(\widetilde{X}_\star \in A,\, \widetilde{h}_\star + \Phi^{D,\widetilde{D}}(\widetilde{X}_\star) > t\Big) \underset{t\to\infty}{\longrightarrow} E\Big(\int_A \mathrm{e}^{\alpha \Phi^{D,\widetilde{D}}(x)} \psi^{\widetilde{D}}(x)\mathrm{d}x\Big). \tag{10.36}$$

Hereby we get

$$t^{-1} \mathrm{e}^{\alpha t} P\big(X_\star \in A,\, h_\star > t\big) \underset{t\to\infty}{\longrightarrow} \int_A \psi^D(x)\mathrm{d}x\,, \tag{10.37}$$

where

$$\psi^D(x) := \psi^{\widetilde{D}}(x)\mathrm{e}^{\frac{1}{2}\alpha^2 C^{D,\widetilde{D}}(x,x)}. \tag{10.38}$$

This is our version of Proposition 10.7 for general $D \in \mathfrak{D}$. Note that ψ^D is not necessarily a probability density.

Since $\frac{1}{2}\alpha^2 g = 2$, from the relation between $C^{D,\widetilde{D}}$, the Green functions and the conformal radius we then get

$$\frac{\psi^D(x)}{r_D(x)^2} = \frac{\psi^{\widetilde{D}}(x)}{r_{\widetilde{D}}(x)^2}, \quad x \in \widetilde{D}. \tag{10.39}$$

From here we find out that the ratio for $\widetilde{D}$ does not change if we shift $\widetilde{D}$ around while keeping $x \in \widetilde{D}$. This implies that the ratio for $\widetilde{D}$ fixed is the same for all $x \in \widetilde{D}$. Consequently, the ratio for D is constant on neighborhoods of x and thus on all connected components. Each connected component can be handled separately due to independence of the DGFF on each of them. Hence, for all D and all $x \in D$,

$$\psi^D(x) = c r_D(x)^2 \tag{10.40}$$

for some $c > 0$ independent of D.

Having extended Proposition 10.7 to all $D \in \mathfrak{D}$, writing $\lambda := \alpha^{-1}\mathrm{e}^{-\alpha t}$, (10.37), Corollary 10.13 and (10.17) then show

$$\frac{E(\widehat{Z}^D(A)[1 - \mathrm{e}^{-\lambda Z^D(D)}])}{\lambda \log(1/\lambda)} \underset{\lambda\downarrow 0}{\longrightarrow} \alpha c_\star \int_A \psi^D(x)\mathrm{d}x \tag{10.41}$$

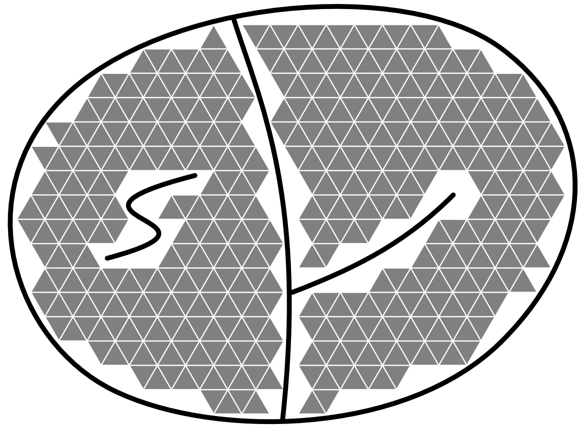

Figure 10.1: A tiling of domain D by equilateral triangles of side-length K^{-1}.

for any $A \subset D$ open with $\mathrm{Leb}(\partial A) = 0$. Using straightforward monotonicity arguments, we get the same for the quantity on the left of (10.27). $\square$

10.3. Connection to Liouville Quantum Gravity

As our final item of concern in this lecture, we wish to explain that the properties listed in Theorem 10.14 actually determine the laws of the Z^D's uniquely. This will then permit us to state that these measures coincide with the critical Liouville Quantum Gravity measures associated with the continuum GFF.

Given an integer $K \geq 1$, consider consider a tiling of the plane by equilateral triangles of side-length K. For a domain $D \in \mathfrak{D}$, let $T^1, \dots, T^{m_K}$ be the triangles in the tiling contained in D, cf Fig. 10.1. Abbreviate

$$\widetilde{D} := \bigcup_{i=1}^{m_K} T^i. \tag{10.42}$$

Given $\delta \in (0,1)$, assume that the triangles we were enumerated so that $i = 1, \dots, n_K$, for some $n_K \leq m_K$, label the triangles that are at least distance δ away from D^{c}. Define $T^\delta_1, \dots, T^\delta_{n_K}$ the the equilateral triangles of side length $(1-\delta)K^{-1}$ that have the same orientation and centers as $T^1, \dots, T^{n_K}$. Recall that the oscillation of a function f on a set A is given by

$$\mathrm{osc}_A f := \sup_{x \in A} f(x) - \inf_{x \in A} f(x) \tag{10.43}$$

We then claim:

Theorem 10.17 *Consider the events $A^i_{K,R}$, $i=1,\dots,n_K$ defined by*

$$A^i_{K,R} := \bigl\{\operatorname{osc}_{T^i_\delta}\Phi^{D,\widetilde{D}} \le R\bigr\} \cap \bigl\{\max_{T^i_\delta}\Phi^{D,\widetilde{D}} \le 2\sqrt{g}\log K - R\bigr\}. \tag{10.44}$$

Then for any family $\{M^D\colon D\in\mathfrak{D}\}$ of random Borel measures satisfying conditions (0-4) of Theorem 10.14 with constant $\hat{c}\in(0,\infty)$ and any $D\in\mathfrak{D}$, the random measure

$$\alpha\hat{c}\psi^D(x)\sum_{i=1}^{n_K} 1_{A^i_{K,R}}\bigl(\alpha\mathrm{Var}(\Phi^{D,\widetilde{D}}(x)) - \Phi^{D,\widetilde{D}}(x)\bigr)\mathrm{e}^{\alpha\Phi^{D,\widetilde{D}}(x)-\frac12\alpha^2\mathrm{Var}(\Phi^{D,\widetilde{D}}(x))}\,1_{T^i_\delta}(x)\,\mathrm{d}x \tag{10.45}$$

tends in law to M^D in the limit as $K\to\infty$, $R\to\infty$ and $\delta\downarrow0$ (in this order). (This holds irrespective of the orientation of the triangular grid.)

As a consequence, we then get:

Corollary 10.18 [Characterization of Z^D-measures] *The laws of $\{Z^D\colon D\in\mathfrak{D}\}$ are determined uniquely by conditions (1-5) up to the choice of the constant $\hat{c}$.*

Proof of Theorem 10.17, sketch. The proof is based on a number of relatively straightforward observations. Denote $\widetilde{D}^\delta := \bigcup_{i=1}^{n_K} T^i_\delta$ and, for $f\in C_c(D)$, let $f_\delta := f1_{\widetilde{D}^\delta}$. Property (1) above then ensures

$$\langle Z^D, f_\delta\rangle \;\underset{\delta\downarrow0}{\overset{\text{law}}{\longrightarrow}}\; \langle Z^D, f\rangle\,. \tag{10.46}$$

So we may henceforth focus on f_δ. Properties (3-4) then give

$$1_{\widetilde{D}^\delta}(x)M^D(\mathrm{d}x) \overset{\text{law}}{=} \sum_{i=1}^{n_K}\mathrm{e}^{\alpha\Phi^{D,\widetilde{D}}(x)}1_{T^\delta_i}(x)\,M^{T_i}(\mathrm{d}x)\,, \tag{10.47}$$

with $M^{T_1},\dots,M^{T_{n_K}}$ and $\Phi^{D,\widetilde{D}}$ all independent. Let $x_1,\dots,x_{n_K}$ be the center points of the triangles $T^1,\dots,T^{n_K}$, respectively. A variation on Exercise 10.9 then shows

$$\limsup_{K\to\infty} P\Bigl(\max_{i=1,\dots,n_K}\Phi^{D,\widetilde{D}}(x_i) > 2\sqrt{g}\log K - c\log\log K\Bigr) = 0. \tag{10.48}$$

for some $c>0$. The first harder bit of the proof is the content of:

Proposition 10.19 *For any $\delta\in(0,1)$ and any $\epsilon>0$,*

$$\lim_{R\to\infty}\limsup_{K\to\infty} P\Bigl(\sum_{i=1}^{n_K}\int_{T^i_\delta} M^{T^i}(\mathrm{d}x)\mathrm{e}^{\alpha\Phi^{D,\widetilde{D}}(x)}1_{\{\operatorname{osc}_{T^i_\delta}\Phi^{D,\widetilde{D}}>R\}} > \epsilon\Bigr) = 0. \tag{10.49}$$

We will not give the proof; instead we refer to Proposition 6.5 of the 2014 joint paper of the lecturer with O. Louidor. We still remark, however, that it is this proposition that forces us to work with triangles. Indeed, in this case the space of piecewise harmonic continuous functions on $\widetilde{D}^\delta$ naturally decomposes into functions that are linear on each T_i, and thus determined by their values at three points in T^i, and functions whose contribution can already be neglected.

The above observations permit us to restrict attention only to those triangles where event $A^i_{K,R}$ occurs. The key point is that $A^i_{K,R}$ forces the field to be part of an equicontinuous family. Let $\mathcal{F}^T_{R,\beta,\delta}$ denote the class of continuous functions $\phi\colon T\to\mathbb{R}$ on triangle T such that

$$\phi(x)\ge\beta\quad\text{and}\quad|\phi(x)-\phi(y)|\le R|x-y|,\qquad x,y\in T_\delta. \tag{10.50}$$

For such functions, property (5) yields:

Proposition 10.20 *Fix $\beta>0$ and $R>0$. For each $\epsilon>0$ there are $\delta_0>0$ and $\lambda_0>0$ such that, for all $\lambda\in(0,\lambda_0)$, all $\delta\in(0,\delta_0)$ and all $f\in\mathcal{F}^T_{R,\beta,\delta}$,*

$$(1-\epsilon)\int_{T_\delta}f(x)\psi^T(x)\,\mathrm{d}x\le\frac{\log E(\mathrm{e}^{-\lambda M^T(f1_{T_\delta})})}{\lambda\log\lambda}\le(1+\epsilon)\int_{T_\delta}f(x)\psi^T(x)\,\mathrm{d}x, \tag{10.51}$$

where $M^T(f1_{T_\delta}):=\int_{T_\delta}M^T(\mathrm{d}x)\,f(x)$.

We will not give a proof and instead refer to Proposition 6.6 of the aforementioned paper. Using this proposition with $\lambda:=K^{-4}\mathrm{e}^{\alpha\Phi^{D,\widetilde D}(x_i)}$, on $A^i_{K,R}$ we then get

$$\begin{aligned}E\biggl(&\exp\Bigl\{-\mathrm{e}^{\alpha\Phi^{D,\widetilde D}(x_i)}M^{T^i}(f1_{T^i_\delta}\mathrm{e}^{\alpha(\Phi^{D,\widetilde D}-\Phi^{D,\widetilde D}(x_i))})\Bigr\}\,\Big|\,\Phi^{D,\widetilde D}\biggr)\\&=\exp\biggl\{(1+\tilde\epsilon)\,K^{-4}\log\bigl(K^{-4}\mathrm{e}^{\alpha\Phi^{D,\widetilde D}(x_i)}\bigr)\int_{T^i_\delta}f(x)\,K^4\psi^{T^i}(x)\mathrm{e}^{\alpha\Phi^{D,\widetilde D}(x)}\,\mathrm{d}x\biggr\}\end{aligned} \tag{10.52}$$

for some $\tilde\epsilon\in[-\epsilon,\epsilon]$ depending only on $\Phi^{D,\widetilde D}$.
Denote by $Z^D_{K,R,\delta}$ the measure in (10.45) and use $M^D_{K,R,\delta}$ be the expression on the right of (10.47) with the sum restricted to i where $A^i_{K,R}$ occurs. Noting that

$$\log(K^{-4}\mathrm{e}^{\alpha\Phi^{D,\widetilde D}(x_i)})=\bigl(1+o(1)\bigr)\alpha\bigl(\Phi^{D,\widetilde D}(x)-\alpha\mathrm{Var}(\Phi^{D,\widetilde D}(x))\bigr) \tag{10.53}$$

with $o(1)\to0$ as $K\to\infty$ uniformly in $x\in T^i_\delta$ and recalling that

$$\psi^{T^i}(x)=\psi^D(x)\mathrm{e}^{-\frac12\alpha^2\mathrm{Var}(\Phi^{D,\widetilde D}(x))},\qquad x\in T^i, \tag{10.54}$$

we thus get

$$E\bigl(\mathrm{e}^{-(1+2\epsilon)\widetilde Z^D_{K,R,\delta}(f)}\bigr)\le E\bigl(\mathrm{e}^{-M^D_{K,R,\delta}(f)}\bigr)\le E\bigl(\mathrm{e}^{-(1-2\epsilon)\widetilde Z^D_{K,R,\delta}(f)}\bigr). \tag{10.55}$$

Since $M^D_{K,R,\delta}(f)$ tends in distribution to $M^D(f)$ in the stated limits, the law of $M^D(f)$ is given by the corresponding limit law of $\widetilde Z^D_{K,R,\delta}(f)$. □

Another consequence of the above theorem is the behavior of the measures under conformal transforms of the underlying domain:

Corollary 10.21 [Conformal invariance] *Let $f\colon D\to f(D)$ be a conformal bijection between admissible domains $D,f(D)\in\mathfrak{D}$. Then*

$$Z^{f(D)}\circ f(\mathrm{d}x)\overset{\text{law}}{=}\bigl|f'(x)\bigr|^4Z^D(\mathrm{d}x) \tag{10.56}$$

In particular, the law of $r_D(x)^{-4}Z^D(\mathrm{d}x)$ is invariant under conformal maps.

This follows, roughly speaking by the fact that the law of $\Phi^{D,\widetilde{D}}$ is conformally invariant and also the observation that a conformal map is locally a dilation and rotation. Hence, the triangles T^i map to near-triangles $f(T^i)$ with the deformation tending to zero with the size of the triangle.

As a final remark we provide identification with a white-noise version of the critical Liouville Quantum Gravity constructed by Duplantier, Rhodes, Sheffield and Vargas. We choose to work with the so called Seneta-Heyde normalization. The specifics are as follows: For $\{B_t \colon t \geq 0\}$ the standard Brownian motion, let $p_t^D(x,y)$ be the transition density from x to y before exiting D. More precisely, letting $\tau_{D^c} := \inf\{t \geq 0 \colon B_t \notin D\}$ we have

$$p_t^D(x,y)\mathrm{d}y := P^x\bigl(B_t \in \mathrm{d}y,\, \tau_{D^c} > t\bigr). \tag{10.57}$$

Writing W for the white noise on $D \times (0,\infty)$ with respect to the Lebesgue measure, consider the Gaussian process

$$\varphi_t(x) := \int_{D\times[\mathrm{e}^{-4t},\infty)} p_{s/2}^D(x,z)W(\mathrm{d}z\,\mathrm{d}s). \tag{10.58}$$

Note that then the Markov property of p^D gives

$$\mathrm{Cov}\bigl(\varphi_t(x),\varphi_t(y)\bigr) = \int_{D\times[\mathrm{e}^{-4t},\infty)} p_s^D(x,y)\mathrm{d}s \underset{t\to\infty}{\longrightarrow} G^D(x,y). \tag{10.59}$$

Define the random measure

$$M_t^D(\mathrm{d}x) := \sqrt{t}\,1_D(x)\psi^D(x)\mathrm{e}^{\alpha\varphi_t(x)-\frac12\alpha^2\mathrm{Var}[\varphi_t(x)]}\,\mathrm{d}x \tag{10.60}$$

We then have:

Theorem 10.22 [Duplantier, Rhodes, Sheffield and Vargas, 2014] *There is a non-vanishing a.s.-finite random measure M_∞^D such that for every $A \subset D$,*

$$M_t^D(A) \underset{t\to\infty}{\longrightarrow} M_\infty^D(A), \quad \text{a.s.} \tag{10.61}$$

The measure M_∞^D is the *critical Liouville Quantum Gravity*. It is a fact that this measure has infinite expectation on any non-empty open subset of D. The measure is thus not really fixed by its expectation and so it is determined only up to a constant. Other definitions exists and, through several contributions by various authors, these are now known to be all equal up to a constant multiple. The measure in (10.45) is another example of this kind, although the uniqueness theorems do not apply to this case. We thus use the opportunity to announce:

Theorem 10.23 [B-Louidor, 2017] *The measures $\{M_\infty^D \colon D \in \mathfrak{D}\}$ obey conditions (1-5) of Theorem 10.14 for some constant $\hat{c}$. In particular, there is $c_\star \in (0,\infty)$ such that*

$$Z^D(\mathrm{d}x) \overset{\text{law}}{=} c_\star M_\infty^D(\mathrm{d}x), \quad D \in \mathfrak{D}. \tag{10.62}$$