The corrector approach to random walk in random environment

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Random walk among random conductances

Uniformly elliptic case

Graph $\mathbb{Z}^d$, edges $\mathbb{B}$ (nearest neighbors only)

i.i.d. *conductances* $(\omega_b \colon b \in \mathbb{B})$; law $\mathbb{P}$, expectation $\mathbb{E}$

Uniform ellipticity $\mathbb{P}(\omega_b \ge \epsilon) = 1$ for some $\epsilon > 0$

Random walk $X_0, X_1, \dots$ with *quenched* law $P_{z,\omega}$

$$P_{z,\omega}(X_{n+1} = x + e | X_n = x) = \frac{\omega_{(x,x+e)}}{\sum_{e' \colon |e'|=1} \omega_{(x,x+e')}} \qquad |e| = 1$$

Initial condition

$$P_{z,\omega}(X_0 = z) = 1$$

Note: *annealed* law $Q(A) = \mathbb{E}_z P_{z,\omega}(A)$ not Markov

Bond percolation on $\mathbb{Z}^d$

Away from uniform ellipticity

Allow $p \stackrel{\text{def}}{=} \mathbb{P}(\omega_b > 0) < 1$ but $p > p_c(d)$ (requires $d \geq 2$)

Case of interest:

$$\omega_b = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathscr{C}_\infty = \mathscr{C}_\infty(\omega)$ be the sites "connected to infinity"

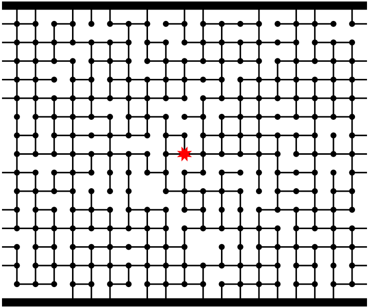
Burton-Keane's Theorem: $\mathscr{C}_\infty$ is connected with probability 1

Denote $\Omega_0 = \{0 \in \mathscr{C}_\infty\}$ and $\mathbb{P}_0(\cdot) = \mathbb{P}(\cdot|\Omega_0)$

A question

Percolation restricted to infinite slab

Is the probability of { walk exits through top side } close to $^1/_2$?



- Trivially true for the *annealed* measure.
- *Quenched* measure: Prove a **Functional CLT**.

Main result

Theorem 1 (Functional CLT for RW on percolation cluster)

Let $d \geq 2$, $p > p_{\rm c}(d)$ and let $\omega \in \Omega_0$. Let $(X_n)_{n\geq 0}$ be the random walk with law $P_{0,\omega}$ and let

$$B_n(t) = \frac{1}{\sqrt{n}}\Big(X_{\lfloor tn\rfloor} + (tn - \lfloor tn\rfloor)(X_{\lfloor tn\rfloor+1} - X_{\lfloor tn\rfloor})\Big), \quad t \geq 0.$$

Then for all $T > 0$ and $\mathbb{P}_0$-a.e. ω, the law of $(B_n(t)\colon 0 \leq t \leq T)$ on $(C[0,T], \mathscr{W}_T)$ converges weakly to the law of an isotropic (non-degenerate) Brownian motion.

Similarly for variants of above RW (lazy walk, continuous time)

Previous results

- **Quenched problem in** $d \geq 4$:

  Sidoravicius & Sznitman (2004)

- **Annealed problem**:

  De Masi & Ferrari & Goldstein & Wick (1989)

- **Directed version**:

  Rassoul-Agha & Sepäläinen (2004)

- **Uniformly elliptic case**:

  Kozlov (1985), Kipnis & Varadhan (1986),
  Sidoravicius & Sznitman (2004), Fontes & Mathieu (2004)

- **Heat-kernel estimates**:

  Nash, Varopoulos, Aronson, . . . , Heicklen & Hoffman
  Mathieu & Remy (2004), Barlow (2004)

Simultaneous results

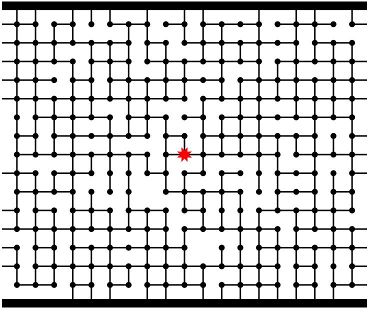
- **Same theorem in** $d = 2, 3$

Mathieu & Piatnitski (2005)

Key word: homogenization theory

Main idea

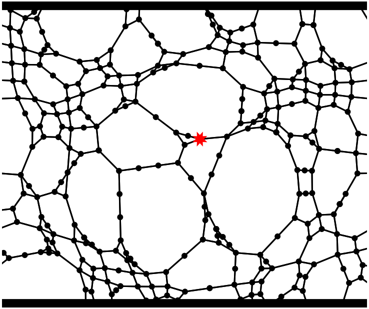
Geometric embedding of $\mathscr{C}_\infty$:



The walk (X_n) is *not* a martingale.

Main idea

Harmonic embedding of $\mathscr{C}_\infty$: $x \mapsto x + \chi(x, \omega)$



The walk $X_n + \chi(X_n, \omega)$ *is* a martingale.

Corrector

Analytical construction

Kozlov, Kipnis & Varadhan, Olla, Mathieu & Piatnitski

Proposition 2 ($d \geq 2$, $p > p_c$)

There is $\chi : \mathbb{Z}^d \times \Omega_0 \to \mathbb{R}^d$ such that, for $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$:

(0) $\chi(0, \omega) = 0$

(1) $x \mapsto x + \chi(x, \omega)$ *is harmonic on* $\mathscr{C}_\infty(\omega)$

(2) χ *is a gradient field on* $\mathscr{C}_\infty$*:*

$$\chi(x, \omega) - \chi(y, \omega) = \chi(x - y, \tau_y \omega), \qquad x, y \in \mathscr{C}_\infty$$

(3) *The gradients of χ are square integrable:*

$$\mathbb{E}_0\big([\chi(e, \omega) - \chi(0, \omega)]^2 1_{\{\omega_e = 1\}} \big) < C, \qquad |e| = 1$$

Sketch of proof I.

L^2-calculus on Ω

Unit vectors $\mathcal{B} = \{\pm e_1, \dots, \pm e_d\}$
Vector field (flow) $v \colon \Omega \times \mathcal{B} \to \mathbb{R}^d$
Consistency: $v(\omega, -b) = -v(\tau_{-b}\omega, b)$
Inner product on $L^2(\Omega \times \mathcal{B})$:

$$(v, w) = \frac{1}{2}\,\mathbb{E}_0\Big[\sum_{b \in \mathcal{B}} \omega_b v(\omega, b) w(\omega, b)\Big]$$

Gradient field: For $\phi \colon \Omega \to \mathbb{R}^d$ let

$$(\nabla\phi)(\omega, b) = \phi(\tau_b\omega) - \phi(\omega)$$

Natural L^2-subspace

$$L^2_\nabla = \overline{\{\nabla\phi \colon \phi\text{-local}\}} \subset L^2(\Omega \times \mathcal{B})$$

Sketch of proof II.
Orthogonal decomposition

Fact: $w \in (L^2_\nabla)^\perp \Leftrightarrow \operatorname{div} w = 0$ (conserved flow)

$$(\operatorname{div} w)(\omega) = \sum_{b \in \mathcal{B}} \omega_b v(\omega, b)$$

Now take $g(\omega, b) = b$ and define $\chi = \chi(b, \omega)$ by

$$\chi = \operatorname{proj}_{L^2_\nabla}(-g)$$

Then $g + \chi \in (L^2_\nabla)^\perp$, i.e., $\operatorname{div}(g + \chi) = 0$. This gives

$$\sum_{b \in \mathcal{B}} \omega_b \big(b + \chi(b, \omega)\big) = 0$$

$g + \chi$ obeys cycle condition $\Rightarrow$ can be extended to $\mathscr{C}_\infty$ $\square$

Deformed random walk

The listed properties make

$$M_n = X_n + \chi(X_n, \omega)$$

an L^2-martingale.

Ergodic theorem: $\mathscr{F}_n = \sigma(M_1, \ldots, M_n)$

$$\frac{1}{n}\sum_{k=0}^{n-1} E_{0,\omega}\big(|M_{k+1} - M_k|^2 | \mathscr{F}_k\big) \underset{n\to\infty}{\longrightarrow} \mathbb{E}_0 E_{0,\omega}(|M_1|^2)$$

Lindenberg-Feller Martingale CLT $\Rightarrow$

The deformed walk scales to Brownian motion

Controlling the deformation

$d = 2$ for now

Need to show that

$$\max_{1 \le k \le n} \left| \chi(X_k, \omega) \right| = o(\sqrt{n}).$$

Since $M_n = O(\sqrt{n})$, it suffices to prove:

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Proposition 3 ($d = 2$)

For $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$,

$$\lim_{n\to\infty} \max_{\substack{x\in\mathscr{C}_\infty(\omega)\\ |x|\le n}} \frac{|\chi(x,\omega)|}{n} = 0.$$

Some ergodic theory
Induced shift

For $\omega \in \Omega_0$, let $(x_n)_{n\in\mathbb{Z}}$ be the intersections of $\mathscr{C}_\infty(\omega)$ with x-axis labeled so that $x_n < x_{n+1}$ and $x_0 = 0$.

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Consider the *induced shift* $\sigma\colon \Omega_0 \to \Omega_0$

$$\sigma(\omega) = \tau_{x_1(\omega)}(\omega), \qquad \omega \in \Omega_0.$$

Standard arguments show:

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Standard arguments show:

Lemma 4 ($d \geq 2$)
σ is $\mathbb{P}_0$-preserving and ergodic.

Along coordinate axes

Now set

$$\Psi(\omega) = \chi\big(x_1(\omega), \omega\big) - \chi(0, \omega)$$

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Then

$$\chi\big(x_n(\omega), \omega\big) = \sum_{k=1}^{n} \Psi \circ \sigma^k(\omega)$$

But $\Psi \in L^1$ (Antal-Pisztora) and

$$\mathbb{E}_0(\Psi) = 0$$

(Ψ is gradient) so the Ergodic Theorem implies:

Corollary 5 ($d \geq 2$)

For $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$,

$$\lim_{n\to\infty} \frac{\chi(x_n(\omega), \omega)}{n} = 0.$$

Weaving webs of goodness

Good lines and sites

Let $K, \epsilon > 0$ and $\omega \in \Omega_0$. The x-axis is called *good in* ω if

$$\left|\chi(x,\omega)\right| \le K + \epsilon|x|$$

for every $x \in \mathscr{C}_\infty$ on x-axis.

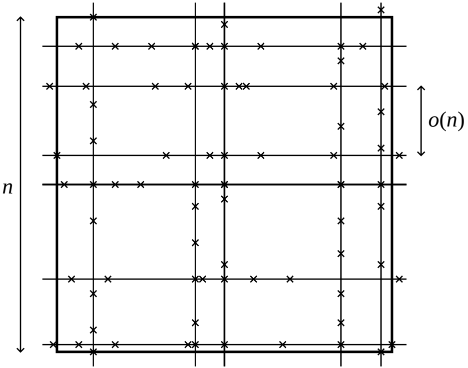
A site $x \in \mathbb{Z}^d$ is called *good in* ω if

- $x \in \mathscr{C}_\infty(\omega)$
- Both x and y-axes are good in $\tau_x(\omega)$.

Weaving webs of goodness
Good grid

For $\mathbb{P}_0$-a.e. ω and all $\epsilon > 0$:

- Origin is good if K is large
- Good sites appear with positive density along both axes



Weaving webs of goodness

Sublinearity everywhere

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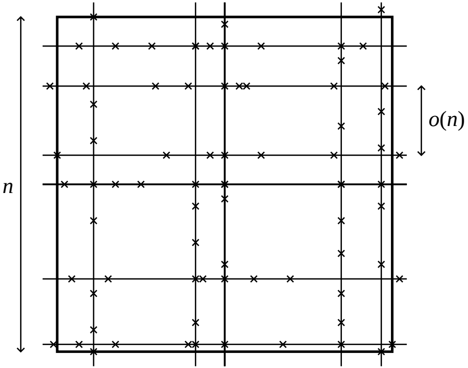
Maximum on good grid: $\leq 2K + 2\epsilon n$.



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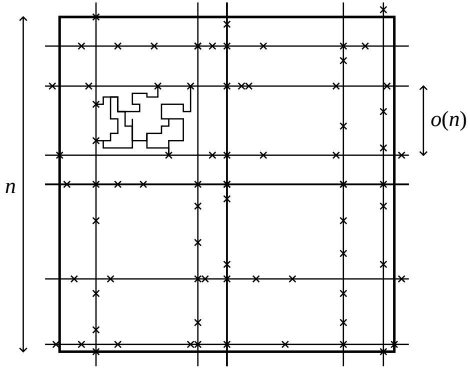

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AND the fact that

$x \mapsto x + \chi(x, \omega)$
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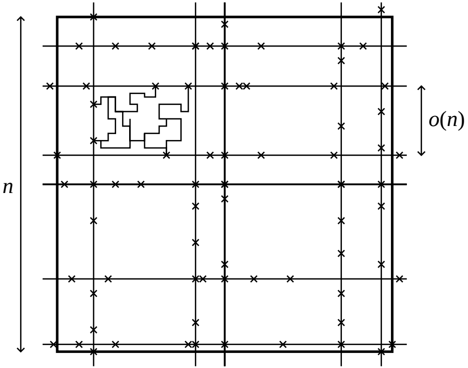

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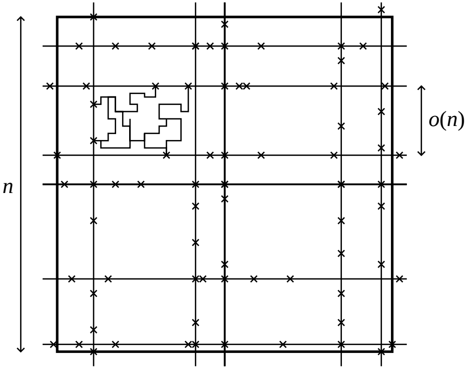


imply:

$$\max_{\substack{x \in \mathscr{C}_\infty(\omega) \\ |x| \leq n}} \big|\chi(x, \omega)\big| \leq 2K + 2\epsilon n + o(n) \quad \square$$

Higher dimensions
A density bound on corrector

Embarrassing fact:

We do not know how to extend this argument to $d \geq 3$

But we can prove:

Proposition 6 ($d \geq 3$)

For $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$ and all $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n^d} \sum_{\substack{x\in\mathscr{C}_\infty(\omega)\\ |x|\leq n}} 1_{\{|\chi(x,\omega)|\geq\epsilon n\}} = 0$$

Higher dimensions
Main idea

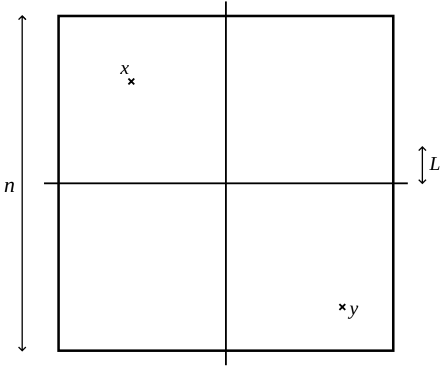
$n \times n$ square in $\mathbb{Z}^3$

WANT:

$$\left|\chi(x,\omega) - \chi(y,\omega)\right| \le \epsilon n$$

for (most of) good

$$x, y \in \mathscr{C}_\infty \cap \text{ square}$$



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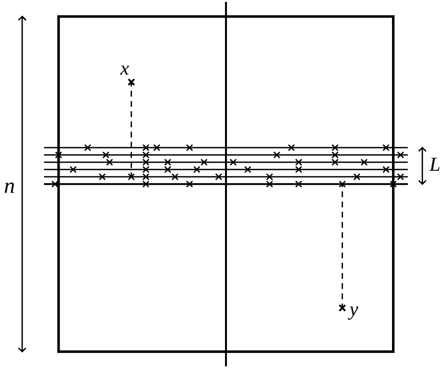

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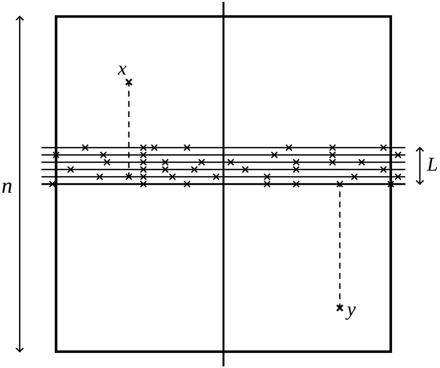
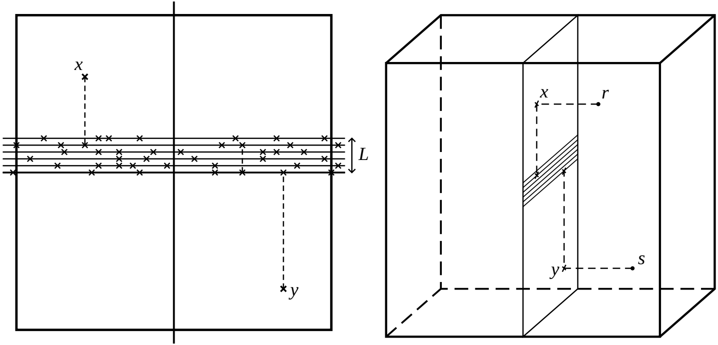


For L large x and y are connected by path shorter than $4n$

Higher dimensions
Main idea

Finally, perform induction on dimension:



Final touches

To finish, we prove tightness using

Theorem 7 (Barlow 2004)

For $\mathbb{P}_0$-a.e. ω and all $x \in \mathscr{C}_\infty(\omega)$,

$$P_{0,\omega}(X_n = x) \le \frac{c_1}{n^{d/2}} \exp\Big\{-c_2 \frac{|x|^2}{n}\Big\},$$

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and focus on finite-dimensional distributions.

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From Proposition 6, we then have

$$\frac{|\chi(X_n, \omega)|}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } P_{0,\omega}\text{-probability}$$

i.e., $X_n/\sqrt{n} = M_n/\sqrt{n} + o(1)$. This implies the CLT in $d \ge 3$. □

Future research
Everybody welcome

Limit laws:

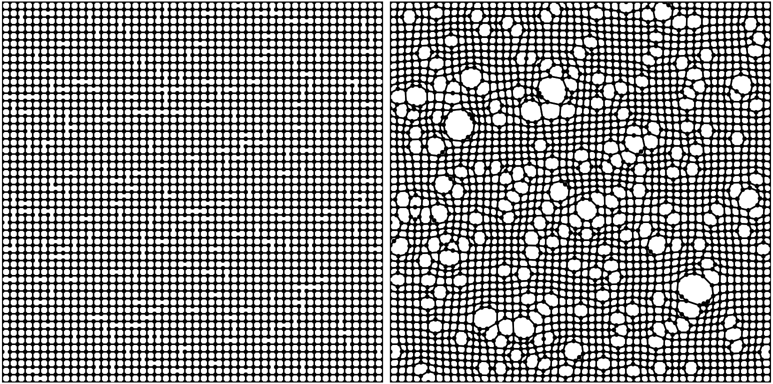
- Maximum bound on corrector in $d \geq 3$
- Other graphs, e.g., Voronoi percolation
- Long-range percolation (stable processes)
- Beyond reversible environments (loop representation)

Corrector:

- A.s. uniqueness $\leftrightarrow$ sublinear harmonic functions
- Scaling limit (Gaussian free field/tightness)
- Behavior as $p \downarrow p_{\rm c}$

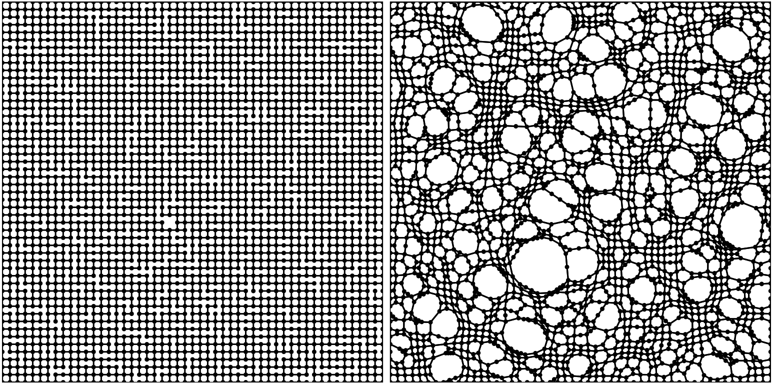
Some figures

Percolation cluster and its deformation: $p = 0.95$



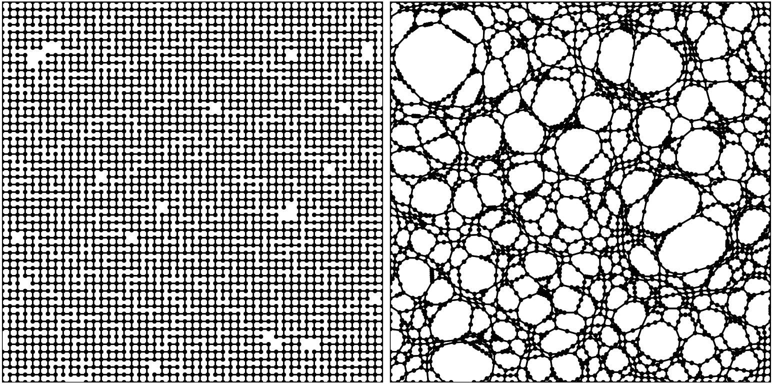
Some figures

Percolation cluster and its deformation: $p = 0.85$



Some figures

Percolation cluster and its deformation: $p = 0.75$



Some figures

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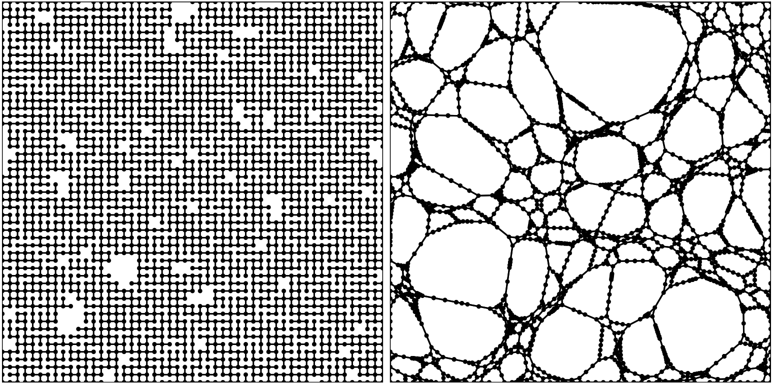

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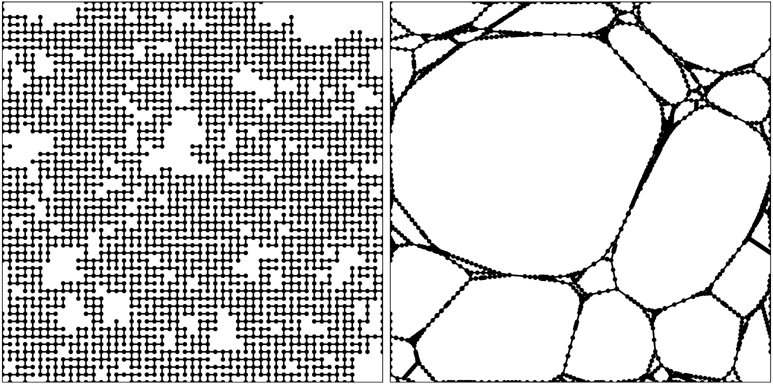

THE END

Slides available from:
http://www.math.ucla.edu/~biskup/talks.html