Phase Coexistence of Gradient Gibbs Measures

Marek Biskup

(UCLA)

joint work with
Roman Kotecký
(Prague & Warwick)

Gradient Measures:

Random field: $(\phi_x)_{x\in\mathbb{Z}^d}$, $\phi_x \in \mathbb{R}$

Potential: $V : \mathbb{R} \to \mathbb{R}$, even, diverges at $\pm\infty$

Gibbs measure: $\Lambda \subset \mathbb{Z}^d$, b.c.

$$P_\Lambda(\mathrm{d}\phi_\Lambda) = \frac{1}{Z_\Lambda} \exp\Big\{-\beta \sum_{\langle x,y\rangle} V(\phi_y - \phi_x)\Big\} \mathrm{d}\phi_\Lambda$$

Example: $V(\eta) = \frac{1}{2}\eta^2$ (Gaussian free field)

Localization vs delocalization

Gradient projections:

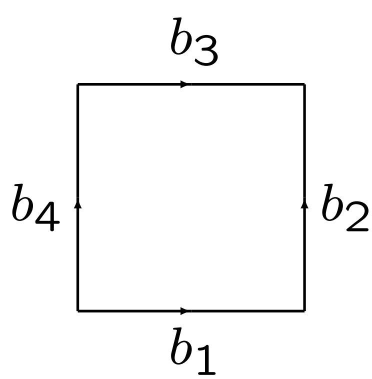
Gradient variables: $b = (x, y)$

$$\eta_b = \phi_y - \phi_x$$

Fact: V sufficient growth at $\pm\infty$
$\Rightarrow \Lambda \to \mathbb{Z}^d$ limit possible
$\Rightarrow$ DLR formalism

gives rise to *Gradient Gibbs Measures (GGM)*

Plaquette constraints:



$$\eta_{b_1} + \eta_{b_2} = \eta_{b_3} + \eta_{b_4}$$

Funaki-Spohn theory :

Tilt (or slope): $u \in \mathbb{R}^d$

$$E_\mu(\eta_b) = b \cdot u$$

$\mathfrak{M}_u$ = set of ergodic, tr.-inv. GGMs with tilt u

Theorem 1 (Funaki & Spohn, CMP 1997)
Suppose V is uniformly strictly convex. Then

$$|\mathfrak{M}_u| = 1 \qquad \forall u \in \mathbb{R}^d$$

Main ideas:

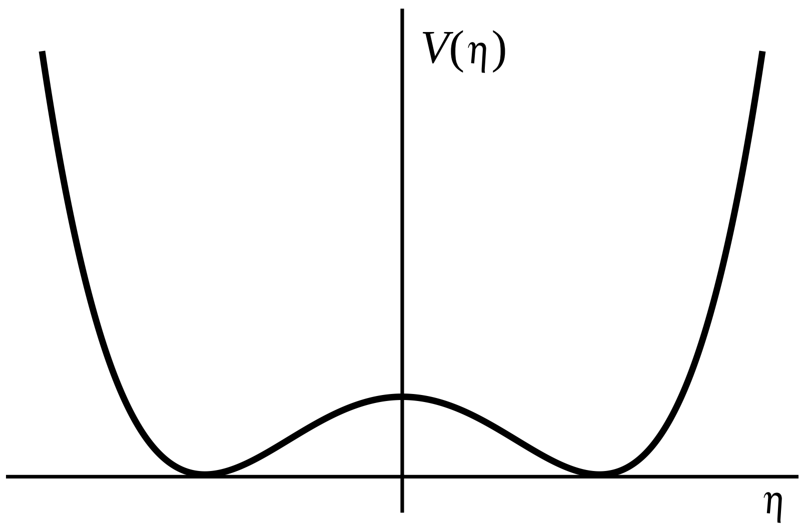
Brascamp-Lieb inequality
coupling via Langevin dynamics

Both need $V''(\eta) \geq c > 0$

Extensions: Sheffield, 2003

Beyond strict convexity I :

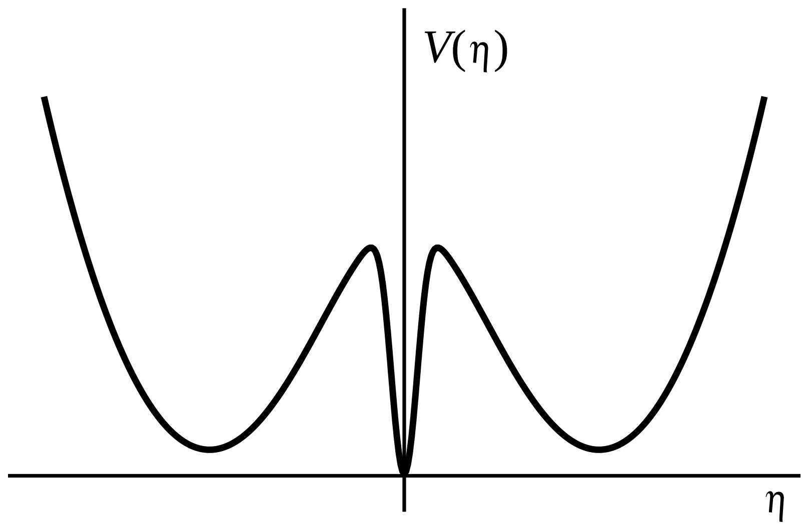
(a)



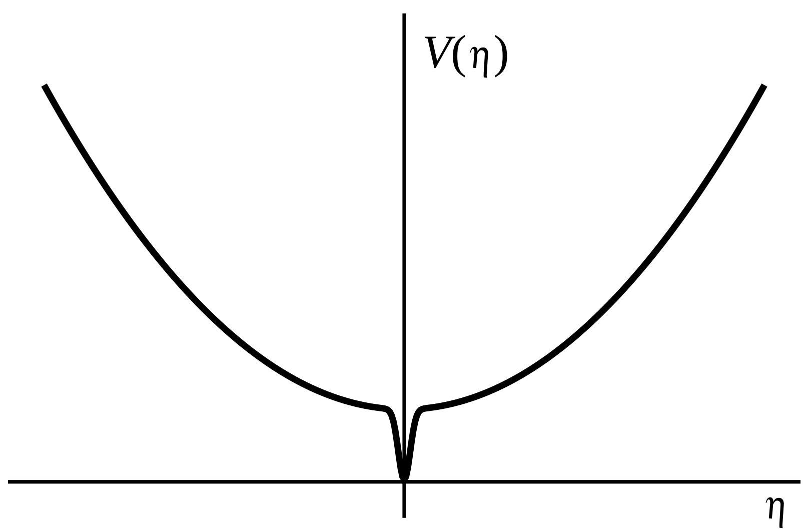
Ground states = ice model = disorder

Beyond strict convexity II:

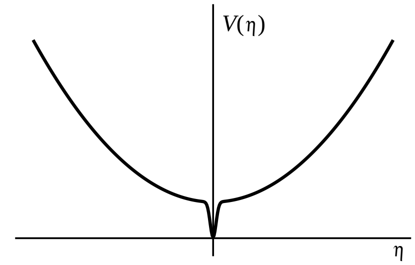
(b)



(c)



$$\mathrm{e}^{-V(\eta)} = p\,\mathrm{e}^{-\frac{1}{2}\kappa_{\mathrm{O}}\eta^2} + (1-p)\mathrm{e}^{-\frac{1}{2}\kappa_{\mathrm{D}}\eta^2}$$



Main results (d=2):

$$\mathrm{e}^{-V_p(\eta)} = p\,\mathrm{e}^{-\frac{1}{2}\kappa_{\mathrm{O}}\eta^2} + (1-p)\mathrm{e}^{-\frac{1}{2}\kappa_{\mathrm{D}}\eta^2}$$

$\mathfrak{M}_{u,p}$ = ergodic GGMs for $V = V_p$ & tilt u

Theorem 2 *Consider model (c) with $\kappa_{\mathrm{O}} \gg \kappa_{\mathrm{D}}$. Then $\exists p_{\mathrm{t}} \in (0,1)$ such that*

$$|\mathfrak{M}_{0,p_{\mathrm{t}}}| \geq 2$$

In fact, $\exists \mu^{\mathrm{ord}}, \mu^{\mathrm{dis}} \in \mathfrak{M}_{0,p_{\mathrm{t}}}$ such that

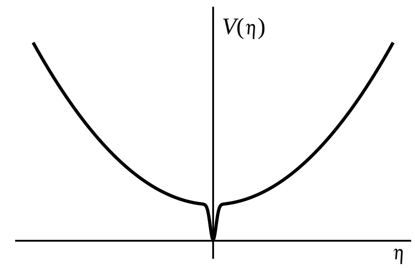
$$\eta_b \sim \begin{cases} \frac{1}{\sqrt{\kappa_{\mathrm{O}}}} & \textit{in } \mu^{\mathrm{ord}} \\ \frac{1}{\sqrt{\kappa_{\mathrm{D}}}} & \textit{in } \mu^{\mathrm{dis}} \end{cases}$$

Order-disorder transition $\frac{1}{\sqrt{\kappa_{\mathrm{O}}}} \ll \frac{1}{\sqrt{\kappa_{\mathrm{D}}}}$

Tools:

graphical representation
chessboard estimates

Graphical representation :



$$\mathrm{e}^{-V_p(\eta)} = \underbrace{p\,\mathrm{e}^{-\frac{1}{2}\kappa_{\mathrm{O}}\eta^2}}_{\text{O-bond}} + \underbrace{(1-p)\mathrm{e}^{-\frac{1}{2}\kappa_{\mathrm{D}}\eta^2}}_{\text{D-bond}}$$

Formally: for each bond

$$\mathrm{e}^{-V(\eta)} = \int_{\{\kappa>0\}} \varrho(\mathrm{d}\kappa)\,\mathrm{e}^{-\frac{1}{2}\kappa\eta^2}$$

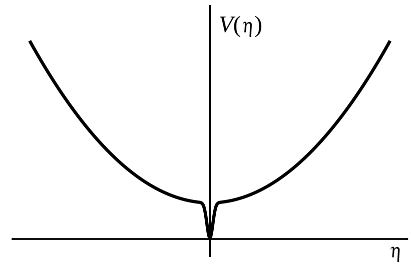
consider joint law of (κ_b, η_b)

Model (c): $\varrho = p\,\delta_{\kappa_{\mathrm{O}}} + (1-p)\delta_{\kappa_{\mathrm{D}}}$

Facts:

Conditional on κ, the η's are Gaussian

Joint measure is reflection positive



Duality (d=2) :

Theorem 3 *Suppose* $\kappa_{\mathrm{O}} \gg \kappa_{\mathrm{D}}$*. Then*

$$\frac{p_{\mathrm{t}}}{1-p_{\mathrm{t}}} = \left(\frac{\kappa_{\mathrm{D}}}{\kappa_{\mathrm{O}}}\right)^{1/4}$$

Consequence of duality:

If p and $p_\star$ are related via

$$\frac{p}{1-p}\,\frac{p_\star}{1-p_\star} = \sqrt{\frac{\kappa_{\mathrm{D}}}{\kappa_{\mathrm{O}}}}$$

Then bond marginals are dual under $\mathrm{O} \leftrightarrow \mathrm{D}$

TRUE on torus w/ proper b.c.

Concluding remarks:

(1) Proof of coexistence: all $d \geq 2$

(2) No control over $u \neq 0$ — absence of RP

(3) Classification of GGMs for given bond marginal

(4) Crossover to 2nd order transition $\kappa_{\mathrm{O}} \gtrsim \kappa_{\mathrm{D}}$