Scaling limit of simple random walk on supercritical percolation clusters

Marek Biskup

(UCLA)

Joint work with
Noam Berger (Caltech)

Bond percolation on $\mathbb{Z}^d$

Let $\mathbb{B}_d$ be the set of n.n. edges in $\mathbb{Z}^d$. Fix $p \in [0, 1]$.

A bond $b \in \mathbb{B}_d$ is *occupied* with probability p and *vacant* with probability $1 - p$, independently of all other bonds.

Formally, $\omega = (\omega_b)_{b \in \mathbb{B}_d}$ i.i.d. 0-1 valued with $\mathbb{P}(\omega_b = 1) = p$.

Percolation transition

Let $\mathscr{C}_\infty = \mathscr{C}_\infty(\omega)$ be the sites "connected to infinity."

Burton-Keane's Theorem: $\mathscr{C}_\infty$ is connected with probability 1.

In $d \geq 2$ there exists $p_c \in (0,1)$ such that

$$\mathbb{P}(0 \in \mathscr{C}_\infty) \begin{cases} = 0, & p < p_c, \\ > 0, & p > p_c. \end{cases}$$

For $p > p_c$ we denote $\Omega_0 = \{0 \in \mathscr{C}_\infty\}$ and $\mathbb{P}_0(\cdot) = \mathbb{P}(\cdot|\Omega_0)$.

Simple random walk on $\mathscr{C}_\infty$

For each $\omega \in \Omega_0$, let $(X_n)_{n\geq 0}$ be the simple random walk on $\mathscr{C}_\infty(\omega)$ started at the origin.

Simple random walk on $\mathscr{C}_\infty$

For each $\omega \in \Omega_0$, let $(X_n)_{n\geq 0}$ be the simple random walk on $\mathscr{C}_\infty(\omega)$ started at the origin.

Explicitly, $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{Z}^d$ with law $P_{0,\omega}$ given by

$$P_{0,\omega}(X_{n+1} = x + e | X_n = x) = \frac{1_{\{\omega_e = 1\}} \circ \tau_x}{\sum_{e' : |e'|=1} 1_{\{\omega_{e'}=1\}} \circ \tau_x}, \quad |e| = 1,$$

and

$$P_{0,\omega}(X_0 = 0) = 1.$$

Here τ_x = shift by x.

Flavors of the problem
I. Lazy vs agile walk

This is the *agile* simple random walk. Another version, the *lazy* walk, is defined as follows:

- At each unit of time, pick one of $2d$ neighbors at random.
- If the bond is occupied, move. Otherwise, stay.

Flavors of the problem
I. Lazy vs agile walk

This is the *agile* simple random walk. Another version, the *lazy* walk, is defined as follows:

- At each unit of time, pick one of $2d$ neighbors at random.
- If the bond is occupied, move. Otherwise, stay.

Same: geometrical image

Different: time parametrization

Flavors of the problem
II. Quenched vs annealed

Quenched problem:
Random walk measure $P_{0,\omega}$ where ω is sampled from a set of full $\mathbb{P}_0$-measure and fixed through all calculations.

Flavors of the problem
II. Quenched vs annealed

Quenched problem:
Random walk measure $P_{0,\omega}$ where ω is sampled from a set of full $\mathbb{P}_0$-measure and fixed through all calculations.

Annealed problem:
Random walk distribution $\mathbb{Q}$ given by

$$\mathbb{Q}(\mathcal{A}) = \mathbb{E}_0\big(P_{0,\omega}(\mathcal{A})\big)$$

Flavors of the problem
II. Quenched vs annealed

Quenched problem:
Random walk measure $P_{0,\omega}$ where ω is sampled from a set of full $\mathbb{P}_0$-measure and fixed through all calculations.

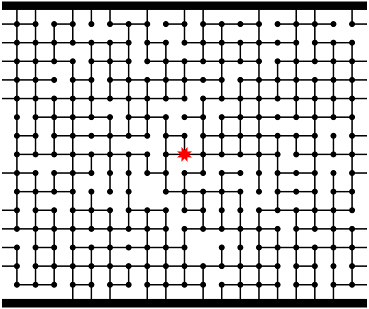
Annealed problem:
Random walk distribution $\mathbb{Q}$ given by

$$\mathbb{Q}(\mathcal{A}) = \mathbb{E}_0\big(P_{0,\omega}(\mathcal{A})\big)$$

Ostensibly different objects

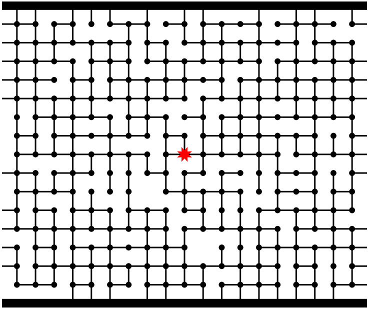
A fundamental question

Is the probability of { walk exits through top side } close to $^1/_2$?



A fundamental question

Is the probability of { walk exits through top side } close to $^1/_2$?



- Trivially true for the *annealed* measure.
- *Quenched* measure: Prove a **Functional CLT**.

Main result

Theorem 1
Let $d \geq 2$, $p > p_{\mathrm{c}}(d)$ and let $\omega \in \Omega_0$. Let $(X_n)_{n\geq 0}$ be the random walk with law $P_{0,\omega}$ and let

$$B_n(t) = \frac{1}{\sqrt{n}}\Big(X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)(X_{\lfloor tn \rfloor + 1} - X_{\lfloor tn \rfloor})\Big), \quad t \geq 0.$$

Then for all $T > 0$ and $\mathbb{P}_0$-a.e. ω, the law of $(B_n(t) \colon 0 \leq t \leq T)$ on $(C[0,T], \mathscr{W}_T)$ converges weakly to the law of an isotropic (non-degenerate) Brownian motion.

A similar result holds for the *lazy* walk as well.

Previous results

Previous results

- **Quenched problem in** $d \geq 4$:

  Sidoravicius & Sznitman (2004)

Previous results

- **Quenched problem in $d \geq 4$:**

  Sidoravicius & Sznitman (2004)

- **Annealed problem**:

  De Masi & Ferrari & Goldstein & Wick (1989)

Previous results

- **Quenched problem in** $d \geq 4$:

  Sidoravicius & Sznitman (2004)

- **Annealed problem**:

  De Masi & Ferrari & Goldstein & Wick (1989)

- **Directed version**:

  Rassoul-Agha & Sepäläinen (2004)

Previous results

- **Quenched problem in** $d \geq 4$:

  Sidoravicius & Sznitman (2004)

- **Annealed problem**:

  De Masi & Ferrari & Goldstein & Wick (1989)

- **Directed version**:

  Rassoul-Agha & Sepäläinen (2004)

- **Walk among random conductances**:

  Kozlov (1985), Kipnis & Varadhan (1986),
  Sidoravicius & Sznitman (2004), Fontes & Mathieu (2004)

Previous results

- **Quenched problem in $d \geq 4$:**

  Sidoravicius & Sznitman (2004)

- **Annealed problem**:

  De Masi & Ferrari & Goldstein & Wick (1989)

- **Directed version**:

  Rassoul-Agha & Sepäläinen (2004)

- **Walk among random conductances**:

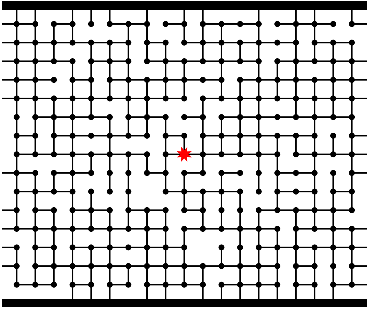
  Kozlov (1985), Kipnis & Varadhan (1986),
  Sidoravicius & Sznitman (2004), Fontes & Mathieu (2004)

- **Heat-kernel estimates**:

  Barlow (2004)

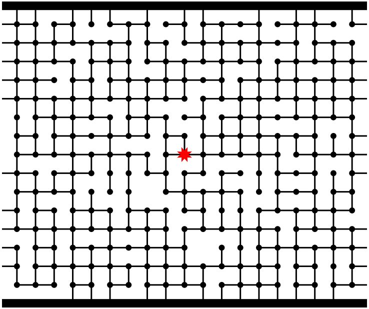
Main idea

Geometric embedding of $\mathscr{C}_\infty$:



Main idea

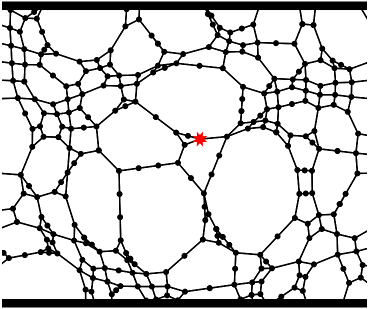
Geometric embedding of $\mathscr{C}_\infty$:



The walk (X_n) is *not* a martingale.

Main idea

Harmonic embedding of $\mathscr{C}_\infty$: $x \mapsto x + \chi(x, \omega)$



The walk $X_n + \chi(X_n, \omega)$ *is* a martingale.

Exit problem revisited

A martingale calculation:

$$P_{0,\omega}(\text{exits thru top}) = \frac{1}{2}\left(1 + \frac{e_2 \cdot \chi(0,\omega)}{\text{width}}\right)$$

Will need to show that $|\chi(0,\omega)| \ll$ width.

Corrector
Probabilistic construction

Natural candidate for the *corrector*:

$$\chi(x, \omega) = \lim_{n\to\infty} \left[E_{x,\omega}(X_n) - E_{0,\omega}(X_n) \right].$$

Corrector
Probabilistic construction

Natural candidate for the *corrector*:

$$\chi(x,\omega) = \lim_{n\to\infty}\big[E_{x,\omega}(X_n) - E_{0,\omega}(X_n)\big].$$

- Trivially harmonic
- Existence of limit unclear

Corrector

Analytic construction (Kipnis & Varadhan)

Proposition 2 ($d \geq 2$)

There exists a function $\chi : \mathbb{Z}^d \times \Omega_0 \to \mathbb{R}^d$ such that, for $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$:

(0) $\chi(0, \omega) = 0$.

(1) $x \mapsto x + \chi(x, \omega)$ *is harmonic on* $\mathscr{C}_\infty(\omega)$.

(2) χ *is a gradient field on* $\mathscr{C}_\infty$:

$$\chi(x, \omega) - \chi(y, \omega) = \chi(x - y, \tau_y \omega), \qquad x, y \in \mathscr{C}_\infty.$$

(3) *The gradients of* χ *are square integrable:*

$$\left\| [\chi(x + e, \omega) - \chi(x, \omega)] 1_{\{x, x+e \in \mathscr{C}_\infty\}} \right\|_2 < C, \qquad |e| = 1.$$

Deformed random walk

The listed properties make

$$M_n = X_n + \chi(X_n, \omega)$$

an L^2-martingale.

Deformed random walk

The listed properties make

$$M_n = X_n + \chi(X_n, \omega)$$

an L^2-martingale.

Martingale CLT + some ergodicity $\Rightarrow$

The deformed walk scales to Brownian motion

Controlling the deformation

$d = 2$ for now

Need to show that

$$\max_{1 \le k \le n} \left|\chi(X_k, \omega)\right| = o(\sqrt{n}).$$

Since $M_n = O(\sqrt{n})$, it suffices to prove:

Controlling the deformation
$d = 2$ for now

Need to show that

$$\max_{1 \le k \le n} \left|\chi(X_k, \omega)\right| = o(\sqrt{n}).$$

Since $M_n = O(\sqrt{n})$, it suffices to prove:

Proposition 3 ($d = 2$)

For $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$,

$$\lim_{n \to \infty} \max_{\substack{x \in \mathscr{C}_\infty(\omega) \\ |x| \le n}} \frac{|\chi(x, \omega)|}{n} = 0.$$

Some ergodic theory
Induced shift

For $\omega \in \Omega_0$, let $(x_n)_{n\in\mathbb{Z}}$ be the intersections of $\mathscr{C}_\infty(\omega)$ with x-axis labeled so that $x_n < x_{n+1}$ and $x_0 = 0$.

Some ergodic theory

Induced shift

For $\omega \in \Omega_0$, let $(x_n)_{n\in\mathbb{Z}}$ be the intersections of $\mathscr{C}_\infty(\omega)$ with x-axis labeled so that $x_n < x_{n+1}$ and $x_0 = 0$.

Consider the *induced shift* $\sigma\colon \Omega_0 \to \Omega_0$

$$\sigma(\omega) = \tau_{x_1(\omega)}(\omega), \qquad \omega \in \Omega_0.$$

Standard arguments show:

Some ergodic theory

Induced shift

For $\omega \in \Omega_0$, let $(x_n)_{n\in\mathbb{Z}}$ be the intersections of $\mathscr{C}_\infty(\omega)$ with x-axis labeled so that $x_n < x_{n+1}$ and $x_0 = 0$.

Consider the *induced shift* $\sigma\colon \Omega_0 \to \Omega_0$

$$\sigma(\omega) = \tau_{x_1(\omega)}(\omega), \qquad \omega \in \Omega_0.$$

Standard arguments show:

Lemma 4 ($d \geq 2$)

σ is $\mathbb{P}_0$-preserving and ergodic.

Along coordinate axes

Now set

$$\Psi(\omega) = \chi\big(x_1(\omega), \omega\big) - \chi(0, \omega)$$

Along coordinate axes

Now set

$$\Psi(\omega) = \chi\big(x_1(\omega), \omega\big) - \chi(0, \omega)$$

Then

$$\chi\big(x_n(\omega), \omega\big) = \sum_{k=1}^{n} \Psi \circ \sigma^k(\omega)$$

Along coordinate axes

Now set

$$\Psi(\omega) = \chi\big(x_1(\omega), \omega\big) - \chi(0, \omega)$$

Then

$$\chi\big(x_n(\omega), \omega\big) = \sum_{k=1}^{n} \Psi \circ \sigma^k(\omega)$$

But $\Psi \in L^1$ (Antal-Pisztora) and

$$\mathbb{E}_0(\Psi) = 0$$

and so the Ergodic Theorem implies:

Along coordinate axes

Now set

$$\Psi(\omega) = \chi\big(x_1(\omega), \omega\big) - \chi(0, \omega)$$

Then

$$\chi\big(x_n(\omega), \omega\big) = \sum_{k=1}^{n} \Psi \circ \sigma^k(\omega)$$

But $\Psi \in L^1$ (Antal-Pisztora) and

$$\mathbb{E}_0(\Psi) = 0$$

and so the Ergodic Theorem implies:

Corollary 5 ($d \geq 2$)

For $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$,

$$\lim_{n\to\infty} \frac{\chi(x_n(\omega), \omega)}{n} = 0.$$

Weaving webs of goodness

Good lines and sites

Let $K, \epsilon > 0$ and $\omega \in \Omega_0$. The x-axis is called *good in* ω if

$$\left|\chi(x, \omega)\right| \leq K + \epsilon|x|$$

for every $x \in \mathscr{C}_\infty$ on x-axis.

Weaving webs of goodness

Good lines and sites

Let $K, \epsilon > 0$ and $\omega \in \Omega_0$. The x-axis is called *good in* ω if

$$\big|\chi(x,\omega)\big| \le K + \epsilon|x|$$

for every $x \in \mathscr{C}_\infty$ on x-axis.

A site $x \in \mathbb{Z}^d$ is called *good in* ω if

- $x \in \mathscr{C}_\infty(\omega)$
- Both x and y-axes are good in $\tau_x(\omega)$.

Weaving webs of goodness

Good grid

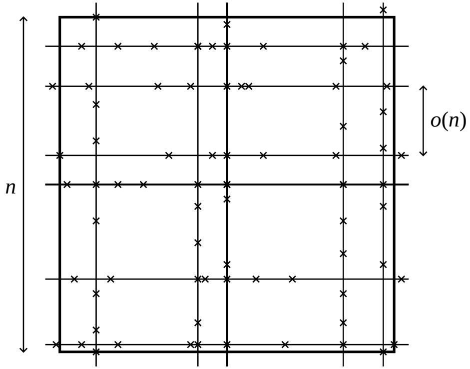
For $\mathbb{P}_0$-a.e. ω and all $\epsilon > 0$:

- Origin is good if K is large
- Good sites appear with positive density along both axes

Weaving webs of goodness
Good grid

For $\mathbb{P}_0$-a.e. ω and all $\epsilon > 0$:

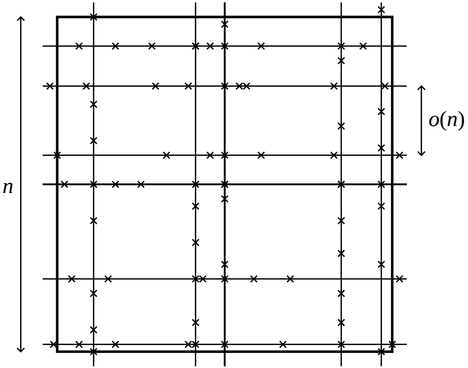
- Origin is good if K is large
- Good sites appear with positive density along both axes



Weaving webs of goodness
Sublinearity everywhere

Weaving webs of goodness
Sublinearity everywhere

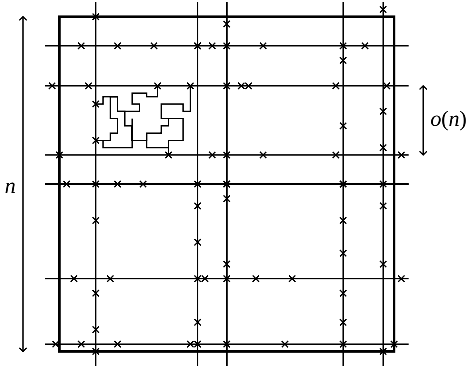
Maximum on good grid: $\leq 2K + 2\epsilon n$.



Weaving webs of goodness

Sublinearity everywhere

Maximum on good grid: $\leq 2K + 2\epsilon n$.

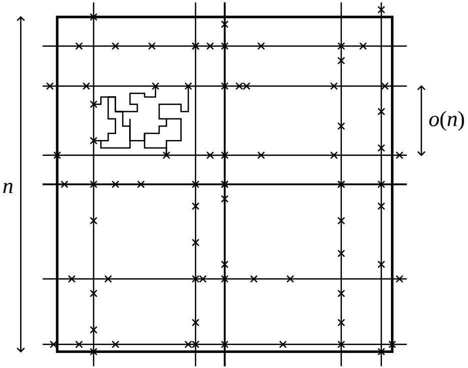


Weaving webs of goodness
Sublinearity everywhere

Maximum on good grid: $\leq 2K + 2\epsilon n$.

AND the fact that

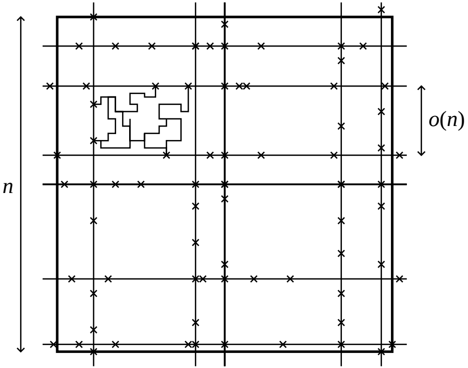
$x \mapsto x + \chi(x, \omega)$
is harmonic



Weaving webs of goodness
Sublinearity everywhere

Maximum on good grid: $\leq 2K + 2\epsilon n$.

AND the fact that



$x \mapsto x + \chi(x,\omega)$
is harmonic

imply:

$$\max_{\substack{x \in \mathscr{C}_\infty(\omega) \\ |x| \leq n}} \big|\chi(x,\omega)\big| \leq 2K + 2\epsilon n + o(n) \qquad \square$$

Higher dimensions
A density bound on corrector

We do not see how to extend this argument to $d \geq 3$.
But we can prove:

Proposition 6 ($d \geq 3$)

For $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$ and all $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{\substack{x\in\mathscr{C}_\infty(\omega) \\ |x|\leq n}} 1_{\{|\chi(x,\omega)|\geq\epsilon n\}} = 0.$$

Higher dimensions
Main idea

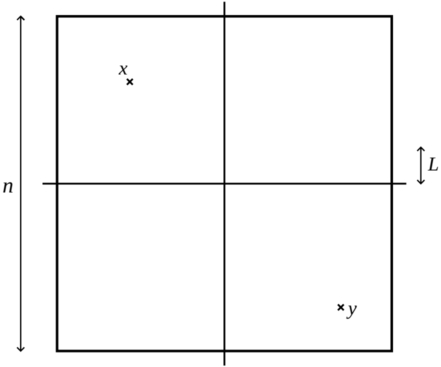
$n \times n$ square in $\mathbb{Z}^3$

WANT:

$$\left|\chi(x,\omega) - \chi(y,\omega)\right| \le \epsilon n$$

for (most of) good

$$x, y \in \mathscr{C}_\infty \cap \text{ square}$$



Higher dimensions
Main idea

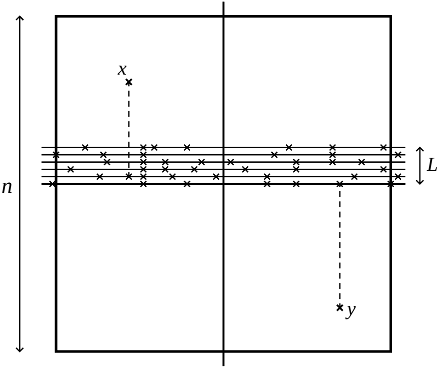
$n \times n$ square in $\mathbb{Z}^3$

WANT:

$$\left|\chi(x,\omega) - \chi(y,\omega)\right| \le \epsilon n$$

for (most of) good

$$x, y \in \mathscr{C}_\infty \cap \text{ square}$$



Higher dimensions
Main idea

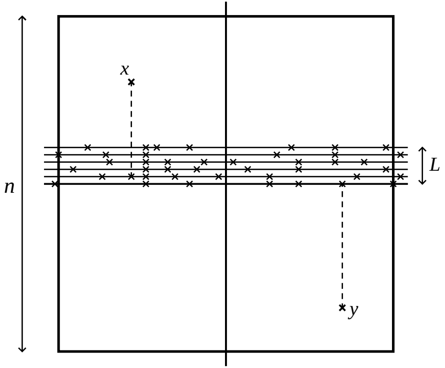
$n \times n$ square in $\mathbb{Z}^3$

WANT:

$$\left|\chi(x,\omega) - \chi(y,\omega)\right| \le \epsilon n$$

for (most of) good

$$x, y \in \mathscr{C}_\infty \cap \text{ square}$$



TRUE FOR:

- Nearly all (good) sites in $\mathscr{C}_\infty \cap$ square
- Nearly fraction $P_\infty = \mathbb{P}(0 \in \mathscr{C}_\infty)$ of $\mathscr{C}_\infty \cap$ cube

Now stack $M \gg 1$ of these squares on top of each other □

Final touches

To finish, we will need:

Theorem 7 (Barlow 2004)

For $\mathbb{P}_0$-a.e. ω and all $x \in \mathscr{C}_\infty(\omega)$,

$$P_{0,\omega}(X_n = x) \le \frac{c_1}{n^{d/2}} \exp\{-c_2 \tfrac{|x|^2}{n}\},$$

once n is sufficiently large.

Final touches

To finish, we will need:

Theorem 7 (Barlow 2004)

For $\mathbb{P}_0$-a.e. ω and all $x \in \mathscr{C}_\infty(\omega)$,

$$P_{0,\omega}(X_n = x) \le \frac{c_1}{n^{d/2}} \exp\{-c_2 \tfrac{|x|^2}{n}\},$$

once n is sufficiently large.

Combined with Proposition 6, we then have

$$\frac{|\chi(X_n,\omega)|}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } P_{0,\omega}\text{-probability.}$$

This implies the CLT in $d \ge 3$. $\square$

Future research
Everybody welcome

Limit laws:

- Maximum bound on corrector in $d \geq 3$
- Other graphs, e.g., Voronoi percolation
- Local CLT
- Long-range percolation (stable processes)

Future research
Everybody welcome

Limit laws:

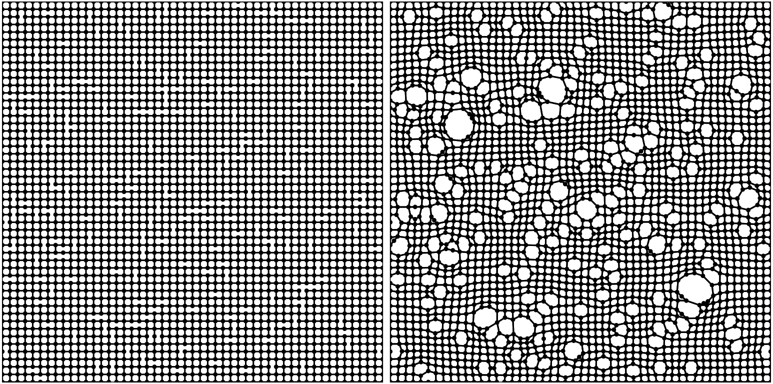
- Maximum bound on corrector in $d \geq 3$
- Other graphs, e.g., Voronoi percolation
- Local CLT
- Long-range percolation (stable processes)

Corrector:

- Actual size
- Scaling limit (Gaussian free field?)
- Behavior as $p \downarrow p_{\mathrm{c}}$

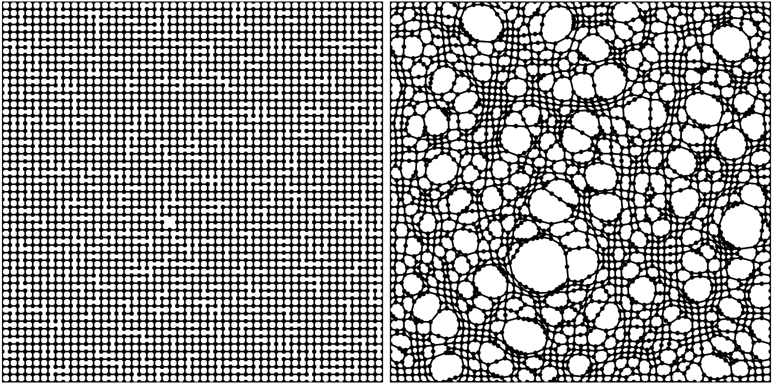
Playing with corrector

Percolation cluster and its deformation: $p = 0.95$



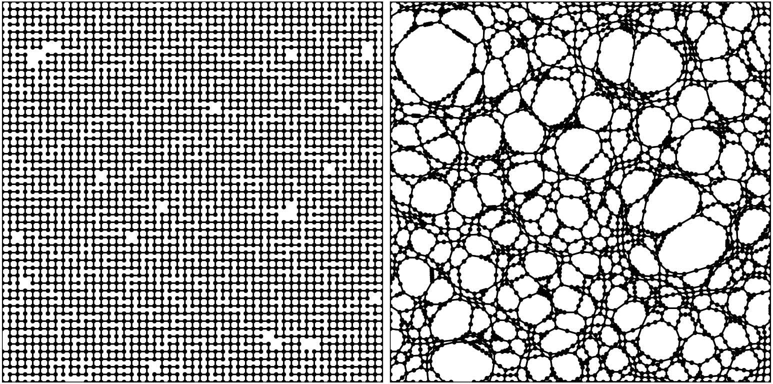
Playing with corrector

Percolation cluster and its deformation: $p = 0.85$



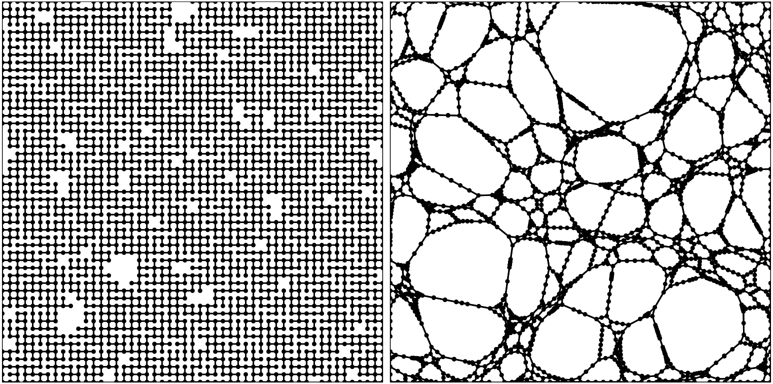
Playing with corrector

Percolation cluster and its deformation: $p = 0.75$



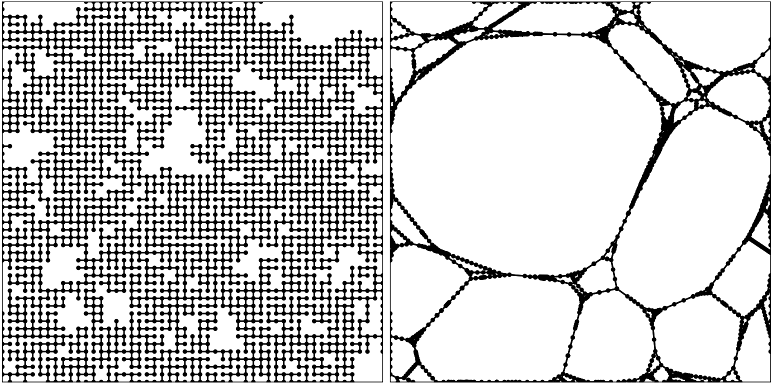
Playing with corrector

Percolation cluster and its deformation: $p = 0.65$



Playing with corrector

Percolation cluster and its deformation: $p = 0.55$



THE END