A perturbative proof of the Lee-Yang Circle Theorem

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based on joint work with

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PLAN

(1) Historical perspective: theory of phase transitions

(2) Lee-Yang Circle Theorem (and proof)

(3) Perturbative approach

PHASE TRANSITIONS

Examples:

(1) Water freezes & boils

(2) Permanent magnetism

(3) Superconductivity, etc.

Common feature: collective phenomenon

ISING MODEL

- $\Lambda \subset \mathbb{Z}^d$ finite set
- $\sigma_\Lambda \in \{-1, +1\}^\Lambda$ spin configuration
- Probability measure
$$\mathbb{P}(\{\sigma_\Lambda\}) = \frac{1}{\mathcal{Z}_\Lambda(J,h)} \Big(\prod_{\langle x,y \rangle} e^{J\sigma_x\sigma_y} \Big) \prod_{x\in\Lambda} e^{h\sigma_x}$$
- Partition function
$$\mathcal{Z}_\Lambda(J,h) = \sum_{\sigma_\Lambda} \Big(\prod_{\langle x,y \rangle} e^{J\sigma_x\sigma_y} \Big) \prod_{x\in\Lambda} e^{h\sigma_x}$$
- Parameters:
  $J \geq 0$ (coupling constant)
  $h \in \mathbb{R}$ (magnetic field)
- Plus-minus symmetry: $\sigma \leftrightarrow -\sigma$ & $h \leftrightarrow -h$

PHASE TRANSITION IN ISING MODEL

- Quantity of interest

$$m_\Lambda(J,h) = \frac{\mathrm{d}}{\mathrm{d}h}\frac{1}{|\Lambda|}\log \mathcal{Z}_\Lambda(J,h)$$

- Thermodynamic limit

$$m_\star(J,h) = \lim_{\Lambda \nearrow \mathbb{Z}^d} m_\Lambda(J,h)$$

- Symmetry: $m_\star(J,-h) = -m_\star(J,h)$

- Phase transition: In $d \geq 2$ $\exists J_\mathsf{c} \in (0,\infty)$ such that

$$\lim_{h\downarrow 0} m_\star(J,h) \begin{cases} > 0 & J > J_\mathsf{c} \\ = 0 & J < J_\mathsf{c} \end{cases}$$

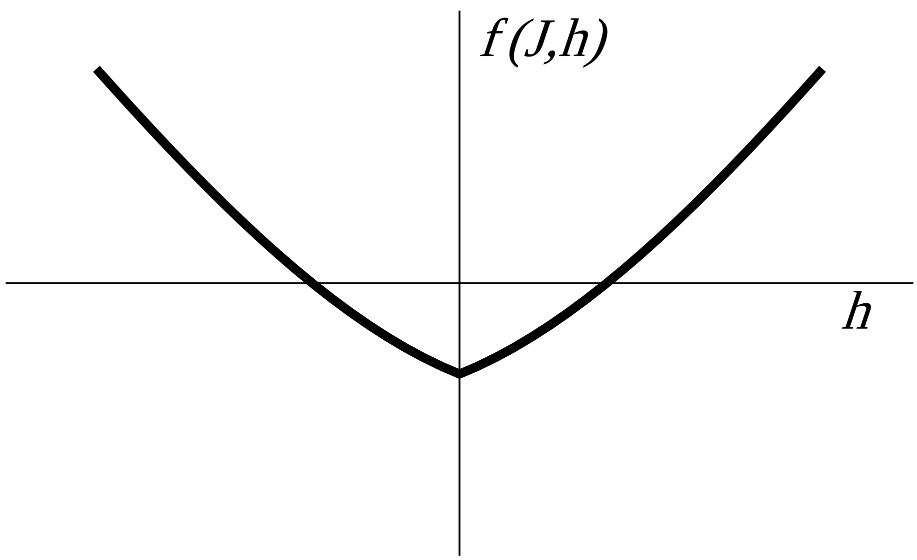
($J_\mathsf{c} = \infty$ in $d = 1$)

ORIGINS OF SINGULARITY

Free energy/pressure

$$f(J,h) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \mathcal{Z}_\Lambda(J,h)$$

Non-analyticity at $h = 0$ when $J > J_c$:



But $h \mapsto \frac{1}{|\Lambda|} \log \mathcal{Z}_\Lambda(J,h)$ (real) analytic $\forall h$!

LEE & YANG'S IDEA

Let $z = e^{2h}$ and fix $J \geq 0$. Then

$$\mathcal{Z}_\Lambda(J,h) = z^{-|\Lambda|/2} Q_{|\Lambda|}(z)$$

where $Q_{|\Lambda|}(z)$ is a polynomial in z with positive coefficients.

- Non-analyticity caused by complex zeros of $Q_{|\Lambda|}$ wandering onto the (physical part of) real axis.

- Thus, $h \mapsto f(J,h)$ has analytic continuation into regions where $Q_{|\Lambda|}$ has no zeros for any Λ.

LEE-YANG CIRCLE THEOREM

Theorem 1 (Lee & Yang, 1952) *For every $J \geq 0$ and all finite $\Lambda \subset \mathbb{Z}^d$, all zeros of $Q_{|\Lambda|}$ lie on the unit circle in $\mathbb{C}$.*

GENERALIZED SETUP

More general setting:

$$Q_n(z_1, \ldots, z_n) = \sum_{S \subset \{1,\ldots,n\}} z^S \prod_{\substack{k \in S \\ \ell \notin S}} A_{k,\ell}$$

where $z^S = \prod_{k \in S} z_k$.

Theorem 2 (Lee & Yang, 1952) *Suppose that for all* $k, \ell = 1, \ldots, n$ *the coefficients* $(A_{k,\ell})$ *obey*

(1) $A_{k,\ell} = A_{\ell,k}$

(2) $A_{k,\ell} \in [-1, 1]$.

Then $|z_1|, \ldots, |z_{n-1}| \geq 1$ *and* $Q_n(z_1, \ldots, z_n) = 0$ *imply that* $|z_n| \leq 1$.

Lemma 3 *For any* $z_1,\dots,z_n \in \mathbb{C}$ *we have*

$$Q_n\Big(\frac{1}{z_1^*},\dots,\frac{1}{z_n^*}\Big) = \frac{1}{(z_1\dots z_n)^*}\,Q_n(z_1,\dots,z_n)^*$$

In particular, if

$$|z_1| = |z_2| = \dots = |z_{n-1}| = 1$$

and $Q_n(z_1,\dots,z_n) = 0$*, then* $|z_n| = 1$*.*

Proof. The first line follows from $A_{k,\ell} \in \mathbb{R}$ and the fact that Q_n is linear in each variable. To get the rest we note that $z \mapsto 1/z^*$ is identity map on $\{z\colon |z| = 1\}$. Thus $1/z_k^* = z_k$ for every $k = 1,\dots,n-1$ and therefore

$$\frac{1}{z_n^*} = z_n.$$

i.e., $|z_n|^2 = 1$. □

HOMOGENEITY RELATION

Lemma 4 *Suppose that* $A_{k,\ell} \neq 0$*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}z_n} Q_n(z_1, \ldots, z_n) \\ = (A_{n,1} \ldots A_{n,n-1})\, Q_{n-1}\Big(\frac{z_1}{A_{n,1}}, \ldots, \frac{z_{n-1}}{A_{n,n-1}}\Big).$$

Proof. The derivative forces $n \in S$ and so the coefficient of z^S contains all $A_{n,\ell}$ with $\ell \notin S$. Taking out $A_{n,1} \ldots A_{n,n-1}$, we have to divide each z_k by $A_{n,k}$ to compensate. $\square$

NOTE: Two-body interaction essential

PROOF OF THEOREM 2

Continuity: Assume that $A_{k,\ell} \in (-1, 1) \setminus \{0\}$.

Induction argument: Holds for $n = 1$ because $Q_1(z_1) = 1 + z_1$.

Now suppose Theorem 2 holds up to $n-1$ and fix $z_1, \ldots, z_{n-2}$ outside the open unit disc.

Define a rational function $\phi\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that for each $z \in \overline{\mathbb{C}}$,

$$Q_n(z_1, \ldots, z_{n-2}, \phi(z), z) = 0.$$

The proof hinges on the fact that $|\phi(z)| < 1$ for $|z|$ sufficiently large.

$|\phi(z)| < 1$ FOR z LARGE

- $\phi(z)$ bounded for $z \to \infty$ ($A_{k,\ell} \neq 0$ implies that all coefficients nonzero).

- Hence, as $z \to \infty$ we must have

$$\frac{\mathsf{d}}{\mathsf{d}z_n} Q_n(z_1, \ldots, \phi(z), z_n) \to 0$$

Letting $z_{n-1} = \phi(\infty)$, by Lemma 4 we thus have

$$Q_{n-1}\left(\frac{z_1}{A_{n,1}}, \ldots, \frac{z_{n-1}}{A_{n,n-1}}\right) = 0.$$

But $|z_k/A_{n,k}| > |z_k| \geq 1$ and so by induction assumption

$$|z_{n-1}| < \left|\frac{z_{n-1}}{A_{n,n-1}}\right| \leq 1,$$

i.e. $|\phi(\infty)| < 1$.

BACK TO THE PROOF

Let now $z_1, \ldots, z_n \in \mathbb{C}$ be such that

$$|z_1|, \ldots, |z_{n-1}| \geq 1$$

and $Q_n(z_1, \ldots, z_n) = 0$.

Suppose now that $|z_n| > 1$. Then we define $z_{n-1}(\lambda) = \phi(\lambda z_n)$ and increase λ from 1 to ∞. By previous reasoning, $z_{n-1}(\lambda)$ must visit unit disc before λ reaches ∞. Stop when unit circle hit.

Do this for all $z_1, \ldots, z_{n-1}$ to produce a collection $\tilde{z}_1, \ldots, \tilde{z}_n$ with

$$|\tilde{z}_1| = \cdots = |\tilde{z}_{n-1}| = 1 < |\tilde{z}_n| < \infty$$

and $Q_n(\tilde{z}_1, \ldots, \tilde{z}_n) = 0$. This is in contradiction with Lemma 3. $\square$

DEFICIENCIES

(1) Restricted to two-body interactions.

(2) No info where the zeros are.

(3) Too dependent on symmetries.

PERTURBATIVE APPROACH

Restricted to:

(1) $d \geq 2$ (based on phase transition techniques)

(3) $\Lambda =$ lattice torus (periodic b.c.)

(4) $J \gg 1$ to enable contour arguments.

Notation:

- $\Lambda_L =$ lattice torus of $L \times \ldots L$ sites
- $Z_L(z) = \mathcal{Z}_{\Lambda_L}(J, h)$ for $z = e^h$

REPRESENTATION OF Z_L

Theorem 5 (BBCKK, 2003) *Let $d \geq 2$ and $J \gg 1$. Then there exist functions $\zeta_\pm \colon \mathbb{C} \to \mathbb{C}$ such that Ξ_L defined by*

$$Z_L(z) = \zeta_+(z)^{L^d} + \zeta_-(z)^{L^d} + \Xi_L(z)$$

satisfies, for some $\tau > 0$,

$$|\Xi_L(z)| \leq e^{-\tau L} \max\Big\{|\zeta_+(z)^{L^d}|, |\zeta_-(z)^{L^d}|\Big\}$$

for all $z \in \mathbb{C}$ and all L sufficiently large.

Moreover, we have

(1) $\zeta_\pm$ are C^2 everywhere, with ζ_+ analytic on $\{z\colon |\zeta_+(z)| > |\zeta_-(z)|\}$ and vice versa.

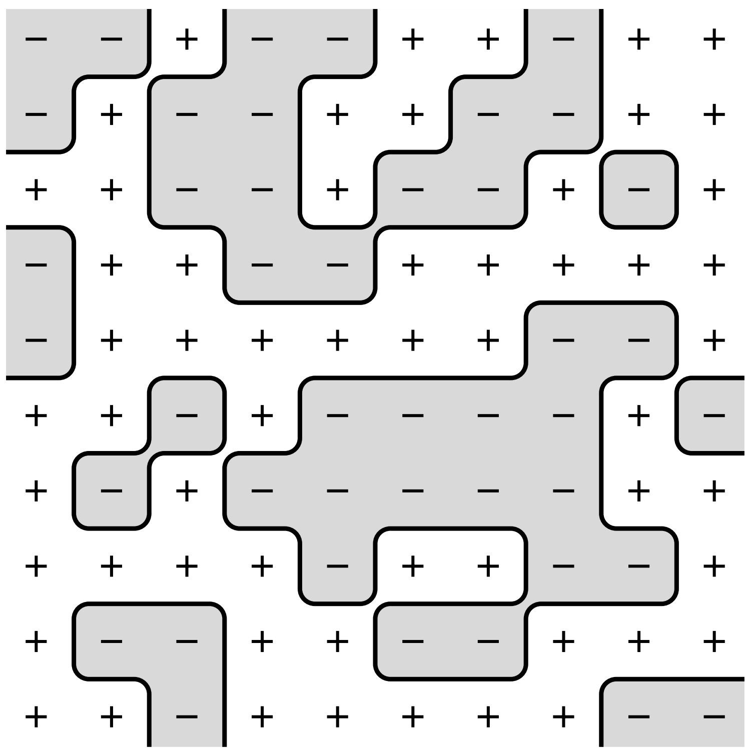
(2) $\zeta_\pm(z) = z^{\pm 1} \exp\{s(z)\}$ where

$$|s(z)|,\ |\partial_z s(z)|,\ |\partial_{\bar{z}} s(z)| \leq e^{-c_1 J}$$

for some $c_1 > 0$ and all $z \in \mathbb{C}$.

IDEA OF PROOF

Contour representation:



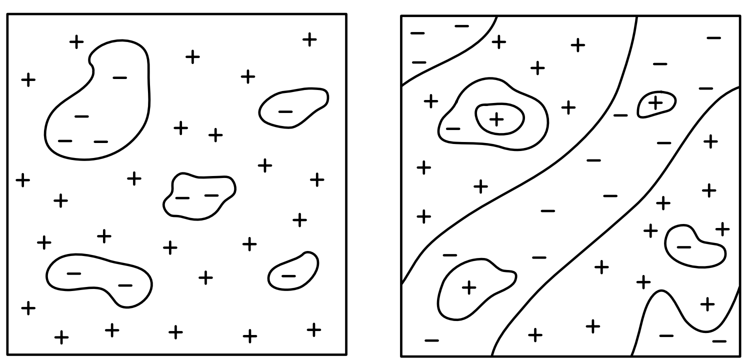
Each contour γ "costs" $e^{-2J|\gamma|}$.

MORE DETAILS

No contours ($J = \infty$)

$$Z_L(z) = z^{L^d} + z^{-L^d}$$

For J very large, we decompose Z_L as follows:



LOCALIZING ZEROS

Theorem 6 (BBCKK, 2003) *There exist constants $C, L_0 \in (0,1)$ such that for all $L \geq L_0$, all zeros of Z_L*

(1) are non-degenerate

(2) lie within $Ce^{-\tau L}$ of the solutions to

$$|\zeta_+(z)| = |\zeta_-(z)|$$

$$L^d(\arg\zeta_+(z) - \arg\zeta_-(z)) = \pi \mod 2\pi$$

(3) lie on the unit circle in $\mathbb{C}$ with neighboring zeros further than $O(L^{-d})$ apart.