Scaling limit of simple random walk on supercritical percolation clusters

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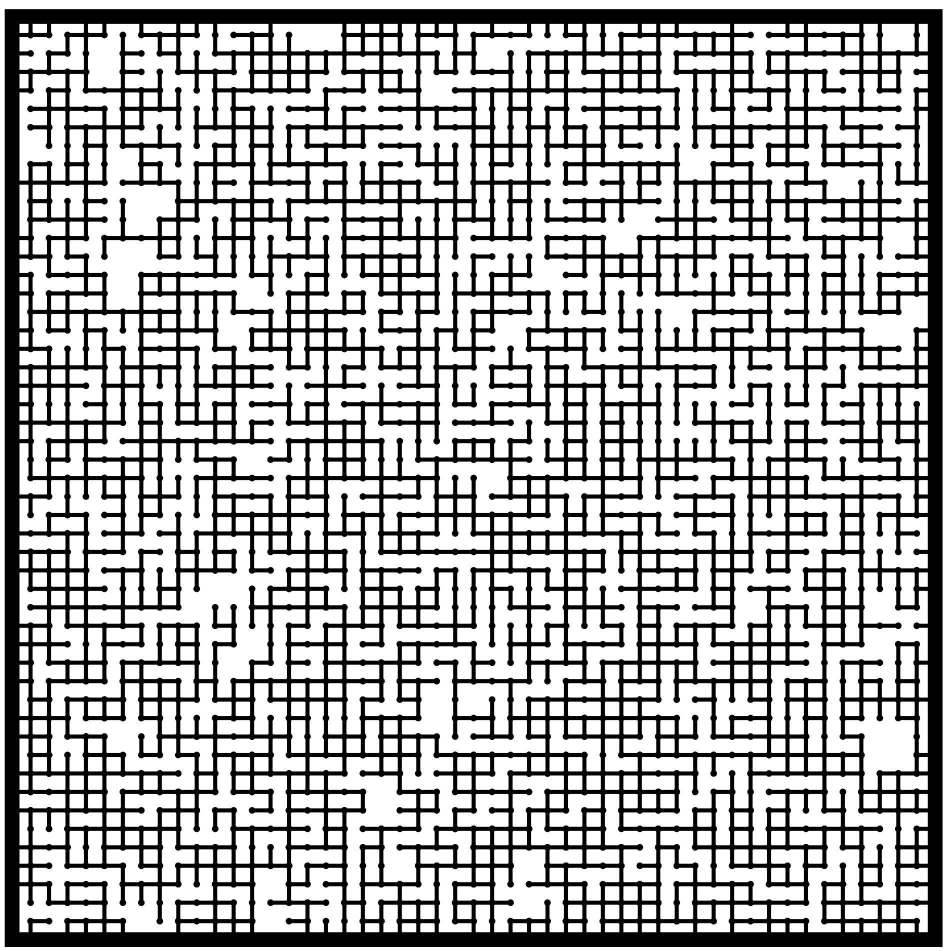
Bond percolation on $\mathbb{Z}^d$:

Assume: $d \geq 2$ & $p > p_c(d)$

Let $\mathscr{C}_\infty$ = unique infinite cluster

Conditional measure $\mathbb{P}_0(-) = \mathbb{P}(-|0 \in \mathscr{C}_\infty)$



Simple random walk on $\mathscr{C}_\infty$:

Let $X_0, X_1, \dots$ be Markov chain with law $P_{0,\omega}$:

$$P_{0,\omega}(X_{n+1} = x + e | X_n = x)$$

$$= \frac{1_{\{\omega_e = 1\}} \circ \tau_x}{\sum_{e' :\, |e'|=1} 1_{\{\omega_{e'} = 1\}} \circ \tau_x}$$

for $|e| = 1$ and

$$P_{0,\omega}(X_0 = 0) = 1.$$

Here $(\tau_x \omega)_y = \omega_{y-x}$

This is *agile* walk.

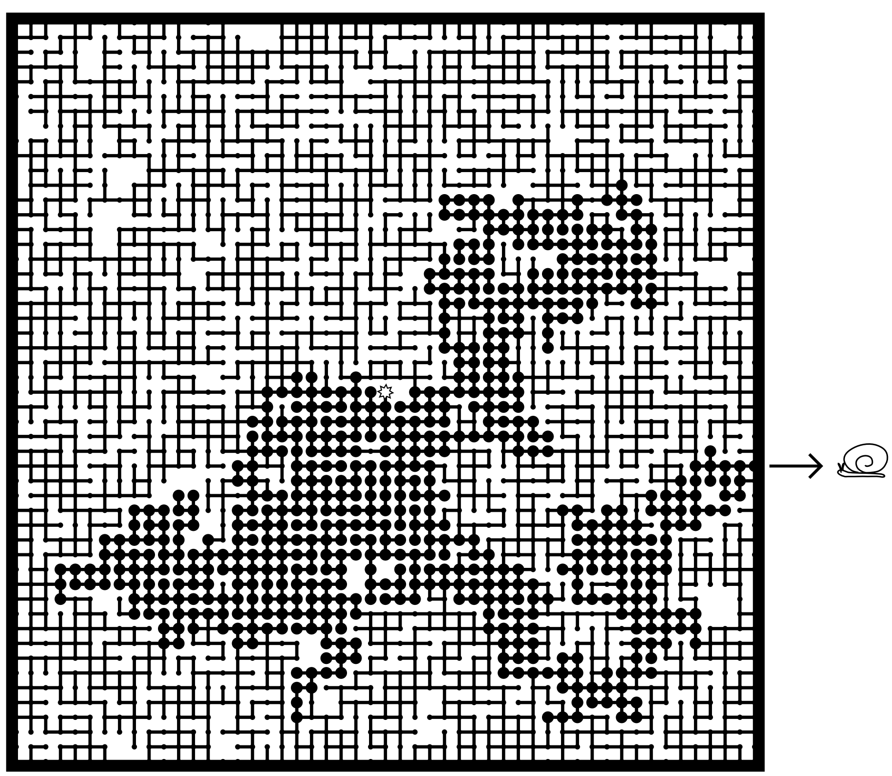
Lazy walk: Picks one of $2d$ neighbors; move suppressed if no bond to it.

Main questions:

- Exit point distribution

- Exit time

- Scaling limit of the path



Main result :

Will prove functional CLT for the path:

Theorem 1 ($d \geq 2$ & $p > p_c(d)$) *For $t \geq 0$, let*

$$B_n^\omega(t) = \frac{1}{\sqrt{n}}\Big(X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)(X_{\lfloor tn \rfloor + 1} - X_{\lfloor tn \rfloor})\Big)$$

Then $\forall T > 0$ and $\mathbb{P}_0$-a.e. ω, the law of

$$(B_n^\omega(t) \colon 0 \leq t \leq T)$$

converges, as $n \to \infty$, to the law of an isotropic and non-degenerate Brownian motion.

Holds for both *lazy* and *agile* walk
(with different diffusion constants)

Previous results:

- **Quenched problem in $d \geq 4$:**
  Sidoravicius-Sznitman (2004)

- **Annealed problem**:
  De Masi & Ferrari & Goldstein & Wick (1989)

- **Directed version**:
  Rassoul-Agha & Sepäläinen (2004)

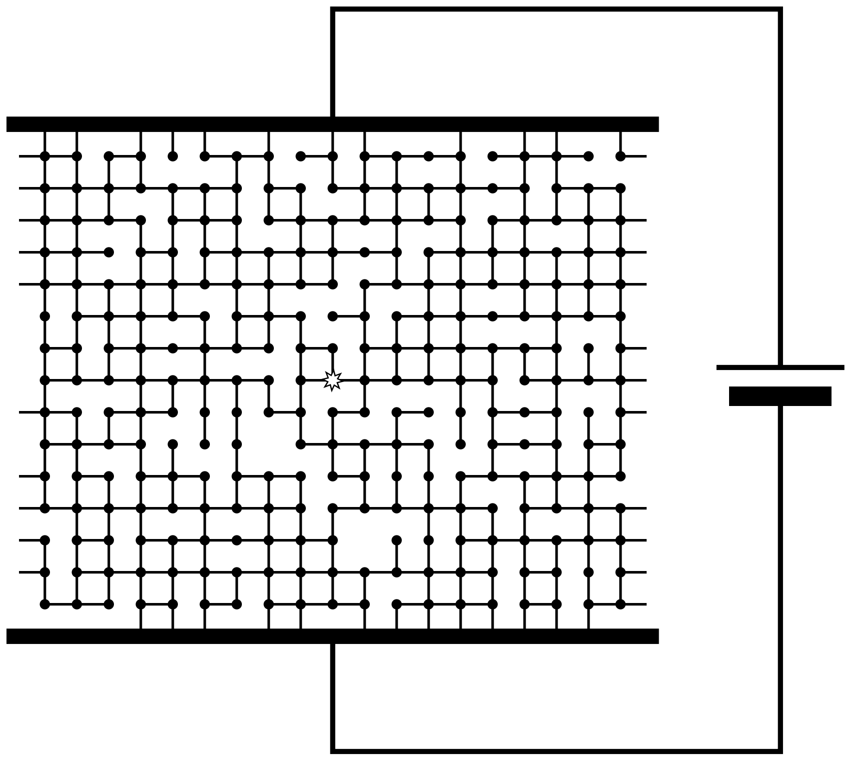
- **Walk among random conductances**:
  Kozlov (1985), Kipnis & Varadhan (1986)
  Sidoravicius & Sznitman (2004)
  Fontes & Mathieu (2004)

- **Heat-kernel estimates**:
  Barlow (2004),. . .

Main idea by example:

Hitting problem: $P_{0,\omega}$(top hit before bottom)

Electrostatic problem: potential at the origin

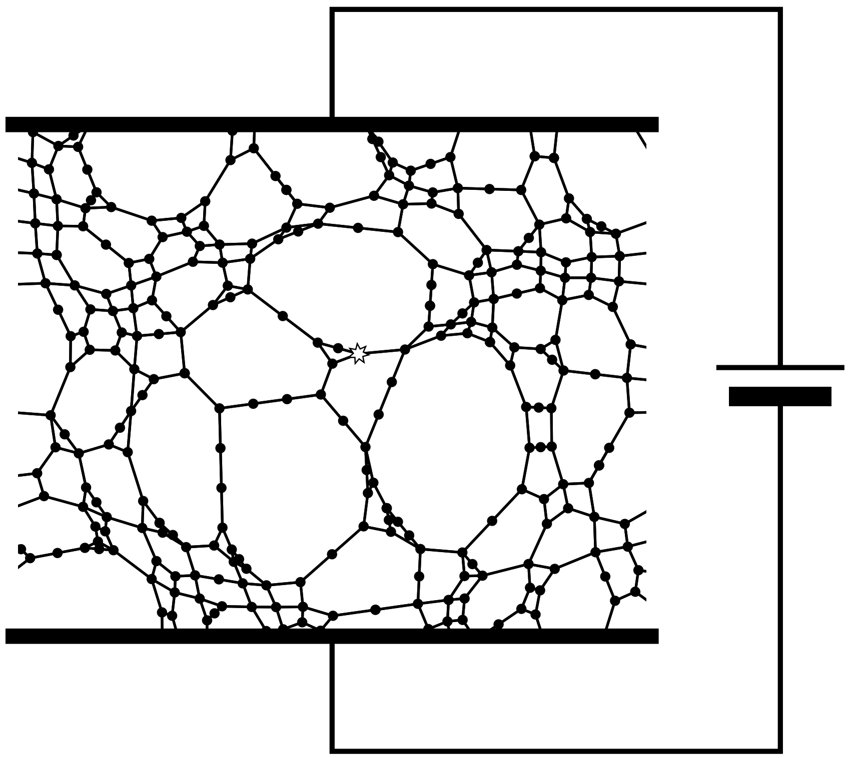


Geometric embedding is bad:

- Position function $g(x) = x$ not harmonic
- Random walk (X_n) not martingale

Main idea—“solution” :

Harmonic embedding better:



New position $x + \chi(x, \omega)$. Then

$$P_{0,\omega}(\text{exits thru top}) = \frac{1}{2} + \frac{e_2 \cdot \chi(0, \omega)}{\text{width}}$$

“Usual” behavior: need $|\chi(0, \omega)| \ll$ width

Corrector :

Deformation expressed by the *corrector* $\chi(\cdot, \omega)$.

Finite volume: Solve Dirichlet problem

Problem: Define χ in infinite volume

Natural candidate:

$$\chi(x, \omega) = \lim_{n \to \infty} \Big[E_{x,\omega}(X_n) - E_{0,\omega}(X_n) \Big].$$

- Trivially harmonic

- Existence of limit unclear

Analytic construction:

Essentially due to Kipnis & Varadhan

Proposition 2 ($d \geq 2$) $\exists\, \chi\colon \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$ *such that, for $\mathbb{P}_0$-a.e. ω:*

(0) $\chi(0,\omega) = 0$

(1) $x \mapsto x + \chi(x,\omega)$ *is harmonic on* $\mathscr{C}_\infty(\omega)$

(2) χ *is a gradient field on* $\mathscr{C}_\infty$*:*

$$\chi(x,\omega) - \chi(y,\omega) = \chi(x-y, \tau_y\omega), \qquad x,y \in \mathscr{C}_\infty$$

(3) The gradients of χ *are square integrable:*

$$\big\| \chi(e,\omega)\, 1_{\{\omega_e = 1\}} \big\|_2 < C, \qquad |e| = 1$$

Deformed random walk :

The listed properties make

$$M_n = X_n + \chi(X_n, \omega)$$

an L^2-martingale.

Martingale CLT + some ergodicity $\Rightarrow$

Deformed walk scales to Brownian motion

To control the deformation, need

$$\max_{1 \le k \le n} \frac{\chi(X_n, \omega)}{\sqrt{n}} \underset{n \to \infty}{\longrightarrow} 0$$

in probability, for $\mathbb{P}_0$-a.e. ω.

Max bound on corrector:

Since $M_n = O(\sqrt{n})$, it suffices to prove:

Proposition 3 ($d = 2$) *For $\mathbb{P}_0$-a.e. ω,*

$$\lim_{n\to\infty} \max_{\substack{x\in\mathscr{C}_\infty(\omega)\\ |x|\le n}} \frac{|\chi(x,\omega)|}{n} = 0.$$

Strategy of proof:

(1) Along coordinate axes:

Label points on x-axis that are in $\mathscr{C}_\infty$:

$$\dots, x_{-2}(\omega), x_{-1}(\omega), x_0(\omega), x_1(\omega), \dots$$

Fact: $(x_{n+1} - x_n)_{n\in\mathbb{Z}}$ stationary & ergodic

Lemma 4 **($d \geq 2$)** *For $\mathbb{P}_0$-a.e. ω,*

$$\lim_{n\to\infty} \frac{\chi(x_n(\omega), \omega)}{n} = 0$$

(2) Good points:

Definition 5 *Let $K, \epsilon > 0$. Then 0 is* good *in ω if $\forall n \in \mathbb{Z}$*

$$\Bigl|\chi(x_n(\omega), \omega)\Bigr| \leq K + \epsilon |x_n(\omega)|$$

A site $x \in \mathscr{C}_\infty$ is good *in ω if 0 is good in $\tau_x\omega$*

Proof of Lemma 4:

Set

$$\Psi(\omega) = \chi\Big(x_1(\omega), \omega\Big) - \chi(0, \omega)$$

If $\sigma(\omega) = \tau_{x_1(\omega)}\omega$, then

$$\frac{\chi(x_n(\omega), \omega)}{n} = \frac{1}{n} \sum_{k=1}^{n} \Psi \circ \sigma^{k}(\omega)$$

Now $\chi \in L^2$ and so Antal & Pisztora gives

$$\Psi \in L^1$$

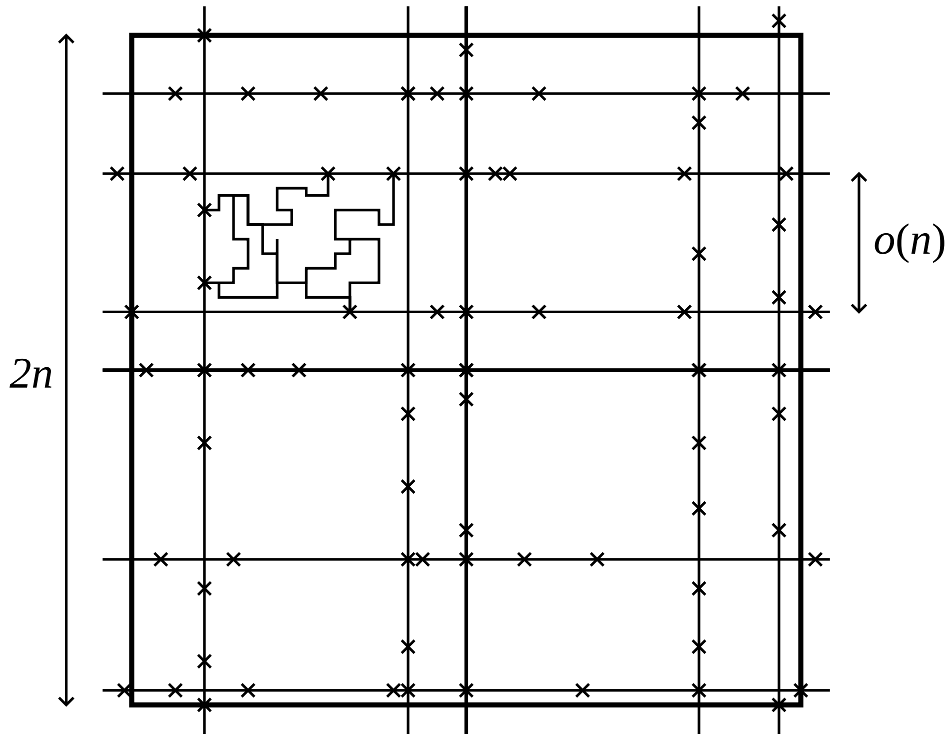
But

$$\mathbb{E}_0(\Psi) = 0$$

and so we just apply the Ergodic Theorem

Weaving webs of goodness:

Good points on coordinate axes ⇒ good grid



Maximum on good grid: $\le 2K + 2\epsilon n$

Harmonicity of $x \mapsto x + \chi(x,\omega)$:

$$\max_{\substack{x\in\mathscr{C}_\infty(\omega)\\ |x|\le n}} |\chi(x,\omega)| \le 2K + 2\epsilon n + o(n)$$

Above 2 dimensions :

Extension of max-bound to $d \geq 3$ unclear
But we can prove:

Proposition 6 *For* $\mathbb{P}_0$*-a.e.* ω *and all* $\epsilon > 0$*:*

$$\limsup_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{\substack{x\in\mathscr{C}_\infty(\omega) \\ |x|\le n}} 1_{\{|\chi(x,\omega)|\ge\epsilon n\}} = 0$$

This is sufficient to control the deformation

Details: [BB05] at math.PR/0503576

Future research:

Limit laws:

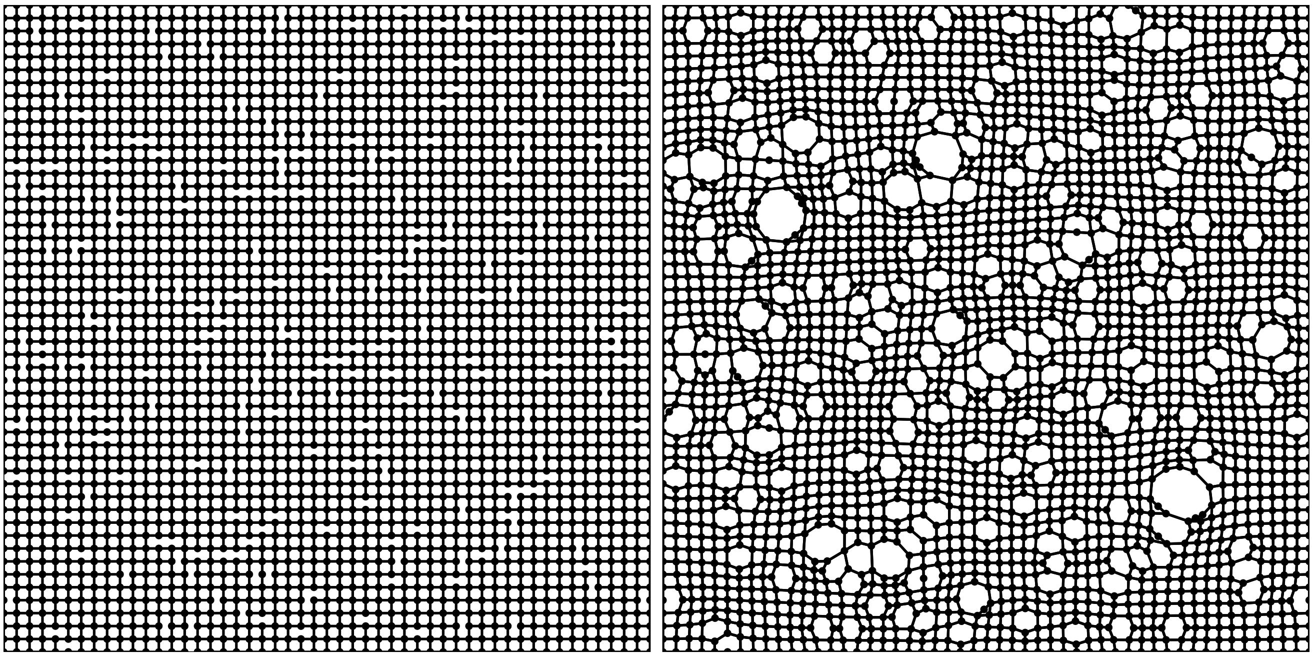
- Max bound on corrector in $d \geq 3$
- Other geometries (Voronoi percolation)
- Local CLT
- Long-range percolation (stable processes)

Corrector:

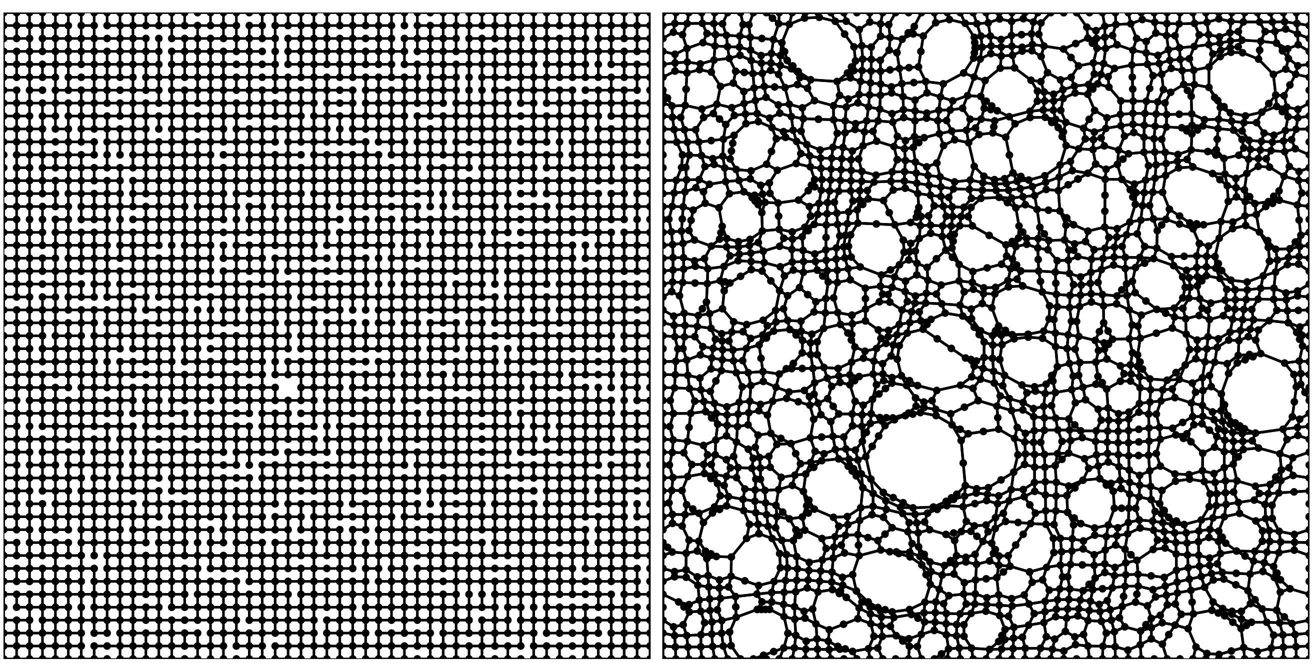
- Actual size
- Scaling limit
- Behavior as $p \downarrow p_c$

Playing with corrector:

$50 \times 50,\ p = 0.95$

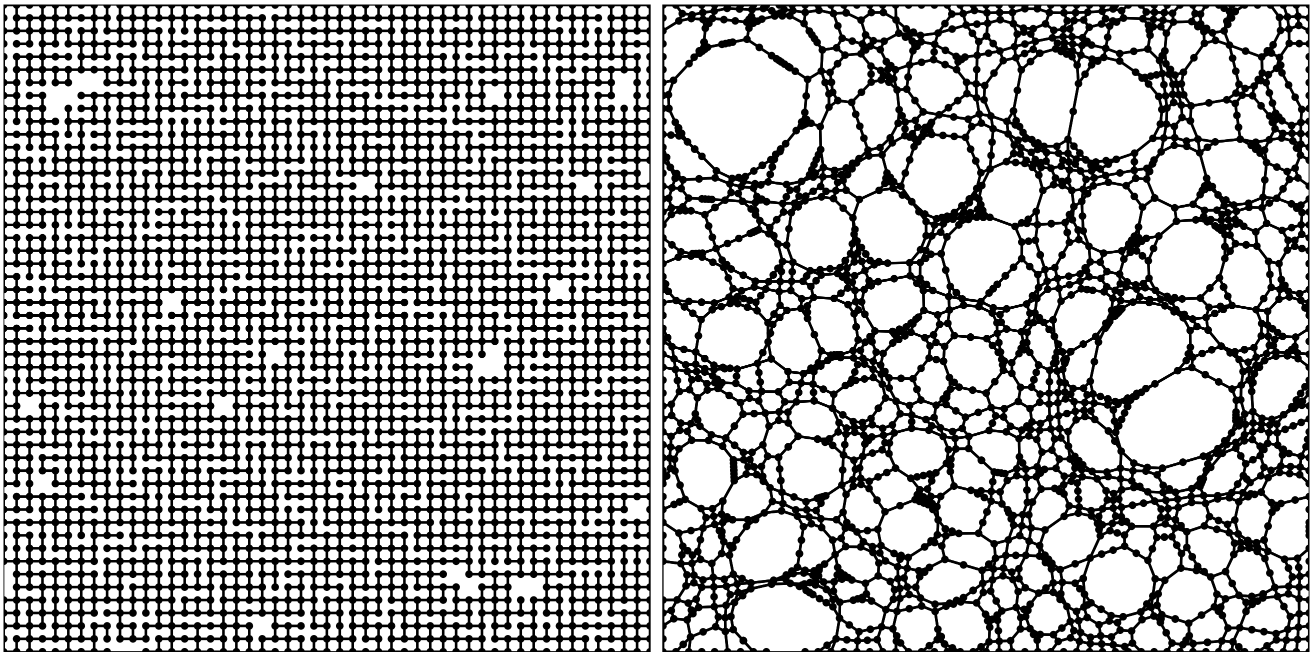


$50 \times 50,\ p = 0.85$

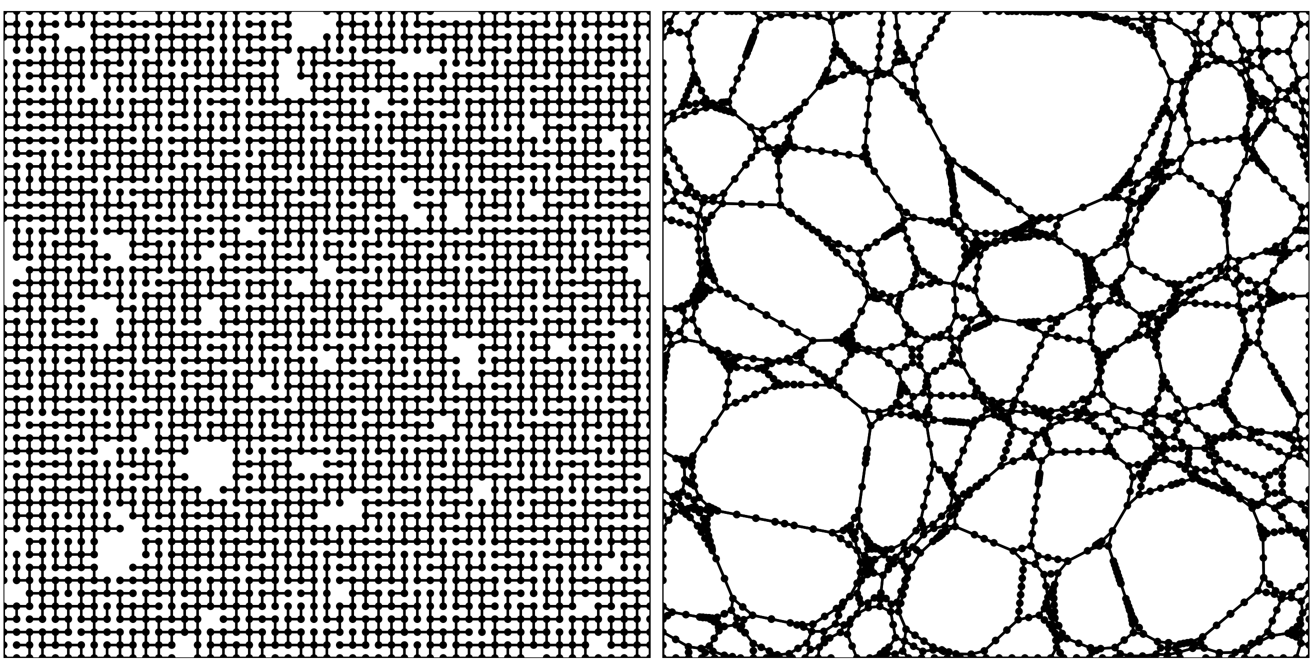


Playing with corrector:

50 × 50, $p = 0.75$



50 × 50, $p = 0.65$



BON APPETITE