ANOMALOUS HEAT-KERNEL DECAY FOR RANDOM WALK AMONG BOUNDED RANDOM CONDUCTANCES

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ABSTRACT. We consider the nearest-neighbor simple random walk on $\mathbb{Z}^d$, $d \geq 2$, driven by a field of bounded random conductances $\omega_{xy} \in [0,1]$. The conductance law is i.i.d. subject to the condition that the probability of $\omega_{xy} > 0$ exceeds the threshold for bond percolation on $\mathbb{Z}^d$. For environments in which the origin is connected to infinity by bonds with positive conductances, we study the decay of the $2n$-step return probability $\mathsf{P}^{2n}_\omega(0,0)$. We prove that $\mathsf{P}^{2n}_\omega(0,0)$ is bounded by a random constant times $n^{-d/2}$ in $d = 2, 3$, while it is $o(n^{-2})$ in $d \geq 5$ and $O(n^{-2}\log n)$ in $d = 4$. By producing examples with anomalous heat-kernel decay approaching $1/n^2$ we prove that the $o(n^{-2})$ bound in $d \geq 5$ is the best possible. We also construct natural n-dependent environments that exhibit the extra $\log n$ factor in $d = 4$.

1. INTRODUCTION

Random walk in reversible random environments is one of the best studied subfields of random motion in random media. In continuous time, such walks are usually defined by their generators $\mathcal{L}_\omega$ which are of the form

$$(\mathcal{L}_\omega f)(x) = \sum_{y \in \mathbb{Z}^d} \omega_{xy}\bigl[f(y) - f(x)\bigr], \tag{1.1}$$

where (ω_{xy}) is a family of random (non-negative) conductances subject to the symmetry condition $\omega_{xy} = \omega_{yx}$. The sum $\pi_\omega(x) = \sum_y \omega_{xy}$ defines an invariant, reversible measure for the corresponding continuous-time Markov chain. The discrete-time walk shares the same reversible measure and is driven by the transition matrix

$$\mathsf{P}_\omega(x,y) = \frac{\omega_{xy}}{\pi_\omega(x)}. \tag{1.2}$$

In most situations ω_{xy} are non-zero only for nearest neighbors on $\mathbb{Z}^d$ and are sampled from a shift-invariant, ergodic or even i.i.d. measure $\mathbb{P}$ (with expectation henceforth denoted by $\mathbb{E}$).

Two general classes of results are available for such random walks under the additional assumptions of uniform ellipticity,

$$\exists \alpha > 0 : \quad \mathbb{P}(\alpha < \omega_b < {}^1\!/_\alpha) = 1, \tag{1.3}$$

and the boundedness of the jump distribution,

$$\exists R < \infty : \quad |x| \geq R \quad \Rightarrow \quad \mathsf{P}_\omega(0,x) = 0, \quad \mathbb{P}\text{-a.s.} \tag{1.4}$$

First, as proved by Delmotte [7], one has the standard, local-CLT like decay of the heat kernel (c_1, c_2 are absolute constants):

$$\mathsf{P}^n_\omega(x,y) \leq \frac{c_1}{n^{d/2}} \exp\Bigl\{-c_2 \frac{|x-y|^2}{n}\Bigr\}. \tag{1.5}$$

Second, an annealed invariance principle holds in the sense that the law of the paths under the measure integrated over the environment scales to a non-degenerate Brownian motion (Kipnis and Varadhan [16]). A quenched invariance principle can also be proved by invoking techniques of homogenization theory (Sidoravicius and Sznitman [23]).

Once the assumption of uniform ellipticity is relaxed, matters get more complicated. The most-intensely studied example is the simple random walk on the infinite cluster of supercritical bond percolation on $\mathbb{Z}^d$, $d \geq 2$. This corresponds to $\omega_{xy} \in \{0,1\}$ i.i.d. with $\mathbb{P}(\omega_b = 1) > p_{\text{c}}(d)$ where $p_{\text{c}}(d)$ is the percolation threshold. Here an annealed invariance principle has been obtained by De Masi, Ferrari, Goldstein and Wick [8, 9] in the late 1980s. More recently, Mathieu and Remy [20] proved the on-diagonal (i.e., $x = y$) version of the heat-kernel upper bound (1.5)—a slightly weaker version of which was also obtained by Heicklen and Hoffman [15]—and, soon afterwards, Barlow [2] proved the full upper and lower bounds on $\mathsf{P}^n_\omega(x,y)$ of the form (1.5). (Both of these results hold for n exceeding some random time defined relative to the environment in the vicinity of x and y.) Heat-kernel upper bounds were then used in the proofs of quenched invariance principles by Sidoravicius and Sznitman [23] for $d \geq 4$, and for all $d \geq 2$ by Berger and Biskup [4] and Mathieu and Piatnitski [19].

Notwithstanding our precise definition (1.3), the case of supercritical percolation may still be regarded as uniformly elliptic because the conductances on the percolation cluster are still uniformly bounded away from zero and infinity. It is thus not clear what phenomena we might encounter if we relax the uniform ellipticity assumption in an essential way. A number of quantities are expected (or can be proved) to vary continuously with the conductance distribution, e.g., the diffusive constant of the limiting Brownian motion. However, this may not apply to asymptotic statements like the heat-kernel bound (1.5).

In a recent paper, Fontes and Mathieu [10] studied continuous-time random walk on $\mathbb{Z}^d$ with conductances given by

$$\omega_{xy} = \omega(x) \wedge \omega(y) \tag{1.6}$$

for some i.i.d. random variables $\omega(x) > 0$. For these cases it was found that the annealed heat kernel, $\mathbb{E}[P_{\omega,0}(X_t = 0)]$, where $P_{\omega,0}$ is the law of the walk started at the origin and $\mathbb{E}$ is the expectation with respect to the environment, exhibits an *anomalous decay* for

environments with too heavy lower tails at zero. Explicitly, from [10, Theorem 4.3] we have

$$\mathbb{E}\big[P_{\omega,0}(X_t = 0)\big] = t^{-(\gamma\wedge\frac{d}{2})+o(1)}, \qquad t \to \infty, \tag{1.7}$$

where $\gamma > 0$ characterizes the lower tail of the ω-variables,

$$\mathbb{P}\big(\omega(x) \le s\big) \sim s^{\gamma}, \qquad s \downarrow 0. \tag{1.8}$$

As for the quenched problem, for $\gamma < d/2$, [10, Theorem 5.1] provides a lower bound on the diagonal heat-kernel decay exponent (a.k.a. spectral dimension):

$$\mathbb{P}\big[P_{\omega,0}(X_t = 0) \le t^{-\alpha}\big] \underset{t\to\infty}{\longrightarrow} 1 \tag{1.9}$$

for every $\alpha < \alpha_0$ where

$$\alpha_0 = \frac{d}{2}\frac{1+\gamma}{1+d/2}. \tag{1.10}$$

But, since $\alpha_0 < d/2$, this does not rule out the usual diffusive scaling. Nevertheless, as $\alpha_0 > \gamma$ for $\gamma < d/2$, the annealed and quenched heat-kernel decay at different rates.

The reason why the annealed heat kernel may decay slower than usual can be seen rather directly from the following argument: The quenched probability that the walk does not even leave the origin up to time t is $\mathrm{e}^{-t\pi_\omega(0)}$. By $\pi_\omega(0) \le 2d\omega(0)$, we have

$$\mathbb{E}\big[P_{\omega,0}(X_t = 0)\big] \ge \mathbb{E}\mathrm{e}^{-2d\omega(0)t}. \tag{1.11}$$

For $\omega(0)$ with the tail (1.8), this yields a lower bound of $t^{-\gamma}$. (A deeper analysis shows that this is actually a dominating strategy [10].) A similar phenomenon can clearly be induced for ω_{xy} that are i.i.d. with a sufficiently heavy tail at zero, even though then the correspondence of the exponents in (1.7–1.8) will take a slightly different form.

The fact that the dominating strategy is so simple makes one wonder how much of this phenomenon is simply an artifact of taking the annealed average. Of not much help in this matter is the main result (Theorem 3.3) of Fontes and Mathieu [10] which shows that the mixing time for the random walk on the largest connected component of a torus will exhibit anomalous (quenched) decay once $\gamma < d/2$. Indeed, the mixing time is by definition dominated by the worst-case local configurations that one can find *anywhere* on the torus and thus the reasoning we used to explain the anomalous decay of the annealed heat kernel applies here as well.

The main goal of this paper is to provide universal upper bounds on the *quenched* heat kernel and support them by examples exhibiting the corresponding lower bounds. Somewhat surprisingly, and unlike for the annealed heat kernel, the existence of anomalous quenched heat-kernel decay turns out to be dimension dependent.

2. Main results

We will work with a collection of bounded, nearest-neighbor conductances $(\omega_b) \in \Omega = [0,1]^{\mathbb{B}}$ where b ranges over the set $\mathbb{B}$ of unordered pairs of nearest neighbors in $\mathbb{Z}^d$. The law $\mathbb{P}$ of the ω's will be i.i.d. subject to the condition that the bonds with positive conductances percolate. Given ω, we use $\mathscr{C}_\infty = \mathscr{C}_\infty(\omega)$ to denote the set of sites that

have a path to infinity along bonds with positive conductances. It is well known that $\mathscr{C}_\infty$ is connected with probability one.

The main result of this paper is as follows:

Theorem 2.1 *Let $d \geq 2$ and consider a collection $\omega = (\omega_b)$ of i.i.d. conductances in $[0,1]$ with $\mathbb{P}(\omega_b > 0) > p_c(d)$ where $p_c(d)$ is the threshold for bond percolation on $\mathbb{Z}^d$. For almost every $\omega \in \{0 \in \mathscr{C}_\infty\}$, there is $C = C(\omega) < \infty$ such that*

$$\mathsf{P}^n_\omega(0,0) \leq C(\omega) \begin{cases} n^{-d/2}, & d = 2,3, \\ n^{-2}\log n, & d = 4, \\ n^{-2}, & d \geq 5, \end{cases} \tag{2.1}$$

for all $n \geq 1$. In fact, for $d \geq 5$, almost surely

$$\lim_{n\to\infty} n^2\, \mathsf{P}^n_\omega(0,0) = 0. \tag{2.2}$$

Note that these estimates imply that the random walk is almost surely transient in all dimensions $d \geq 3$. This is of course a consequence of the fact—to be exploited in more depth later—that under $p > p_c(d)$ one has an infinite cluster of bonds with conductances bounded strictly from below. Then a.s. transience in $d \geq 3$ follows by monotonicity in conductances and the result of Grimmett, Kesten and Zhang [14]. (Recurrence in $d = 1,2$ is inferred directly from the monotonicity of this notion in the conductances.)

To show that our general upper bound in $d \geq 5$ represents a real phenomenon, we state the existence of appropriate examples:

Theorem 2.2 *(1) Let $d \geq 5$ and $\kappa > 1/d$. There exists an i.i.d. law $\mathbb{P}$ on bounded, nearest-neighbor conductances with $\mathbb{P}(\omega_b > 0) > p_c(d)$ and a random variable $C = C(\omega)$ such that for almost every $\omega \in \{0 \in \mathscr{C}_\infty\}$,*

$$\mathsf{P}^{2n}_\omega(0,0) \geq\ C(\omega)\frac{\mathrm{e}^{-(\log n)^\kappa}}{n^2}, \qquad n \geq 1. \tag{2.3}$$

(2) Let $d \geq 5$. For every increasing sequence $\{\lambda_n\}_{n=1}^\infty$, $\lambda_n \to \infty$, there exists an i.i.d. law $\mathbb{P}$ on bounded, nearest-neighbor conductances with $\mathbb{P}(\omega_b > 0) > p_c(d)$ and an a.s. positive random variable $C = C(\omega)$ such that for almost every $\omega \in \{0 \in \mathscr{C}_\infty\}$,

$$\mathsf{P}^n_\omega(0,0) \geq \frac{C(\omega)}{\lambda_n n^2} \tag{2.4}$$

along a subsequence that does not depend on ω.

The upper bounds in Theorem 2.1 can be extended to more general shift-invariant, ergodic environments under suitable assumptions on their percolation properties. In particular, it follows that for the Fontes-Mathieu example (1.7–1.8) no anomaly occurs for the quenched heat kernel in dimensions $d = 2,3$. On the other hand, Theorem 2.2 can be specialized to the case (1.6) with i.i.d. $\omega(x)$'s and, in $d \geq 5$, we can produce anomalous decay as soon as the tails of ω at zero are sufficiently heavy. (The constructions in the

proof of Theorem 2.2 actually work for all $d \geq 2$ but the result is of course interesting only for $d \geq 5$.)

The distributions that we use in part (1) of Theorem 2.2 have a tail near zero of the general form

$$\mathbb{P}(\omega_{xy} < s) \approx |\log(s)|^{-\theta} \tag{2.5}$$

with $\theta > 0$. Presumably, one can come up with examples of distributions that exhibit "anomalous" behavior and have the power law tail,

$$\mathbb{P}(\omega_{xy} < s) \approx s^{\gamma}, \tag{2.6}$$

for some $\gamma > 0$. However, the construction seems to require subtle control of heat-kernel *lower* bounds which go beyond the estimates that can be easily pulled out from the literature.

As we will see in the proofs, the underlying idea of all examples in Theorem 2.2 is the same: The walk finds a *trap* which, by our specific choice, is a "strong" edge that can be reached only by crossing an edge of strength of order $1/n$. Such traps allow the walk to get stuck for time of order n and thus improve its chances to make it back to the origin at the required time. To enter and exit the trap, the walk has to make two steps over the $O(1/n)$-edge; these are responsible for the overall n^{-2}-decay. Of course, in $d = 2, 3$ this cannot compete with the "usual" decay of the heat kernel and so we have to go to $d \geq 4$ to make this strategy dominant.

The upper bound in (2.2) and the lower bound in (2.4) show that the $1/n^2$ decay in $d \geq 5$ is never achieved, but can be approached arbitrary closely. We believe the same holds also for $d = 4$ for the decay rate $n^{-2}\log n$. We demonstrate the reason for our optimism by proving a lower bound for environments where the aforementioned traps occur with a positive density:

Theorem 2.3 *Let $d \geq 4$ and let $p > p_c(d)$. Sample a percolation configuration $\tilde{\omega}$ with parameter p. For each $n \geq 1$ consider the i.i.d. environment $\omega^{(n)}$ defined from $\tilde{\omega}$ by putting $\omega_b^{(n)} = 1$ on occupied bonds and $\omega_b^{(n)} = 1/n$ on vacant bonds. For a.e. $\tilde{\omega}$ in which 0 has an occupied path to infinity, there is $C(\tilde{\omega}) > 0$ such that for all $n \geq 1$,*

$$\mathsf{P}^{2n}_{\omega^{(n)}}(0,0) \geq C(\tilde{\omega}) \begin{cases} n^{-2}, & d \geq 5, \\ n^{-2}\log n, & d = 4. \end{cases} \tag{2.7}$$

We conclude with a remark concerning the path properties of the above random walk. As mentioned previously, heat-kernel estimates of the form (1.5) have been crucial for the proof of the quenched invariance principle for simple random walk on supercritical percolation clusters in $d \geq 3$. (The $d = 2$ argument of Berger and Biskup [4] actually avoids these bounds by appealing to the nearest-neighbor structure of the walk and to an underlying maximum principle.) The absence of "usual" decay might suggest difficulty in following the same strategy. Notwithstanding, using truncation to a "strong component," a version of which is invoked also in the present paper, this problem can be circumvented and the corresponding quenched invariance principle proved (Mathieu [18], Biskup and Prescott [5]).

Thus there are i.i.d. environments for which one has a functional CLT *without* a local CLT. This should not be too surprising as a CLT describes the typical behavior whereas the heat-kernel decay, and a local-CLT, describe rare events. Naturally, a CLT is much more robust than its local counterpart.

Theorem 2.1 is proved in Sect. 3 while Theorems 2.2-2.3 are proved in Sect. 4. The Appendix (Sect. 5) contains a self-contained proof of the isoperimetric inequality on the supercritical percolation cluster that we need in the proof of Theorem 2.1.

3. HEAT-KERNEL UPPER BOUNDS

Here we will prove the heat-kernel bounds from Theorem 2.1. The general strategy of our proof is as follows: For every $\alpha > 0$, we use $\mathscr{C}_{\infty,\alpha} = \mathscr{C}_{\infty,\alpha}(\omega)$ to denote the set of all sites in $\mathbb{Z}^d$ that have a path to infinity along edges with conductances at least α. Clearly, $\mathscr{C}_{\infty,\alpha}$ is a subgraph of $\mathscr{C}_\infty$; we will sometimes refer to $\mathscr{C}_{\infty,\alpha}$ as the *strong component*. We first prove the "standard" heat-kernel decay for the Markov chain obtained by recording the position of the random walk when it is on the strong component $\mathscr{C}_{\infty,\alpha}$ for an appropriately chosen α. Then we control the difference between the time scales for the two walks using rather straightforward estimates.

3.1 Coarse-grained walk.

The i.i.d. nature of the measure $\mathbb{P}$ ensures there is an a.s.-unique infinite connected component $\mathscr{C}_\infty$ of bonds with positive conductances. Given $z \in \mathscr{C}_\infty$, we define the random walk $X = (X_n)$ as a Markov chain on $\mathscr{C}_\infty$ with transition probabilities

$$P_{\omega,z}(X_{n+1} = y | X_n = x) = \mathsf{P}_\omega(x,y) = \frac{\omega_{xy}}{\pi_\omega(x)} \tag{3.1}$$

and initial condition

$$P_{\omega,z}(X_0 = z) = 1. \tag{3.2}$$

We use $E_{\omega,z}$ to denote expectation with respect to $P_{\omega,z}$. (Note the typographical distinction between the path distribution $P_{\omega,z}$, the heat kernel P_ω, and the law of the environment $\mathbb{P}$.)

Next we will disregard bonds whose conductance is less than some small positive number α which is chosen so that the remaining bonds still form an infinite component—to be denoted by $\mathscr{C}_{\infty,\alpha}$. We quote Proposition 2.2 from Biskup and Prescott [5]:

Lemma 3.1 *Let $d \geq 2$ and $p = \mathbb{P}(\omega_b > 0) > p_c(d)$. Then there exists $c(p,d) > 0$ such that if α satisfies*

$$\mathbb{P}(\omega_b \geq \alpha) > p_c(d) \tag{3.3}$$

and

$$\mathbb{P}(0 < \omega_b < \alpha) < c(p,d) \tag{3.4}$$

then $\mathscr{C}_{\infty,\alpha}$ is nonempty and $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha}$ has only finite components a.s. In fact, if $\mathcal{F}_x$ is the set of sites (possibly empty) in the finite component of $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha}$ containing x, then

$$\mathbb{P}\bigl(x \in \mathscr{C}_\infty \;\&\; \operatorname{diam} \mathcal{F}_x \geq n\bigr) \leq C\mathrm{e}^{-\eta n}, \qquad n \geq 1, \tag{3.5}$$

*for some $C < \infty$ and $\eta > 0$. Here "*diam*" is the diameter in the ℓ_∞-distance on $\mathbb{Z}^d$.*

Given $z \in \mathscr{C}_{\infty,\alpha}$ we consider the *coarse grained* random walk $\hat{X} = (\hat{X}_\ell)$—started at z—which records the successive visits of $X = (X_n)$ to $\mathscr{C}_{\infty,\alpha}$. Explicitly, let $T_1, T_2, \dots$ denote the times X takes between the successive steps of $\hat{X}$, i.e., $T_{\ell+1} = \inf\{n > 0 \colon X_{T_\ell+n} \in \mathscr{C}_{\infty,\alpha}\}$ with $T_0 = 0$. Note that, as all components of $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha}$ are finite, $T_\ell < \infty$ a.s. for all ℓ. Then

$$\hat{X}_\ell = X_{T_1+\cdots+T_\ell}, \qquad \ell \geq 1. \tag{3.6}$$

Let $\hat{\mathsf{P}}_\omega(x,y)$ denote the transition probability of the random walk $\hat{X}$,

$$\hat{\mathsf{P}}_\omega(x,y) = P_{\omega,x}(X_{T_1} = y), \qquad x, y \in \mathscr{C}_{\infty,\alpha}. \tag{3.7}$$

As is easy to check, the restriction of the measure π_ω to $\mathscr{C}_{\infty,\alpha}$ is invariant and reversible for the Markov chain on $\mathscr{C}_{\infty,\alpha}$ induced by $\hat{\mathsf{P}}_\omega$.

Consider the quantities

$$\hat{\omega}_{xy} = \pi_\omega(x)\hat{\mathsf{P}}_\omega(x,y), \qquad x, y \in \mathscr{C}_{\infty,\alpha}. \tag{3.8}$$

We may think of $\hat{X}$ as the walk on $\mathscr{C}_{\infty,\alpha}$ with the weak components "re-wired" by putting a bond with conductance $\hat{\omega}_{xy}$ between any pair of sites (x,y) on their (strong) boundary. By Lemma 3.1, all weak components are finite and everything is well defined.

Our first item of business is to show that $\hat{X}$ obeys the standard heat-kernel bound:

Lemma 3.2 *For almost every $\omega \in \{0 \in \mathscr{C}_{\infty,\alpha}\}$ and every $x \in \mathscr{C}_{\infty,\alpha}(\omega)$ there exists random variable $C(\omega,x) < \infty$ such that*

$$\hat{\mathsf{P}}^n_\omega(x,y) \leq \frac{C(\omega,x)}{n^{d/2}}, \qquad n \geq 1. \tag{3.9}$$

We remark that the reversibility of the random walk, and the fact that $\pi_\omega \geq \alpha$ on $\mathscr{C}_{\infty,\alpha}$, imply that $\hat{\mathsf{P}}^n_\omega(x,y)$ may also be bounded in terms of $C(\omega,y)$. Note that, unlike for P, the powers for which $\hat{\mathsf{P}}^n(x,y)$ is non-zero are not necessarily tied to the parity of $y-x$.

Lemma 3.2 will be implied by the fact that the Markov chain $\hat{X}$ obeys the "usual" d-dimensional isoperimetric inequality. The connection between isoperimetric inequalities and heat-kernel decay can be traced back to the work on elliptic PDEs done by Nash, Moser and others. In its geometric form it was first proved using Sobolev inequalities (Varopoulos [24]). Alternative approaches use Nash inequalities (Carlen, Kusoka and Stroock [6]), Faber-Krahn inequalities (Grigor'yan [12], Goel, Montenegro and Tetali [11]) and evolving sets (Morris and Peres [17]). The paper [17] will serve us as a convenient reference.

Consider a Markov chain on a countable state-space V with transition probability denoted by $\mathsf{P}(x,y)$ and invariant measure denoted by π. Define $\mathsf{Q}(x,y) = \pi(x)\mathsf{P}(x,y)$ and for each $S_1, S_2 \subset V$, let

$$\mathsf{Q}(S_1,S_2) = \sum_{x\in S_1}\sum_{y\in S_2} \mathsf{Q}(x,y). \tag{3.10}$$

For each $S \subset V$ with $\pi(S) \in (0,\infty)$ we define

$$\Phi_S = \frac{\mathsf{Q}(S, S^{\mathrm{c}})}{\pi(S)} \tag{3.11}$$

and use it to define the isoperimetric profile

$$\Phi(r) = \inf\bigl\{\Phi_S\colon\, \pi(S) \le r\bigr\}. \tag{3.12}$$

(Here $\pi(S)$ is the measure of S.) It is easy to check that we may restrict the infimum to sets S that are connected in the graph structure induced on V by P.

The following summarizes Theorem 2 of [17]: Suppose that $\mathsf{P}(x,x) \ge \gamma$ for some $\gamma \in (0, 1/2]$ and all $x \in V$. Let $\epsilon > 0$ and $x, y \in V$. Then

$$\mathsf{P}^n(x,y) \le \epsilon\pi(y) \tag{3.13}$$

for all n such that

$$n \ge 1 + \frac{(1-\gamma)^2}{\gamma^2} \int_{4[\pi(x)\wedge\pi(y)]}^{4/\epsilon} \frac{4}{u\Phi(u)^2}\,\mathrm{d}u. \tag{3.14}$$

Note that, to prove the "usual" decay $\mathsf{P}^n(x,y) \le cn^{-d/2}$, it suffices to show that the bound $\Phi(r) \le cr^{-1/d}$ holds for r sufficiently large.

We will adapt this machinery to the following setting

$$V = \mathscr{C}_{\infty,\alpha}(\omega), \quad \mathsf{P} = \hat{\mathsf{P}}_\omega^2 \quad \text{and} \quad \pi = \pi_\omega, \tag{3.15}$$

with the objects in (3.10–3.12) denoted by $\widehat{\mathsf{Q}}_\omega$, $\widehat{\Phi}_S^{(\omega)}$ and $\widehat{\Phi}_\omega(r)$. However, to estimate $\widehat{\Phi}_\omega(r)$ we will replace $\hat{\mathsf{P}}_\omega$ by the Markov chain with transition probabilities

$$\tilde{\mathsf{P}}_\omega(x,y) = \frac{\omega_{xy}\,1_{\{\omega_{xy}\ge\alpha\}}}{\tilde{\pi}_\omega(x)}, \qquad |x-y| = 1, \tag{3.16}$$

i.e., the random walk on $V = \mathscr{C}_{\infty,\alpha}$ that can only use edges physically present in the infinite cluster. The quantity

$$\tilde{\pi}_\omega(x) = \sum_y \omega_{xy}\,1_{\{\omega_{xy}\ge\alpha\}} \tag{3.17}$$

denotes the corresponding stationary measure. We will use $\widetilde{\mathsf{Q}}_\omega$, $\widetilde{\Phi}_S^{(\omega)}$ and $\widetilde{\Phi}_\omega(r)$ to denote the objects in (3.10–3.12) for this Markov chain.

Lemma 3.3 *There exists a constant $c > 0$ depending only on d and α such that for any finite set $\Lambda \subset \mathscr{C}_{\infty,\alpha}$,*

$$\widehat{\Phi}_\Lambda^{(\omega)} \ge c\widetilde{\Phi}_\Lambda^{(\omega)}. \tag{3.18}$$

Proof. The stationary measures π_ω and $\tilde{\pi}_\omega$ compare via

$$\pi_\omega(x) \ge \tilde{\pi}_\omega(x) \ge \frac{\alpha}{2d}\pi_\omega(x) \tag{3.19}$$

Restricting $\hat{\mathsf{P}}^2_\omega(x,y)$ to transitions with $T_2 = 2$ shows

$$\hat{\mathsf{P}}^2_\omega(x,y) \geq \sum_z \frac{\omega_{xz}\,1_{\{\omega_{xz}\geq\alpha\}}}{\pi_\omega(x)}\,\frac{\omega_{zy}\,1_{\{\omega_{zy}\geq\alpha\}}}{\pi_\omega(z)} \geq ({}^{\alpha}/_{2d})^2\tilde{\mathsf{P}}^2_\omega(x,y). \tag{3.20}$$

It follows that (3.18) holds with $c = ({}^{\alpha}/_{2d})^3$. □

Our next step involves extraction of appropriate bounds on surface and volume terms. As the infimum in (3.12) can always be restricted to connected subsets of the Markov graph, and since the Markov graph underlying the quantity $\widetilde{\Phi}_\omega(r)$ is just the infinite cluster $\mathscr{C}_{\infty,\alpha}$, we can restrict our attention to subsets of $\mathscr{C}_{\infty,\alpha}$ that are connected in the usual sense.

Lemma 3.4 *Let $\theta > 0$, $d \geq 2$ and let α be as above. Then there exists a constant $c > 0$ and random variable $R_1 = R_1(\omega)$ with $\mathbb{P}(R_1 < \infty) = 1$ such that for a.e. $\omega \in \{0 \in \mathscr{C}_{\infty,\alpha}\}$ and all $R \geq R_1(\omega)$ the following holds: For any connected $\Lambda \subset \mathscr{C}_{\infty,\alpha} \cap [-R,R]^d$ with*

$$\tilde{\pi}_\omega(\Lambda) \geq R^\theta \tag{3.21}$$

we have

$$\widetilde{\mathsf{Q}}_\omega(\Lambda, \mathscr{C}_{\infty,\alpha}\setminus\Lambda) \geq c\pi_\omega(\Lambda)^{\frac{d-1}{d}}. \tag{3.22}$$

Proof. Since $\mathscr{C}_{\infty,\alpha}$ has the law of the infinite bond-percolation cluster, we will infer this from isoperimetry for the percolation cluster; cf. Theorem 5.1. Let $\partial^{\omega,\alpha}\Lambda$ denote the set of edges in $\mathscr{C}_{\infty,\alpha}$ with one endpoint in Λ and the other in $\mathscr{C}_{\infty,\alpha}\setminus\Lambda$. We claim that

$$\widetilde{\mathsf{Q}}_\omega(\Lambda, \mathscr{C}_{\infty,\alpha}\setminus\Lambda) \geq \frac{\alpha^2}{2d}\,|\partial^{\omega,\alpha}\Lambda| \tag{3.23}$$

and

$$\tilde{\pi}_\omega(\Lambda) \leq 2d|\Lambda|. \tag{3.24}$$

Since Λ obeys the conditions of Theorem 5.1, once $R \gg 1$, we have $|\partial^{\omega,\alpha}\Lambda| \geq c_2|\Lambda|^{\frac{d-1}{d}}$, cf. equation (5.2) in Sect. 5. Then (3.22) will follow from (3.23–3.24).

It remains to prove (3.23–3.24). The bound (3.24) is implied by $\tilde{\pi}_\omega(x) \leq 2d$. For (3.23), since $\tilde{\mathsf{P}}^2_\omega$ represents two steps of a random walk, we get a lower bound on $\widetilde{\mathsf{Q}}_\omega(\Lambda, \mathscr{C}_{\infty,\alpha}\setminus\Lambda)$ by picking a site $x \in \Lambda$ which has a neighbor $y \in \Lambda$ that has a neighbor z on the outer boundary of Λ. The relevant contribution is bounded as

$$\tilde{\pi}_\omega(x)\tilde{\mathsf{P}}^2_\omega(x,z) \geq \tilde{\pi}_\omega(x)\frac{\omega_{xy}}{\tilde{\pi}_\omega(x)}\frac{\omega_{yz}}{\tilde{\pi}_\omega(y)} \geq \frac{\alpha^2}{2d}. \tag{3.25}$$

Once Λ has at least two elements, we can do this for (y,z) ranging over all bonds in $\partial^{\omega,\alpha}\Lambda$, so summing over $(y,z) \in \partial^{\omega,\alpha}\Lambda$ we get (3.23). □

Now we are finally ready to estimate the decay of $\hat{\mathsf{P}}^n_\omega(x,y)$:

Proof of Lemma 3.2. It clearly suffices to prove this for $x = 0$. Pick $\theta \in (0, {}^1/_2)$ and let R be the largest ℓ_∞-distance the walk X can go on $\mathscr{C}_\infty$ by time $T_1 + \cdots + T_{2n}$, i.e., by the time $\hat{X}$ makes $2n$ steps. Lemma 3.1 tells us that the largest jump $\hat{X}$ can make in a box of side length n^2 is $O(\log n)$, and so $R = O(n\log n)$. As the walk will not leave the

box $[-R,R]^d$ by time n, we may restrict the infimum defining $\widehat{\Phi}_\omega(r)$ to sets Λ entirely contained in $[-R,R]^d$. (This can be formally achieved also by modifying the Markov chain "outside" $[-R,R]^d$.) Moreover, invoking (3.18) we can instead estimate $\widetilde{\Phi}_\omega(r)$ which allows us to restrict to $\Lambda \subset \mathscr{C}_{\infty,\alpha} \cap [-R,R]^d$ that are connected in the usual graph structure on $\mathscr{C}_{\infty,\alpha}$.

We will now derive a bound on $\widetilde{\Phi}^{(\omega)}_\Lambda$ for connected $\Lambda \subset \mathscr{C}_{\infty,\alpha}(\omega) \cap [-R,R]^d$. Henceforth c denotes a generic constant. If $\pi_\omega(\Lambda) \geq R^\theta$, then (3.19) and (3.22) imply

$$\widetilde{\Phi}^{(\omega)}_\Lambda \geq c\pi_\omega(\Lambda)^{-1/d}. \tag{3.26}$$

On the other hand, for $\pi_\omega(\Lambda) < R^\theta$ the bound (3.23) yields

$$\widetilde{\Phi}^{(\omega)}_\Lambda \geq c\pi_\omega(\Lambda)^{-1} \geq cR^{-\theta}. \tag{3.27}$$

From Lemma 3.3 we conclude that

$$\widehat{\Phi}_\omega(r) \geq c\widetilde{\Phi}_\omega(r) \geq c(r^{-1/d} \wedge R^{-\theta}) \tag{3.28}$$

once $R \geq R_1(\omega)$. The crossover between the two regimes occurs when $r = R^{d\theta}$ which (due to $\theta < 1/2$) is much less than $4/\epsilon$ once $\epsilon \approx n^{-d/2}$. The relevant integral is thus bounded by

$$\int_{4[\pi(x)\wedge\pi(y)]}^{4/\epsilon} \frac{4}{u\widehat{\Phi}_\omega(u)^2}\,\mathrm{d}u \leq c_1 R^{2\theta}\log R + c_2\epsilon^{-2/d} \leq c_3\epsilon^{-2/d} \tag{3.29}$$

for some constants $c_1, c_2, c_3 > 0$. Setting ϵ proportional to $n^{-d/2}$ and noting $\gamma \geq (\alpha/2d)^2$, the right-hand side is less than n and $\mathsf{P}^n(0,x) \leq cn^{-d/2}$ for each $x \in \mathscr{C}_\infty \cap [-R,R]^d$. As $\mathsf{P}^n(0,x) = 0$ for $x \notin [-R,R]^d$, the bound holds in general. This proves the claim for even n; for odd n we just concatenate this with a single step of the random walk. □

3.2 Integral bound.

We now want to link the estimates on $\hat{\mathsf{P}}$ to a heat-kernel type bound for the walk X. Specifically, we will prove the following estimate:

Proposition 3.5 *For almost every $\omega \in \{0 \in \mathscr{C}_{\infty,\alpha}\}$, there exists a constant $C = C(\omega) < \infty$ such that for every $\ell \geq 1$ and every $n \geq 1$,*

$$P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_\ell \geq n) \leq C(\omega)\frac{\ell^{1-d/2}}{n}. \tag{3.30}$$

and, in fact,

$$\lim_{n\to\infty} n\,P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_\ell \geq n) = 0 \quad \text{a.s.} \tag{3.31}$$

In order to prove this claim, we will need to occasionally refer to the Markov chain on environments "from the point of view of the particle." Let τ_x be the shift by x on Ω and let $\Omega_\alpha = \{0 \in \mathscr{C}_{\infty,\alpha}\}$. We define a random shift $\tau_{\hat{X}_1} : \Omega_\alpha \to \omega$ by sampling $\hat{X}_1$ for

the given ω and applying τ_x with $x = \hat{X}_1$. This random map induces a Markov chain on $\Omega_\alpha^{\mathbb{Z}}$ via the iterated action of $\tau_{\hat{X}_1}$. Define the measure

$$\mathbb{Q}_\alpha(\mathrm{d}\omega) = Z\pi_\omega(0)\,\mathbb{P}(\mathrm{d}\omega|0 \in \mathscr{C}_{\infty,\alpha}) \tag{3.32}$$

where $Z^{-1} = \mathbb{E}(\pi_\omega(0)|0 \in \mathscr{C}_{\infty,\alpha})$. Let $E_{\mathbb{Q}_\alpha}$ denote expectation with respect to $\mathbb{Q}_\alpha$. We recall the following standard facts whose proof can be found in, e.g., [4, Section 3]:

Lemma 3.6 (Ergodicity of Markov chain on environments) *The measure $\mathbb{Q}_\alpha$ is stationary and ergodic with respect to the Markov shift $\tau_{\hat{X}_1}$ on environments. In particular, if $f \in L^1(\Omega, \mathbb{P})$ then for $\mathbb{Q}_\alpha$-a.e. ω and for $P_{\omega,0}$-a.e. trajectory $\hat{X} = (\hat{X}_1, \hat{X}_2, \dots)$,*

$$\lim_{\ell\to\infty} \frac{1}{\ell} \sum_{j=0}^{\ell-1} f(\tau_{\hat{X}_j}\omega) = E_{\mathbb{Q}_\alpha}(f). \tag{3.33}$$

The convergence occurs also in L^1 (i.e., under expectation $E_{0,\omega}$ and, if desired, also $E_{\mathbb{Q}_\alpha}$).

Recall our notation $\mathcal{F}_y$ for the finite component of $\mathscr{C}_\infty \setminus \mathscr{C}_{\infty,\alpha}$ containing y. For $x \in \mathscr{C}_{\infty,\alpha}$, let

$$\mathcal{G}'_x = \bigcup_{y:\ \omega_{xy}>0} \mathcal{F}_y \tag{3.34}$$

and let $\mathcal{G}_x$ denote the union of $\mathcal{G}'_x$ with all of its neighbors on $\mathscr{C}_{\infty,\alpha}$. We will refer to this set as the *weak component* incident to x. Note that $\mathcal{G}_x$ is the set of vertices that can be visited by the walk X started at x by the time X steps again onto the strong component.

Lemma 3.7 *Recall that $E_{\mathbb{Q}_\alpha}$ denotes expectation with respect to $\mathbb{Q}_\alpha$ and let $|\mathcal{G}_x|$ be the number of sites in $\mathcal{G}_x$. Under the conditions of Lemma 3.1, we have $E_{\mathbb{Q}_\alpha}|\mathcal{G}_0| < \infty$.*

Proof. This is an immediate consequence of (3.5). □

Next we will estimate the expected time the random walk hides in such a component:

Lemma 3.8 (Hiding time estimate) *Let $d \geq 2$ and set $c = 4d\alpha^{-1}$. Then for all $x \in \mathbb{Z}^d$ and all ω such that $x \in \mathscr{C}_{\infty,\alpha}$ and $\mathcal{G}_x$ is finite, we have*

$$E_{\omega,x}(T_1) \leq c|\mathcal{G}_x|. \tag{3.35}$$

Proof. Fix $x \in \mathscr{C}_{\infty,\alpha}$ and let $\mathcal{G}_x$ be its incident weak component which we regard as a finite graph. Add a site Δ to this graph and connect it by an edge to every site of $\mathcal{G}_x$ that has a strong bond to $\mathscr{C}_{\infty,\alpha} \setminus \mathcal{G}_x$. (Here Δ represents the rest of $\mathscr{C}_{\infty,\alpha}$; note that multiple edges between Δ and sites of $\mathcal{G}_x$ are possible.) Equip each such edge with the corresponding conductance and call the resulting finite graph $\mathcal{H}_x$. Clearly, the random walk on $\mathcal{H}_x$ started at x and the corresponding random walk on $\mathscr{C}_{\infty,\alpha}$ have the same law until they first hit Δ (i.e., leave $\mathcal{G}_x$). In particular, T_1 for the walk on $\mathscr{C}_{\infty,\alpha}$ is stochastically dominated by S_x, the first time the walk on $\mathcal{H}_x$ returns back to its starting point.

Notice that $x \mapsto \pi_\omega(x)$ is an invariant measure of the walk on $\mathcal{H}_x$ provided we set

$$\pi_\omega(\Delta) = \sum_{x\in\mathcal{G}_x} \sum_{y\in\mathscr{C}_{\infty,\alpha}\setminus\mathcal{G}_x} \omega_{xy}. \tag{3.36}$$

Standard Markov chain theory tells us that $z \mapsto (\tilde{E}_z S_z)^{-1}$, where $\tilde{E}_z$ is the expectation with respect to the walk on $\mathcal{H}_x$ started at z, is an invariant distribution and

$$\tilde{E}_x S_x = \frac{\pi_\omega(\mathcal{H}_x)}{\pi_\omega(x)}. \tag{3.37}$$

But $x \in \mathscr{C}_{\infty,\alpha}$ implies that $\pi_\omega(x) \geq \alpha$ while the bound $\omega_{yz} \leq 1$ yields

$$\pi_\omega(\Delta) \leq \pi_\omega(\mathcal{G}_x) \leq 2d|\mathcal{G}_x| \tag{3.38}$$

and

$$\pi_\omega(\mathcal{H}_x) \leq 4d|\mathcal{G}_x|. \tag{3.39}$$

It follows that $E_{\omega,x}T_1 \leq \tilde{E}_x S_x \leq (4d/\alpha)|\mathcal{G}_x|$. □

Proof of Proposition 3.5. For simplicity of the notation, let us assume that ℓ is even; otherwise, replace all occurrences of $\ell/2$ by $\lceil \ell/2 \rceil$. By reversibility of $\hat{X}$, if $k < \ell$,

$$P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_k \geq n/2) = P_{\omega,0}(\hat{X}_\ell = 0,\, T_\ell + \cdots + T_{\ell-k+1} \geq n/2). \tag{3.40}$$

This means that the probability of interest is bounded by twice the quantity on the left with $k = \ell/2$. Chebyshev's inequality then yields

$$\begin{aligned} P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_\ell \geq n) &\leq 2P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_{\ell/2} \geq n/2) \\ &\leq \frac{4}{n} E_{\omega,0}\bigl(1_{\{\hat{X}_\ell = 0\}}(T_1 + \cdots + T_{\ell/2})\bigr). \end{aligned} \tag{3.41}$$

Conditioning on the position of $\hat{X}$ at the times before and after T_j we then get

$$\begin{aligned} &P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_\ell \geq n) \\ &\qquad \leq \sum_{j=1}^{\ell/2} \sum_{x,y} \frac{4}{n} P_{\omega,0}(\hat{X}_{j-1} = x)\, E_{\omega,x}(T_1\, 1_{\{\hat{X}_1 = y\}})\, P_{\omega,y}(\hat{X}_{\ell-j} = 0). \end{aligned} \tag{3.42}$$

The calculation now proceeds by inserting uniform bounds for the last two terms on the right-hand side, and resumming the result using a stationarity argument.

Since $\ell - j \geq \ell/2$, reversibility and Lemma 3.2 tell us

$$P_{\omega,y}(\hat{X}_{\ell-j} = 0) = \frac{\pi_\omega(0)}{\pi_\omega(y)} P_{\omega,0}(\hat{X}_{\ell-j} = y) \leq \frac{c}{\ell^{d/2}} \tag{3.43}$$

uniformly in $y \in \mathscr{C}_{\infty,\alpha}$ for some absolute constant c. Furthermore, Lemma 3.8 gives

$$\sum_y E_{\omega,x}(T_1\, 1_{\{\hat{X}_1 = y\}}) = E_{\omega,x}(T_1) \leq c|\mathcal{G}_x| \tag{3.44}$$

where $\mathcal{G}_x$ is the weak component incident to x. Rewriting the sum over j as an ergodic average, Lemma 3.6 with $f(\omega) = |\mathcal{G}_0|$ and Lemma 3.7 now show that, for all $k \geq 1$,

$$\sum_{j=1}^{k} \sum_x P_{\omega,0}(\hat{X}_{j-1} = x)\, |\mathcal{G}_x| = E_{\omega,0}\biggl(\sum_{j=0}^{k-1} |\mathcal{G}_{\hat{X}_j}| \biggr) \leq C(\omega)k \tag{3.45}$$

for a random constant $C(\omega)$. Using (3.43–3.45) in (3.42), the desired bound (3.30) follows.

In order to prove the convergence to zero in (3.31), we note that

$$\sum_{n=1}^{\infty} P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_\ell \geq n) = E_{\omega,0}\bigl(1_{\{\hat{X}_\ell = 0\}}\,(T_1 + \cdots + T_\ell)\bigr). \tag{3.46}$$

The argument (3.42–3.45) shows that the expectation on the right is finite a.s. Since $n \mapsto P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_\ell \geq n)$ is non-increasing, the claim follows by noting that, for any non-increasing non-negative sequence (a_n) with $\limsup_{n\to\infty} na_n > 0$, the sum $\sum_{n\geq 1} a_n$ diverges. □

3.3 Proof of the upper bound.

To turn (3.30) into the proof of Theorem 2.1, we will also need the following standard fact from Markov chain theory:

Lemma 3.9 *The sequence $n \mapsto \mathsf{P}_\omega^{2n}(0,0)$ is decreasing.*

Proof. Let $\langle f, g\rangle_\omega = \sum_{x\in\mathbb{Z}^d} \pi_\omega(x) f(x) g(x)$ denote a scalar product in $L^2(\mathbb{Z}^d, \pi_\omega)$. Then

$$\mathsf{P}_\omega^{2n}(0,0) = \langle \delta_0, \mathsf{P}_\omega^{2n}\delta_0\rangle_\omega. \tag{3.47}$$

Since P_ω is self-adjoint and $\|\mathsf{P}_\omega\|_2 \leq 1$, the sequence of operators P_ω^{2n} is decreasing. □

Now we put everything together and prove the desired heat-kernel upper bounds:

Proof of Theorem 2.1(1). Introduce the random variable

$$R_n = \sup\{\ell \geq 0\colon T_1 + \cdots + T_\ell \leq n\}. \tag{3.48}$$

The fact that $0 \in \mathscr{C}_{\infty,\alpha}(\omega)$ yields

$$\sum_{m\geq n} P_{\omega,0}(X_m = 0,\, R_m = \ell) = P_{\omega,0}(\hat{X}_\ell = 0,\, T_1 + \cdots + T_\ell \geq n). \tag{3.49}$$

Proposition 3.5 now implies

$$\sum_{n\leq m<2n} P_{\omega,0}(X_m = 0,\, R_m = \ell) \leq C(\omega)\frac{\ell^{1-d/2}}{n}. \tag{3.50}$$

By summing over $\ell = 1, \ldots, 2n$ and using that $R_m \leq 2n$ once $m \leq 2n$ we derive

$$\sum_{n\leq m<2n} \mathsf{P}_\omega^m(0,0) \leq \tilde{C}(\omega)\begin{cases} n^{1-d/2}, & d = 2,3,\\ n^{-1}\log n, & d = 4,\\ n^{-1}, & d \geq 5,\end{cases} \tag{3.51}$$

where $\tilde{C}$ is proportional to C. By Lemma 3.9, $\mathsf{P}_\omega^{2m}(0,0)$ is decreasing in m and so the sum on the left is bounded below by $\frac{1}{2}n\mathsf{P}_\omega^{2n}(0,0)$. From here the claim follows. □

Proof of Theorem 2.1(2). By (3.31), for each fixed $\ell \geq 1$ the sum in (3.50) multiplied by n tends to zero as $n \to \infty$. As $\ell^{1-d/2}$ is summable in $d \geq 5$, the uniform bound (3.50) shows the same holds even under the sum over $\ell \geq 1$. □

4. Examples with slow decay

Here we provide proofs of Theorems 2.2 and 2.3. The underlying ideas are very similar, but the proof of Theorem 2.2 is technically easier.

4.1 Anomalous decay in $d \geq 5$.

The proof of Theorem 2.2 will be based on the following strategy: Suppose that in a box of side length ℓ_n there exists a configuration where a strong bond is separated from other sites by bonds of strength $1/n$, and (at least) one of these "weak" bonds is connected to the origin by a "strong" path not leaving the box. Then the probability that the walk is back to the origin at time n is bounded below by the probability that the walk goes directly towards the above pattern (this costs $\mathrm{e}^{O(\ell_n)}$ of probability) then crosses the weak bond (which costs $1/n$), spends time $n - 2\ell_n$ on the strong bond (which costs only $O(1)$ of probability), then crosses a weak bond again (another factor of $1/n$) and then heads towards the origin to get there on time (another $\mathrm{e}^{O(\ell_n)}$ term). The cost of this strategy is $O(1)\mathrm{e}^{O(\ell_n)}n^{-2}$ so if $\ell_n = o(\log n)$ then we get leading order n^{-2}.

Proof of Theorem 2.2(1). Our task is to construct environments for which (2.3) holds. For $\kappa > 1/d$ let $\epsilon > 0$ be such that $(1 + 4d\epsilon)/d < \kappa$. Let $\mathbb{B}$ denote the set of edges in $\mathbb{Z}^d$ and let $\mathbb{P}$ be an i.i.d. conductance law on $\{2^{-N}\colon N \geq 0\}^{\mathbb{B}}$ such that:

$$\mathbb{P}(\omega_b = 1) > p_{\mathrm{c}}(d) \tag{4.1}$$

and

$$\mathbb{P}(\omega_b = 2^{-N}) = cN^{-(1+\epsilon)}, \qquad N \geq 1, \tag{4.2}$$

where $c = c(\epsilon)$ is adjusted so that the distribution is normalized. Let $\hat{\mathrm{e}}_1$ denote the unit vector in the first coordinate direction. Define the scale

$$\ell_N = N^{(1+4d\epsilon)/d} \tag{4.3}$$

and, given $x \in \mathbb{Z}^d$, let $A_N(x)$ be the event that the configuration near x, $y = x + \hat{\mathrm{e}}_1$ and $z = x + 2\hat{\mathrm{e}}_1$ is as follows (see the comments before this proof):

(1) $\omega_{yz} = 1$ and $\omega_{xy} = 2^{-N}$, while every other bond emanating out of y or z has $\omega_b \leq 2^{-N}$.
(2) x is connected to the boundary of the box of side length $(\log \ell_N)^2$ centered at x by bonds with conductance one.

Since bonds with $\omega_b = 1$ percolate and since $\mathbb{P}(\omega_b \leq 2^{-N}) \sim N^{-\epsilon}$, we have

$$\mathbb{P}\bigl(A_N(x)\bigr) \geq cN^{-[1+(4d-2)\epsilon]}. \tag{4.4}$$

Now consider a grid $\mathbb{G}_N$ of sites in $[-\ell_N, \ell_N]^d \cap \mathbb{Z}^d$ that are spaced by distance $2(\log \ell_N)^2$. The events $\{A_N(x)\colon x \in \mathbb{G}_N\}$ are independent, so

$$\mathbb{P}\Bigl(\bigcap_{x\in\mathbb{G}_N} A_N(x)^{\mathrm{c}}\Bigr) \leq \exp\Bigl\{-c\Bigl(\frac{\ell_N}{(\log \ell_N)^2}\Bigr)^d N^{-[1+(4d-2)\epsilon]}\Bigr\} \leq \mathrm{e}^{-cN^{\epsilon}} \tag{4.5}$$

and the intersection occurs only for finitely many N.

By the stretched-exponential decay of truncated connectivities (Grimmett [13, Theorem 8.65]), every connected component of side length $(\log \ell_N)^2$ in $[-\ell_N, \ell_N]^d \cap \mathbb{Z}^d$ will eventually be connected to the largest connected component in $[-2\ell_N, 2\ell_N]^d \cap \mathbb{Z}^d$. We conclude that there exists $N_0 = N_0(\omega)$ with $\mathbb{P}(N_0 < \infty) = 1$ such that once $N \geq N_0$, the event $A_N(x)$ occurs for some even-parity site $x = x_N(\omega) \in [-\ell_N, \ell_N]^d \cap \mathbb{Z}^d$ that is connected to 0 by a path, Path_N, in $[-2\ell_N, 2\ell_N]^d$, on which only the last N_0 edges—namely, those close to the origin—may have conductance smaller than one.

We are now ready to employ the above strategy. Suppose $N \geq N_0$ and let n be such that $2^N \leq 2n < 2^{N+1}$. Let x_N be the site in $[-\ell_N, \ell_N]^d \cap \mathbb{Z}^d$ for which $A_N(x)$ occurs and let r_N be the length of Path_N. Let $\alpha = \alpha(\omega)$ be the minimum of ω_b for b within N_0 steps of the origin. The passage from 0 to x_N in time r_N has probability at least $\alpha^{N_0}(2d)^{-r_N}$, while staying on the bond (y, z) for time $2n - 2r_N - 2$ costs an amount which is bounded independently of ω. The transitions across (x, y) cost order 2^{-N} each. Hence we have

$$\mathsf{P}^{2n}_\omega(0,0) \geq c\alpha^{2N_0}(2d)^{-2r_N}2^{-2N}. \tag{4.6}$$

By the comparison of the graph-theoretic distance and the Euclidean distance (Antal and Pisztora [1]), we have $r_N \leq c\ell_N$ once N is sufficiently large. Since n is of order 2^N we are done. □

The argument for the second part follows very much the same strategy:

Proof of Theorem 2.2(2). Let (λ_n) be a sequence in the statement and suppose, without loss of generality, that $\lambda_1 \gg 1$. Let

$$q_n = \left(\frac{1}{2}\frac{\log \lambda_n}{\log(2d)}\right)^{1/4} \tag{4.7}$$

and let $\{n_k\}$ be even numbers chosen as follows:

$$1 - q_{n_1}^{-1} > p_c \quad \text{and} \quad q_{n_{k+1}} > 2q_{n_k}. \tag{4.8}$$

Define an i.i.d. law $\mathbb{P}$ on $(\{1\} \cup \{n_k \colon k \geq 1\})^{\mathbb{B}}$ as follows:

$$\mathbb{P}(\omega_b = 1) = 1 - q_{n_1}^{-1} \quad \text{and} \quad \mathbb{P}(\omega_b = 1/n_k) = q_{n_k}^{-1} - q_{n_{k+1}}^{-1}. \tag{4.9}$$

Let $\mathcal{C}_\infty$ denote the (a.s. unique) infinite connected component of edges with conductance one.

By following the argument in the proof of Theorem 2.2(1), for almost every ω and every k large enough, we can find $x \in \mathcal{C}_\infty$ such that:

(1) For $y = x + \hat{e}_1$ and $z = x + 2\hat{e}_1$, we have $\omega_{y,z} = 1$, and all other bonds emanating from y and z are of conductance $1/n_k$.
(2) The chemical distance between x and the closest point in $\mathcal{C}_\infty$ to the origin is less than $q_{n_k}^4$.

Explicitly, set $\ell_N = \theta q_{n_k}^4$ for some constant θ and let $A_n(x)$ be the event that (1) holds and x is connected to the boundary of the box $x + [-(\log \ell_N)^2, (\log \ell_N)^2]^d$ by edges with strength one. Then $\mathbb{P}(A_N(x)) \geq cq_{n_k}^{-4d+2} = c\ell_N^{-d+\delta}$ for $\delta = (2d)^{-1}$. Plugging this in (4.5)

results in a sequence that is summable on k (note that q_k increase exponentially). Percolation arguments, and the choice of θ, then ensure that (most of) the x's where $A_N(x)$ occurs have a strong connection near the origin of length at most $q_{n_k}^4$.

The argument leading to (4.6)—with r_N replaced by $q_{n_k}^4$—now gives

$$\mathsf{P}_\omega^{n_k}(0,0) \geq c\alpha^{2N_0} \frac{(2d)^{-2q_{n_k}^4}}{n_k^2}. \tag{4.10}$$

By the choice of q_n, we are done. □

4.2 Time-dependent environments.

Here we will prove Theorem 2.3. Let $\mathbb{P}$ be the Bernoulli measure on $\mathbb{B}$ with parameter $p > p_c(d)$. Let $\mathcal{C}_\infty$ denote the infinite component of occupied bonds. We define $\omega_b = 1$ on occupied bonds and $\omega_b = 1/n$ on vacant bonds. The proof proceeds via three lemmas:

Lemma 4.1 *Let $Y = (Y_1, \dots, Y_n)$ be the first n steps of the random walk on environment ω conditioned to avoid bonds with $\omega_b = 1/n$. Let $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_n)$ be the simple random walk on the percolation cluster of $\omega_b = 1$. Then the the corresponding path measures are absolutely continuous with respect to each other and the Radon-Nikodym derivatives are (essentially) bounded away from zero and infinity, uniformly in n and $\omega \in \{0 \in \mathcal{C}_\infty\}$.*

Proof. Fix a sequence of sites $x_1, \dots, x_n \in \mathcal{C}_\infty$ such that $\omega_{x_i,x_{i+1}} = 1$ for all $i = 1, \dots, n-1$. Then the probability that $\tilde{X}$ executes this sequence is $\prod_{i=0}^n d(x_i)^{-1}$, where $d(x)$ is the degree of the percolation cluster at x. For Y we get $C_n \prod_{i=0}^{n-1} \pi_\omega(x_i)^{-1}$, where C_n^{-1} is the probability that the unconditioned random walk X has not used a weak bond in its first n-th steps. Since

$$\pi_\omega(x) - d(x) = O(1/n), \tag{4.11}$$

the ratio of the products is bounded away from zero and infinity uniformly in n and the points $x_1, \dots, x_n$. But both path distributions are normalized and so C_n is bounded as well. □

Next we provide a lower bound on the probability that the walk X visits a given site in n steps. Let S_x be the first visit of X to x,

$$S_x = \inf\{n \geq 0 \colon X_n = x\}. \tag{4.12}$$

Then we have:

Lemma 4.2 *For a.e. $\omega \in \{0 \in \mathcal{C}_\infty\}$ there is $C = C(\omega) > 0$ and a constant $n_0 < \infty$ such that for all $n \geq n_0$ and all $x \in \mathcal{C}_\infty$ satisfying $|x| \leq \sqrt{n}$, we have*

$$P_{\omega,0}(S_x \leq n) \geq C(\omega)|x|^{-(d-2)}. \tag{4.13}$$

Proof. The choice of the conductance values ensures that the probability that X stays on $\mathcal{C}_\infty$ for the first n steps is uniformly positive. Conditioning on this event, and applying Lemma 4.1, it thus suffices to prove (4.13) for the walk $\tilde{X}$. The proof makes use

of Barlow's heat-kernel bounds for the random walk on percolation cluster; cf [2, Theorem 1].

Consider the continuous time version $\tilde{X}'$ of the walk $\tilde{X}$, i.e., $\tilde{X}'$ executes the same steps but at times that are i.i.d. exponential. By integrating the heat-kernel bounds we get that the expected amount of time $\tilde{X}'$ spends at x up to time $n/2$ is at least $C(\omega)|x|^{-(d-2)}$. A similar calculation shows that the expected time the walk $\tilde{X}'$ spends at x *conditioned on it hitting* x is uniformly bounded. Therefore the probability of $\tilde{X}'$ hitting x before time $n/2$ is at least $C(\omega)|x|^{-(d-2)}$. To get back to $\tilde{X}$, we need to subtract the probability that by continuous time $n/2$ the walk $\tilde{X}'$ did more than n discrete steps, which is less than e^{-cn}. As $|x| \le \sqrt{n}$, this cannot compete with $|x|^{-(d-2)}$ once n is sufficiently large. □

We now define the notion of a *trap* which is similar to that underlying the event $A_N(x)$ in the proof of Theorem 2.2. Explicitly, a trap is the triple of sites x, y, z with $y = x + \hat{\mathrm{e}}_1$ and $z = x + 2\hat{\mathrm{e}}_1$ such that $x \in \mathcal{C}_\infty$ and such that all bonds emanating out of y and z are weak except the bond between them. Let $T(x)$ be the event that a trap occurs at x.

Lemma 4.3 *For a.e. $\omega \in \{0 \in \mathcal{C}_\infty\}$ there is $c < \infty$ and $n_1(\omega) < \infty$ such that*

$$\sum_{\substack{x:\, |x|\le\sqrt{n} \\ T(x) \text{ occurs}}} |x|^{-(2d-4)} \ge \begin{cases} c, & d \ge 5, \\ c \log n, & d = 4, \end{cases} \tag{4.14}$$

for all $n \ge n_1$.

Proof. This is a consequence of the Spatial Ergodic Theorem. Indeed, let $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ and note that the fraction of Λ_L occupied by $\{x \in \Lambda_L \colon T(x) \text{ occurs}\}$ converges a.s. to $\rho = \mathbb{P}(T(0)) > 0$. But then also the corresponding fraction in the annuli $\Lambda_{2^{k+1}} \setminus \Lambda_{2^k}$ converges a.s. to ρ. In particular, there is $k_0 = k_0(\omega)$ such that this fraction exceeds $\rho/2$ for all $k \ge k_0$. Now take n and find k so that $2^k \le \sqrt{n} \le 2^{k+1}$. Bounding $|x| \le 2^{k+1}$ on the k-th annulus, we get

$$\sum_{\substack{x:\, |x|\le\sqrt{n} \\ T(x) \text{ occurs}}} |x|^{-(2d-4)} \ge \sum_{\ell=k_0}^{k} \frac{\rho}{2} \frac{|\Lambda_{2^{\ell+1}} \setminus \Lambda_{2^\ell}|}{(2^{\ell+1})^{2d-4}}. \tag{4.15}$$

As $|\Lambda_{2^{\ell+1}} \setminus \Lambda_{2^\ell}| \ge (2^\ell)^d$, the result follows. □

We are now ready to prove the heat-kernel lower bounds (2.7):

Proof of Theorem 2.3. Pick $\omega \in \{0 \in \mathcal{C}_\infty\}$ and let x be a trap (i.e., event $T(x)$ occurs and y and z are the endpoints of the "trapped" strong edge) with $|x| < \frac{1}{4}\sqrt{n}$. Let $U(x, k, \ell)$ be the event that the random walk starts at the origin, hits x for the first time at time k, crosses the edge (y, z), spends time $2n - k - \ell - 2$ on this edge and then exits, and then arrives back to the origin in ℓ units of time. Clearly,

$$P_{\omega,0}\big(U(x,k,\ell)\big) \ge P_{\omega,0}(S_x = k)\, \frac{c}{n} \Big(1 - \frac{\tilde{c}}{n}\Big)^{n-k-\ell-2} \frac{c}{n}\, P_{\omega,x}(S_0 = \ell) \tag{4.16}$$

where c and $\tilde{c}$ are constants depending only on dimension. Reversibility tells us

$$P_{\omega,x}(S_0 = \ell) = P_{\omega,0}(S_x = \ell)\frac{\pi_\omega(0)}{\pi_\omega(x)} \geq cP_{\omega,0}(S_x = \ell) \tag{4.17}$$

and so

$$P_{\omega,0}\bigl(U(x,k,\ell)\bigr) \geq cn^{-2}\,P_{\omega,0}(S_x = k)P_{\omega,0}(S_x = \ell). \tag{4.18}$$

Denote

$$U(x) = \bigcup_{\substack{1\leq k\leq n/5\\ 1\leq \ell\leq n/5}} U(x,k,\ell). \tag{4.19}$$

Using the disjointness of $U(x,k,\ell)$ for different k and ℓ and invoking Lemma 4.2,

$$P_{\omega,0}\bigl(U(x)\bigr) \geq C(\omega)\,n^{-2}|x|^{-(2d-4)}. \tag{4.20}$$

But, for n large enough, the events $\{U(x)\colon\ x \text{ is a trap}\}$ are disjoint because the restriction $k,\ell < n/5$ makes the walk spend more than half of its time at the strong bond constituting the trap. (This bond determines the trap entrance/exit point x.) Hence,

$$\mathsf{P}^{2n}_\omega(0,0) \geq P_{\omega,0}\biggl(\bigcup_{x\colon |x|<\frac12\sqrt{n}} U(x)\biggr) \geq C(\omega)\,n^{-2}\sum_{\substack{x\colon |x|\leq\frac12\sqrt{n}\\ T(x)\text{ occurs}}} |x|^{-(2d-4)}. \tag{4.21}$$

Applying Lemma 4.3, the desired claim is proved. □

5. Appendix: Isoperimetry on percolation cluster

In this section we give a proof of isoperimetry of the percolation cluster which were needed in the proof of Lemma 3.4. Consider bond percolation with parameter p and let $\mathcal{C}_\infty$ denote the a.s. unique infinite cluster. For $\Lambda \subset \mathbb{Z}^d$ let $\partial\Lambda$ denote the set of edges between Λ and $\mathbb{Z}^d \setminus \Lambda$ and let $\partial^\omega\Lambda$ denote those edges in $\partial\Lambda$ that are occupied. We call Λ *ω-connected* if every two sites in Λ can be connected by a finite path that uses only the sites in Λ and whose every bond is occupied in ω. Then we have:

Theorem 5.1 *For all $d \geq 2$ and $p > p_c(d)$, there are positive and finite constants $c_1 = c_1(d,p)$ and $c_2 = c_2(d,p)$ and an a.s. finite random variable $R_0 = R_0(\omega)$ such that for each $R \geq R_0$ and each ω-connected Λ satisfying*

$$\Lambda \subset \mathcal{C}_\infty \cap [-R,R]^d \quad \text{and} \quad |\Lambda| \geq (c_1 \log R)^{\frac{d}{d-1}} \tag{5.1}$$

we have

$$|\partial^\omega\Lambda| \geq c_2|\Lambda|^{\frac{d-1}{d}}. \tag{5.2}$$

This claim was the basic technical point of Benjamini and Mossel [3] as well as of many subsequent studies of random walk on percolation cluster. Unfortunately, the proof of [3] for the case $d \geq 3$ and p close to $p_c(d)$ contains a gap. A different proof was recently given in Rau [22, Proposition 1.4] but the argument is quite long and it builds (ideologically) upon a weaker version of (5.2) proved by Mathieu and Remy [20],

whose proof is also rather long. Closely related estimates were derived in Barlow [2], but additional arguments are needed to extract (5.2).

For the convenience of the reader, and future reference, we provide a self-contained (and reasonably short) proof of Theorem 5.1 below. Our arguments are close to those of Benjamini and Mossel [3] and they indicate that the seriousness of the gaps in [3] has been somewhat exaggerated. An independent argument, based on exponential cluster repulsion, has simultaneously been found by Pete [21].

Theorem 5.1 will be a consequence of the following, slightly more general estimate:

Proposition 5.2 *For $d \geq 2$ and $p > p_c(d)$, there are $c_2, c_3, \zeta \in (0,\infty)$ such that for all $t > 0$,*

$$\mathbb{P}\bigl(\,\exists \Lambda \ni 0,\ \omega\text{-connected},\ |\Lambda| \geq t^{\frac{d}{d-1}},\ |\partial^{\omega}\Lambda| < c_2|\Lambda|^{\frac{d-1}{d}}\bigr) \leq c_3 \mathrm{e}^{-\zeta t}. \tag{5.3}$$

Proof of Theorem 5.1 from Proposition 5.2. Using translation invariance, the probability that there exists a set $\Lambda \subset \mathbb{Z}^d \cap [-R,R]^d$ with the properties listed in (5.3) is bounded by a constant times $R^d \mathrm{e}^{-\zeta t}$. This applies, in particular, to sets $\Lambda \subset \mathcal{C}_\infty \cap [-R,R]^d$. Setting $t = c_1 \log R$ for c_1 such that $c_1\zeta > d+1$, this probability is summable on R. By the Borel-Cantelli lemma, the corresponding event occurs only for finitely many R. □

The advantage of the formulation (5.3) is that it links the tail bound on R_0 to the cut-off on the size of $|\Lambda|$. For instance, if we only care for $|\Lambda| \geq R^\theta$ for some $\theta \in (0,d)$, then $\mathbb{P}(R_0 \geq R)$ decays exponentially with $R^{\theta(1-1/d)}$.

As noted by Benjamini and Mossel [3] the proof is quite straightforward in $d = 2$ and in any d once p is close to one. However, to have a proof that works in $d \geq 3$ all the way down to p_c, we will have to invoke the "static" block-renormalization technique (Grimmett [13, Section 7.4]). For each integer $N \geq 1$, consider the cubes

$$B_N(x) = x + \mathbb{Z}^d \cap [0,N]^d \tag{5.4}$$

and

$$\tilde{B}_{3N}(x) = x + \mathbb{Z}^d \cap [-N,2N]^d \tag{5.5}$$

Let $G_N(x)$ be the event such that:

(1) For each neighbor y of x, the side of the block $B_N(Ny)$ adjacent to $B_N(Nx)$ is connected to the opposite side of $B_N(Ny)$ by an occupied path.
(2) Any two occupied paths connecting $B_N(Nx)$ to the boundary of $\tilde{B}_{3N}(Nx)$ are connected by an occupied path using only edges with both endpoints in $\tilde{B}_{3N}(Nx)$.

From Theorem 8.97 and Lemma 7.89 in Grimmett [13] we know that, for each $p > p_c(d)$,

$$\mathbb{P}\bigl(G_N(0)\bigr) \underset{N\to\infty}{\longrightarrow} 1. \tag{5.6}$$

By [13, Theorem 7.65], for each $p \in [0,1]$ there exists $\eta_N(p) \in [0,1]$ with $\eta_N(p) \uparrow 1$ as $p \uparrow 1$ such that the 0-1-valued process $\{1_{G_N(x)} \colon x \in \mathbb{Z}^d\}$ is dominated from below by independent Bernoulli's with parameter $\eta_N(p)$.

Given a finite set $\Lambda \subset \mathbb{Z}^d$, let $\Lambda^{(N)} = \{x \in \mathbb{Z}^d \colon \Lambda \cap B_N(Nx) \neq \varnothing\}$ and define $\overline{\Lambda}^N$ to be the complement of the unique infinite component of $\mathbb{Z}^d \setminus \Lambda^{(N)}$. We will also need a

notation $\partial^*\Delta$ for the inner site-boundary of a set Δ,

$$\partial^*\Delta = \bigl\{x \in \Delta\colon \exists y \in \mathbb{Z}^d \setminus \Delta \text{ with } |x-y| = 1\bigr\}, \tag{5.7}$$

and $\operatorname{diam}\Lambda$ for the diameter of Λ in ℓ_∞-distance on $\mathbb{Z}^d$. The crucial observation—which is where the setting of [3] runs into a problem—is now as follows:

Lemma 5.3 *For $\omega \in \Omega$, let $\Lambda \subset \mathbb{Z}^d$ be ω-connected with $\overline{\Lambda}^N = \Delta$ and $\operatorname{diam}\Lambda \geq 3N$. If*

$$|\partial^\omega\Lambda| < \frac{1}{2\cdot 3^d}|\partial^*\Delta| \tag{5.8}$$

then

$$\bigl|\{x \in \partial^*\Delta\colon G_N(x)^{\rm c} \text{ occurs}\}\bigr| > \frac{1}{2}|\partial^*\Delta|. \tag{5.9}$$

Proof. Let $\Delta = \overline{\Lambda}^N$ and note that $x \in \partial^*\Delta$ implies $x \in \overline{\Lambda}^N$, i.e., $\Lambda \cap B_N(Nx) \neq \emptyset$. We claim that, for each $x \in \partial^*\Delta$,

$$G_N(x) \subset \bigl\{\tilde{B}_{3N}(Nx) \text{ contains an edge in } \partial^\omega\Lambda\bigr\}. \tag{5.10}$$

Indeed, if $G_N(x)$ occurs then, by $\operatorname{diam}\Lambda \geq 3N$, the box $B_N(Nx)$ is connected to a site on the boundary of $\tilde{B}_{3N}(Nx)$ by an occupied path in Λ. As $x \in \partial^*\Delta$ there exists a neighbor $y \in \Delta^{\rm c}$. Part (1) of the definition of $G_N(x)$ ensures that there is another such path "crossing" $B_N(Ny)$; as $\Lambda \cap B_N(Ny) = \emptyset$, this path contains no sites in Λ. By part (2) of the definition of $G_N(x)$, the two paths must be joined by an occupied path in $\tilde{B}_{3N}(Nx)$ which then must contain an edge in $\partial^\omega\Lambda$.

Since each edge in $\partial^\omega\Lambda$ belongs to at most 3^d distinct cubes $\tilde{B}_{3N}(Nx)$ with $x \in \partial^*\Delta$, the number of boundary sites $x \in \partial^*\Delta$ where $G_N(x)$ occurs is bounded by $3^d|\partial^\omega\Lambda|$, i.e.,

$$|\partial^*\Delta| - \bigl|\{x \in \partial^*\Delta\colon G_N(x)^{\rm c} \text{ occurs}\}\bigr| \leq 3^d|\partial^\omega\Lambda|. \tag{5.11}$$

Under the assumption (5.8), this implies (5.9). □

Proof of Proposition 5.2. Abbreviate $c_4 = (2\cdot 3^d)^{-1}$ and fix $\Delta \subset \mathbb{Z}^d$ finite, connected with connected complement. Suppose Λ is ω-connected with $\overline{\Lambda}^N = \Delta$. Then $|\Delta| \geq N^{-d}|\Lambda|$ and, invoking the standard isoperimetry on $\mathbb{Z}^d$,

$$|\partial^*\Delta| \geq c_5|\Delta|^{\frac{d-1}{d}} \geq c_5 N^{1-d}|\Lambda|^{\frac{d-1}{d}}, \tag{5.12}$$

where $c_5 = c_5(d) > 0$. Setting $c_2 = c_4c_5N^{1-d}$ we then have

$$\bigl\{|\partial^\omega\Lambda| < c_2|\Lambda|^{\frac{d-1}{d}}\bigr\} \subset \bigl\{|\partial^\omega\Lambda| < c_4|\partial^*\Delta|\bigr\} \tag{5.13}$$

and also

$$|\partial^*\Delta| \geq c_5 N^{1-d}t \tag{5.14}$$

whenever $|\Lambda| \geq t^{\frac{d}{d-1}}$. We will suppose $t^{\frac{d}{d-1}} \geq (3N)^d$ to enable Lemma 5.3.

Equation (5.13), Lemma 5.3 and the fact that $\{1_{G_N(x)}\colon x \in \mathbb{Z}^d\}$ stochastically dominates site percolation with parameter $\eta_N(p) = 1 - \epsilon_N$ then yield

$$\begin{aligned}\mathbb{P}\bigl(\exists \Lambda \ni 0,\ \omega\text{-connected},\ |\Lambda| \ge t^{\frac{d-1}{d}},\ \overline{\Lambda}^N = \Delta,\ |\partial^\omega \Lambda| < c_2|\Lambda|^{\frac{d-1}{d}}\bigr)\\ \le \mathbb{P}\left(\sum_{x\in\partial^*\Delta} 1_{G_N(x)} \le \frac{1}{2}|\partial^*\Delta|\right) \le 2^{|\partial^*\Delta|}(\epsilon_N)^{\frac{1}{2}|\partial^*\Delta|}. \qquad (5.15)\end{aligned}$$

Here $2^{|\partial^*\Delta|}$ bounds the number of possible subsets $\{x \in \partial^*\Delta\colon G_N(x)^{\mathrm{c}} \text{ occurs}\}$ of $\partial^*\Delta$. To finish the proof, we need to sum over all eligible Δ's.

Let $c_6 = c_6(d)$ be a number such that c_6^n bounds the total number of connected sets $\Delta \subset \mathbb{Z}^d$ with connected complement, containing the origin and having $|\partial^*\Delta| = n$. (The fact that this grows exponentially in n follows from the fact that $\partial^*\Delta$ is connected in an appropriate notion of adjacency on $\mathbb{Z}^d$.) As $\epsilon_N \to 0$ by (5.6), we can find N so that $2c_6\sqrt{\epsilon_N} \le 1/2$. Summing (5.15) over all connected Δ with connected complement that obey (5.14) now gives

$$\begin{aligned}\mathbb{P}\bigl(\exists \Lambda \ni 0,\ \omega\text{-connected},\ |\Lambda| \ge t^{\frac{d}{d-1}},\ |\partial^\omega \Lambda| < c_2|\Lambda|^{\frac{d-1}{d}}\bigr)\\ \le \sum_{n\ge c_5 N^{1-d}t} 2^n(\epsilon_N)^{\frac{1}{2}n}c_6^n \le \sum_{n\ge c_5 N^{1-d}t} 2^{-n} \le 2^{1-\lfloor c_5 N^{1-d}t\rfloor}, \qquad (5.16)\end{aligned}$$

where we also assumed that $2\epsilon_N^{1/2} \le 1$ to get the first inequality. Choosing the constants appropriately, this yields the desired claim. □

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