To appear in Communications in Mathematical Physics

Partition function zeros at first-order phase transitions: A general analysis

M. Biskup[1], **C. Borgs**[2], **J.T. Chayes**[2], **L.J. Kleinwaks**[3], **R. Kotecký**[4]

[1] Department of Mathematics, UCLA, Los Angeles CA 90095-1555, U.S.A.
[2] Microsoft Research, One Microsoft Way, Redmond WA 98052, U.S.A.
[3] Department of Physics, Princeton University, Princeton NJ 08544, U.S.A.
[4] Center for Theoretical Study, Charles University, Prague 110 00, Czech Republic

Received: April 3, 2003 / Accepted: March 31, 2004

Abstract: We present a general, rigorous theory of partition function zeros for lattice spin models depending on one complex parameter. First, we formulate a set of natural assumptions which are verified for a large class of spin models in a companion paper [5]. Under these assumptions, we derive equations whose solutions give the location of the zeros of the partition function with periodic boundary conditions, up to an error which we prove is (generically) exponentially small in the linear size of the system. For asymptotically large systems, the zeros concentrate on phase boundaries which are simple curves ending in multiple points. For models with an Ising-like plus-minus symmetry, we also establish a local version of the Lee-Yang Circle Theorem. This result allows us to control situations when in one region of the complex plane the zeros lie precisely on the unit circle, while in the complement of this region the zeros concentrate on less symmetric curves.

Contents

© 2003 M. Biskup, C. Borgs, J.T. Chayes, L.J. Kleinwaks, R. Kotecký. Reproduction of the entire article for non-commercial purposes is permitted without charge.

1. Introduction

1.1. Motivation. One of the cornerstones of equilibrium statistical mechanics is the notion that macroscopic systems undergo phase transitions as the external parameters change. A mathematical description of phase transitions was given by Gibbs [17] who characterized a phase transition as a point of non-analyticity in thermodynamic functions, e.g., the pressure. This definition was originally somewhat puzzling since actual physical systems are finite, and therefore their thermodynamic functions are manifestly real-analytic. A solution to this contradiction came in two seminal papers by Yang and Lee [25, 42], where it was argued that non-analyticities develop in physical quantities because, as the system passes to the thermodynamic limit, complex singularities of the pressure pinch the physical (i.e., real) domain of the system parameters. Since the pressure is proportional to the logarithm of the partition function, these singularities correspond exactly to the zeros of the partition function.

In their second paper [25], Lee and Yang demonstrated the validity of their theory in a particular example of the Ising model in a complex magnetic field h. Using an induction argument, they proved the celebrated Lee-Yang Circle Theorem which states that, in this model, the complex-e^h zeros of the partition function on any finite graph with free boundary conditions lie on the unit circle. The subject has been further pursued by a number of authors in the following fifty years. Generalizations of the Lee-Yang theorem have been developed [26, 31, 32, 35] and extensions to other complex parameters have been derived (for instance, the Fisher zeros [14] in the complex temperature plane and the zeros of the q-state Potts model in the complex-q plane [40, 41]). Numerous papers have appeared studying the partition function zeros using various techniques including computer simulations [10, 20, 22], approximate analyses [21, 24, 29] and exact solutions of 1D and 2D lattice systems [8, 9, 12, 18, 27, 28, 38, 39]. However, in spite of this progress, it seems fair to say that much of the original Lee-Yang program—namely, to learn about the transitions in physical systems by studying the zeros of partition functions—had remained unfulfilled.

In [2], we outlined a general program, based on Pirogov-Sinai theory [6, 33, 34, 43], to determine the partition function zeros for a large class of lattice models depending on one complex parameter z. The present paper, and its companion [5], give the mathematical details of that program. Our results apply to a host of systems with first-order phase transitions; among others, they can be applied to field-driven transitions in many low-temperature spin systems as well as temperature-driven transitions—for instance, the order-disorder transition in the q-state Potts model with large q or the confinement

Higgs transition in lattice gauge theories. We consider lattice models with a finite number of equilibrium states that satisfy several general assumptions (formulated in detail below). The validity of the assumptions follows whenever a model can be analyzed using a convergent contour expansion based on Pirogov-Sinai theory, even in the complex domain. In the present work, we study only models with periodic boundary conditions, although—with some technically involved modifications—our techniques should allow us to treat also other boundary conditions.

Under our general assumptions, we derive a set of model-specific equations; the solutions of these equations yield the locations of the partition function zeros, up to rigorously controlled errors which are typically exponentially small in the linear size of the system. It turns out that, as the system size tends to infinity, the partition function zeros concentrate on the union of a countable number of simple smooth curves in the complex z-plane. Another outcome of our analysis is a local version of the Lee-Yang Circle Theorem. Whereas the global theorem says that, for models with the full Ising interaction, all partition function zeros lie on the unit circle, our local theorem says that if the model has an Ising-like symmetry in a restricted region of the complex z-plane, the corresponding portion of the zeros lies on a piece of the unit circle. In particular, there are natural examples (see the discussion of the Blume-Capel model in [2]) where only some of the partition function zeros lie on the unit circle, and others lie on less symmetric curves. Our proof indicates that it is just the Ising plus-minus symmetry (and a natural non-degeneracy condition) that makes the Lee-Yang theorem true, which is a fact not entirely apparent in the original derivations of this result.

In addition to being of interest for the foundations of statistical mechanics, our results can often be useful on a practical level—even when the parameters of the model are such that we cannot rigorously verify all of our assumptions. We have found that our equations seem to give accurate locations of finite-volume partition function zeros for system sizes well beyond what can be currently achieved using, e.g., computer-assisted evaluations of these partition functions (see [2] for the example of the three dimensional 25-state Potts model on 1000 sites). Our techniques are also capable of handling situations with more than one complex parameter in the system. However, the actual analysis of the manifolds of partition function zeros may be technically rather involved. Finally, we remark that, in one respect, our program falls short of the ultimate goal of the original Lee-Young program—namely, to describe the phase structure of any statistical-mechanical system directly on the basis of its partition function zeros. Instead, we show that both the location of the partition function zeros and the phase structure are consequences of an even more fundamental property: the ability to represent the partition function as a sum of terms corresponding to different metastable phases. This representation is described in the next section.

1.2. Basic ideas. Here we will discuss the main ideas of our program, its technical difficulties and our assumptions in more detail. We consider spin models on $\mathbb{Z}^d$, with $d \geq 2$, whose interaction depends on a complex parameter z. Our program is based on the fact that, for a large class of such models, the partition function Z_L^{per} in a box of side L and with periodic boundary conditions can be written as

$$Z_L^{\text{per}}(z) = \sum_{m=1}^{r} q_m e^{-f_m(z)L^d} + O(e^{-\text{const}\,L} e^{-f(z)L^d}). \tag{1.1}$$

Here $q_1, \dots, q_r$ are positive integers describing the degeneracies of the phases $1, \dots, r$, the quantities $f_1, \dots, f_r$ are smooth (but not in general analytic) complex functions of the parameter z which play the role of *metastable free energies* of the corresponding phases, and $f(z) = \min_{1\le m\le r} \operatorname{Re} f_m(z)$. The real version of the formula (1.1) was instrumental for the theory of finite-size scaling near first-order phase transitions [7]; the original derivation goes back to [6].

It follows immediately from (1.1) that, asymptotically as L tends to infinity, $Z_L^{\text{per}} = 0$ requires that $\operatorname{Re} f_m(z) = \operatorname{Re} f_{\widetilde m}(z) = f(z)$ for at least two distinct indices m and $\widetilde m$. (Indeed, otherwise the sum in (1.1) would be dominated by a single, non-vanishing term.) Therefore, asymptotically, all zeros of Z_L^{per} concentrate on the set

$$\mathscr{G} = \bigl\{z\colon \text{ there exist } m \ne \widetilde m \text{ with } \operatorname{Re} f_m(z) = \operatorname{Re} f_{\widetilde m}(z) = f(z)\bigr\}. \tag{1.2}$$

Our first concern is the topological structure of $\mathscr{G}$. Let us call a point where $\operatorname{Re} f_m(z) = f(z)$ for at least three different m a *multiple point*; the points $z \in \mathscr{G}$ that are not multiple points are called *points of two-phase coexistence*. Under suitable assumptions on the functions $f_1, \dots, f_r$, we show that $\mathscr{G}$ is a countable union of non-intersecting simple smooth curves that begin and end at multiple points. Moreover, there are only a finite number of multiple points inside any compact subset of $\mathbb{C}$. See Theorem 2.1 for details.

The relative interior of each curve comprising $\mathscr{G}$ consists entirely of the points of two-phase coexistence, i.e., we have $\operatorname{Re} f_m(z) = \operatorname{Re} f_{\widetilde m}(z) = f(z)$ for exactly two indices m and $\widetilde m$. In particular, the sum in (1.1) is dominated by two terms. Supposing for a moment that we can neglect all the remaining contributions, we would have

$$Z_L^{\text{per}}(z) = q_m e^{-f_m(z)L^d} + q_{\widetilde m} e^{-f_{\widetilde m}(z)L^d}, \tag{1.3}$$

and the zeros of Z_L^{per} would be determined by the equations

$$\begin{aligned} \operatorname{Re} f_m(z) &= \operatorname{Re} f_{\widetilde m}(z) + L^{-d}\log(q_m/q_{\widetilde m}) \\ \operatorname{Im} f_m(z) &= \operatorname{Im} f_{\widetilde m}(z) + (2\ell+1)\pi L^{-d}, \end{aligned} \tag{1.4}$$

where ℓ is an integer. The presence of additional terms of course makes the actual zeros only approximate solutions to (1.4); the main technical problem is to give a reasonable estimate of the distance between the solutions of (1.4) and the zeros of Z_L^{per}. In a neighborhood of multiple points, the situation is even more complicated because there the equations (1.4) will not be even approximately correct.

It turns out that the above heuristic argument cannot possibly be converted into a rigorous proof without making serious adjustments to the initial formula (1.1). This is a consequence of subtle analytic properties of the functions f_m. For typical physical systems, the metastable free energy f_m is known to be analytic only in the interior of the region

$$\mathscr{S}_m = \bigl\{z\colon \operatorname{Re} f_m(z) = f(z)\bigr\}. \tag{1.5}$$

On the boundary of $\mathscr{S}_m$, one expects—and in some cases proves [15,19]—the existence of essential singularities. Thus (1.1) describes an approximation of an analytic function, the function Z_L^{per}, by a sum of non-analytic functions, with singularities appearing precisely in the region where we expect to find the zeros of Z_L^{per}! It is easy to construct examples where an arbitrarily small non-analytic perturbation of a complex polynomial *with a degenerate zero* produces extraneous roots. This would not be an issue along the two-phase coexistence lines, where the roots of Z_L^{per} turn out to be non-degenerate, but we would not be able to say much about the roots near the multiple points. In short, we need an approximation that respects the analytic structure of our model.

Fortunately, we do not need to look far to get the desirable analytic counterpart of (1.1). In fact, it suffices to modify slightly the derivation of the original formula. For the benefit of the reader, we will recall the main steps of this derivation: First we use a contour representation of the model—the class of models we consider is characterized by the property of having such a contour reformulation—to rewrite the partition function as a sum over the collections of contours. Then we divide the configurations contributing to Z_L^{per} into $r+1$ categories: Those in which all contours are of diameter smaller than, say, $L/3$ and in which the dominant phase is m, where $m = 1, \dots, r$, and those not falling into the preceding categories. Let $Z_m^{(L)}$ be the partial partition function obtained by summing the contributions corresponding to the configurations in the m-th category, see Fig. 1. It turns out that the error term is still uniformly bounded as in (1.1), so we have

$$Z_L^{\text{per}}(z) = \sum_{m=1}^{r} Z_m^{(L)}(z) + O(e^{-\text{const}\, L} e^{-f(z)L^d}), \tag{1.6}$$

but now the functions $Z_m^{(L)}(z)$ are analytic, and non-zero in a small neighborhood of $\mathscr{S}_m$. (However, the size of the neighborhood shrinks with $L \to \infty$, and one of the challenges of using the formula (1.6) is to cope with this restriction of analyticity.) Moreover, writing

$$Z_m^{(L)}(z) = q_m e^{-f_m^{(L)}(z)L^d} \tag{1.7}$$

and using the contour representation, the functions $f_m^{(L)}$ can be expressed by means of convergent cluster expansions [11, 23]. In particular, they can be shown to converge quickly to the functions f_m as $L \to \infty$.

In this paper, we carry out the analysis of the partition function zeros starting from the representation (1.6). In particular, we formulate minimal conditions (see Assumptions A and B in Sect. 2) on the functions $f_m^{(L)}$ and the error terms that allow us to analyze the roots of Z_L^{per} in great detail. The actual construction of the functions $f_m^{(L)}$ and the proof that they satisfy the required conditions is presented in [3, 4] for the q-state Potts model with one complex external field and q sufficiently large, and in [5] for a general class lattice models with finite number of equilibrium states.

1.3. Discussion of assumptions and results. Here we will describe our main assumptions and indicate how they feed into the proofs of our main theorems. For consistency with the previous sections, we will keep using the functions f_m and $f_m^{(L)}$ even though the assumptions will actually be stated in terms of the associated exponential variables

$$\zeta_m(z) = e^{-f_m(z)} \quad \text{and} \quad \zeta_m^{(L)}(z) = e^{-f_m^{(L)}(z)}. \tag{1.8}$$

The first set of assumptions (Assumption A, see Sect. 2.1) concerns the infinite-volume quantities f_m, and is important for the description of the set of coexistence points $\mathscr{G}$. The functions f_m are taken to be twice differentiable in the variables $x = \text{Re}z$ and $y = \text{Im}z$, and analytic in the interior of the set $\mathscr{S}_m$. If, in addition, $f(z) = \min_m \text{Re} f_m$ is uniformly bounded from above, good control of the two-phase coexistence curves is obtained by assuming that, for any distinct m and $\widetilde{m}$, the difference of the first derivatives

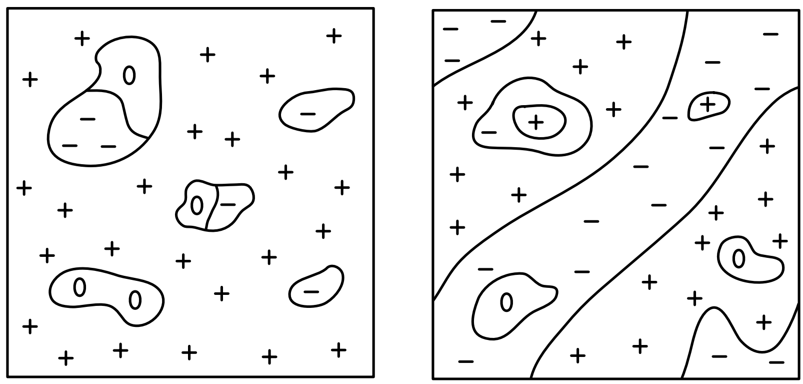

Fig. 1. Schematic examples of configurations, along with their associated contours, which contribute to different terms in the decomposition in (1.6). Here we have a spin model with $r = 3$ equilibrium phases denoted by $+$, $-$ and 0. The configuration on the left has all contours smaller than the cutoff—which we set to $L/3$ where L is the side of the box—and will thus contribute to $Z_+^{(L)}$ because $+$ is the external phase for all external contours. The configuration on the right has long contours and will be assigned to the error term.

of f_m and $f_{\tilde{m}}$ is uniformly bounded from below on $\mathscr{S}_m \cap \mathscr{S}_{\tilde{m}}$. Finally, in order to discuss multiple coexistence points, we need an additional non-degeneracy assumption on the derivatives of the functions f_m for the coexisting phases. Given these assumptions, we are able to give a very precise characterization of the topology of the coexistence set $\mathscr{G}$, see Theorem 2.1.

The second set of assumptions (Assumption B, see Sect. 2.2) is crucial for our results on the partition function zeros, and is formulated in terms of the functions $f_m^{(L)}$. These will be taken to be analytic with a uniform upper bound on the first r derivatives in an order-$(1/L)$ neighborhood of the sets $\mathscr{S}_m$. In this neighborhood, $f_m^{(L)}$ is also assumed to be exponentially close to f_m, with a lower bound on the difference of the first derivatives for any pair $f_m^{(L)}$ and $f_{\tilde{m}}^{(L)}$ in the intersection of the corresponding order-$(1/L)$ neighborhoods. Finally, we need a bound on the error term and its derivatives in an approximation of the form (1.6) where the sum runs only over the dominating terms, i.e., those m for which z lies in the order-$(1/L)$ neighborhood of $\mathscr{S}_m$.

Combining Assumptions A and B, we are able to prove several statements on the location of the partition function zeros. We will start by covering the set of available z-values by sets with a given number of stable (or "almost stable") phases. The covering involves three scale functions, ω_L, γ_L and ρ_L which give rise to three classes of sets: the region where one phase is decisively dominating the others (more precisely, the complement of an $L^{-d}\omega_L$-neighborhood of the set $\mathscr{G}$), a γ_L-neighborhood of sets with two stable phases, excluding a γ_L-neighborhood of multiple points, and the ρ_L-neighborhoods of multiple points. As is shown in Proposition 2.6, for a suitable choice of sequences ω_L, γ_L, and ρ_L, these three sets cover all possibilities.

In each part of the cover, we will control the zeros by a different method. The results of our analysis can be summarized as follows: First, there are no zeros of Z_L^{per} outside an $L^{-d}\omega_L$-neighborhood of the set $\mathscr{G}$. This claim, together with a statement on the maximal possible degeneracy of zeros, is the content of Theorem 2.2. The next theorem, Theorem 2.3, states that in a γ_L-neighborhood of the two-phase coexistence points, excluding a neighborhood of multiple points, the zeros of Z_L^{per} are exponentially close to the solutions of (1.4). In particular, this implies that the zeros are spaced in intervals of order-L^{-d} along the two-phase coexistence curves with the asymptotic density expressed in terms of the difference of the derivatives of the corresponding free energies—a result known in a special case already to Yang and Lee [42]; see Proposition 2.4. The control of the zeros in the vicinity of multiple points is more difficult and the results are less detailed. Specifically, in the ρ_L-neighborhood of a multiple point with q coexisting phases, the zeros of Z_L^{per} are shown to be located within a $L^{-d-d/q}$ neighborhood of the solutions of an explicitly specified equation.

We finish our discussion with a remark concerning the positions of zeros of complex functions of the form:

$$Z_N(z) = \sum_{m=1}^{r} \alpha_m(z)\zeta_m(z)^N, \tag{1.9}$$

where $\alpha_1, \dots, \alpha_r$ and $\zeta_1, \dots, \zeta_r$ are analytic functions of z. Here there is a general theorem, due to Beraha, Kahane and Weiss [1] (generalized recently by Sokal [41]), that the set of zeros of Z_N asymptotically concentrates on the set of z such that either $\alpha_m(z) = 0$ and $|\zeta_m(z)| = \max_k |\zeta_k(z)|$ for some $m = 1, \dots, r$ or $|\zeta_m(z)| = |\zeta_n(z)| = \max_k |\zeta_k(z)|$ for two distinct indices m and n. The present paper provides a substantial extension of this result to situations when analyticity of $\zeta_m(z)$ can be guaranteed only in a shrinking neighborhood of the sets where m is the "dominant" index. In addition, we also provide detailed control of the rate of convergence.

2. Main results

2.1. Complex phase diagram. We begin by abstracting the assumptions on the metastable free energies of the contour model and showing what kind of complex phase diagram they can yield. Throughout the paper, we will assume that a domain $\mathscr{O} \subset \mathbb{C}$ and a positive integer r are given, and use $\mathcal{R}$ to denote the set $\mathcal{R} = \{1, \dots, r\}$. For each $z \in \mathscr{O}$, we let $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ and define, as usual,

$$\partial_z = \tfrac{1}{2}\bigl(\tfrac{\partial}{\partial x} - \mathrm{i}\tfrac{\partial}{\partial y}\bigr) \quad \text{and} \quad \partial_{\bar z} = \tfrac{1}{2}\bigl(\tfrac{\partial}{\partial x} + \mathrm{i}\tfrac{\partial}{\partial y}\bigr). \tag{2.1}$$

Assumption A. There exists a constant $\alpha > 0$ and, for each $m \in \mathcal{R}$, a function $\zeta_m : \mathscr{O} \to \mathbb{C}$, such that the following conditions are satisfied:

(1) The quantity $\zeta(z) = \max_{m\in\mathcal{R}} |\zeta_m(z)|$ is uniformly positive in $\mathscr{O}$, i.e., we have $\inf_{z\in\mathscr{O}} \zeta(z) > 0$.
(2) Each function ζ_m, viewed as a function of two real variables $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, is twice continuously differentiable on $\mathscr{O}$ and it satisfies the Cauchy-Riemann equations $\partial_{\bar z}\zeta_m(z) = 0$ for all $z \in \mathscr{S}_m$, where

$$\mathscr{S}_m = \bigl\{z \in \mathscr{O} : |\zeta_m(z)| = \zeta(z)\bigr\}. \tag{2.2}$$

In particular, ζ_m is analytic on the interior of $\mathscr{S}_m$.
(3) For any pair of distinct indices $m, n \in \mathcal{R}$ and any $z \in \mathscr{S}_m \cap \mathscr{S}_n$ we have

$$\left|\frac{\partial_z\zeta_m(z)}{\zeta_m(z)} - \frac{\partial_z\zeta_n(z)}{\zeta_n(z)}\right| \ge \alpha. \tag{2.3}$$

(4) If $\mathcal{Q} \subset \mathcal{R}$ is such that $|\mathcal{Q}| \geq 3$, then for any $z \in \bigcap_{m \in \mathcal{Q}} \mathscr{S}_m$,

$$v_m(z) = \frac{\partial_z \zeta_m(z)}{\zeta_m(z)}, \quad m \in \mathcal{Q}, \tag{2.4}$$

are the vertices of a strictly convex polygon in $\mathbb{C} \simeq \mathbb{R}^2$.

Remark 1. In (1), we assumed uniform positivity in order to simplify some of our later arguments. However, uniformity in $\mathscr{O}$ can easily be replaced by uniformity on compact sets. Note that Assumptions A3–4 are invariant with respect to conformal transformations of $\mathscr{O}$ because the functions involved in (2.3) and (2.4) satisfy the Cauchy-Riemann conditions. Also note that, by Assumption A3, the length of each side of the polygon from Assumption A4 is at least α; cf Fig. 3.

The indices $m \in \mathcal{R}$ will be often referred to as *phases*. We call a phase m *stable* at z if $z \in \mathscr{S}_m$, i.e., if $|\zeta_m(z)| = \zeta(z)$. For each $z \in \mathscr{O}$ we define

$$\mathcal{Q}(z) = \bigl\{ m \in \mathcal{R} \colon |\zeta_m(z)| = \zeta(z) \bigr\} \tag{2.5}$$

to be the set of phases *stable at* z. If $m, n \in \mathcal{Q}(z)$, then we say that the phases m and n *coexist at* z. The phase diagram is determined by the *set of coexistence points*:

$$\mathscr{G} = \bigcup_{m,n \in \mathcal{R} \colon m \neq n} \mathscr{G}(m,n) \quad \text{with} \quad \mathscr{G}(m,n) = \mathscr{S}_m \cap \mathscr{S}_n. \tag{2.6}$$

If $|\zeta_m(z)| = \zeta(z)$ for at least three distinct $m \in \mathcal{R}$, we call such $z \in \mathscr{O}$ a *multiple point*.

In the following, the phrase *simple arc* denotes the image of $(0, 1)$ under a continuous and injective map while *simple closed curve* denotes a corresponding image of the unit circle $\{z \in \mathbb{C} \colon |z| = 1\}$. A curve will be called *smooth* if it can be parametrized using twice continuously differentiable functions.

Our main result concerning the topology of $\mathscr{G}$ is then as follows.

Theorem 2.1. *Suppose that Assumption A holds and let $\mathscr{D} \subset \mathscr{O}$ be a compact set. Then there exists a finite set of open discs $\mathbb{D}_1, \mathbb{D}_2, \dots, \mathbb{D}_\ell \subset \mathscr{O}$ covering $\mathscr{D}$, such that for each $k = 1, \dots, \ell$, the set $\mathscr{A}_k = \mathscr{G} \cap \mathbb{D}_k$ satisfies exactly one of the following properties:*

(1) $\mathscr{A}_k = \emptyset$.
(2) $\mathscr{A}_k$ is a smooth simple arc with both endpoints on $\partial\mathbb{D}_k$. Exactly two distinct phases coexist along the arc constituting $\mathscr{A}_k$.
(3) $\mathscr{A}_k$ contains a single multiple point z_k with $s_k = |\mathcal{Q}(z_k)| \geq 3$ coexisting phases, and $\mathscr{A}_k \setminus \{z_k\}$ is a collection of s_k smooth, non-intersecting, simple arcs connecting z_k to $\partial\mathbb{D}_k$. Each pair of distinct curves from $\mathscr{A}_k \setminus \{z_k\}$ intersects at a positive angle at z_k. Exactly two distinct phases coexist along each component of $\mathscr{A}_k \setminus \{z_k\}$.

In particular, $\mathscr{G} = \bigcup_{\mathscr{C} \in \mathfrak{C}} \mathscr{C}$, where $\mathfrak{C}$ is a finite or countably-infinite collection of smooth simple closed curves and simple arcs which intersect each other only at the endpoints.

Theorem 2.1 is proved in Sect. 3.2. Further discussion is provided in Sect. 2.4.

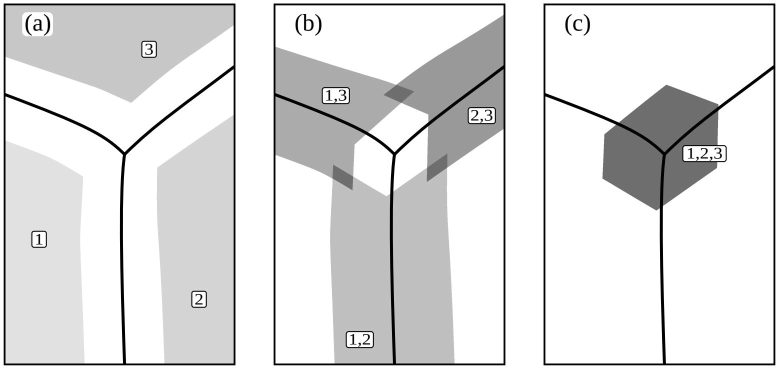

Fig. 2. An illustration of the sets $\mathscr{U}_\epsilon(\mathcal{Q})$ in the vicinity of a multiple point. The thick lines indicate the visible portion of the set of coexistence points $\mathscr{G}$. Three phases, here labeled 1, 2 and 3, are stable at the multiple point. In (a), the three shaded domains represent the sets $\mathscr{U}_\epsilon(\{1\})$, $\mathscr{U}_\epsilon(\{2\})$ and $\mathscr{U}_\epsilon(\{3\})$, with the label indicated by the number in the box. Similarly, in (b) the three regions represent the sets $\mathscr{U}_\epsilon(\{1,2\})$, $\mathscr{U}_\epsilon(\{2,3\})$ and $\mathscr{U}_\epsilon(\{1,3\})$. Finally, (c) contains only one shaded region, representing the set $\mathscr{U}_\epsilon(\{1,2,3\})$. The various regions $\mathscr{U}_\epsilon(\mathcal{Q})$ generously overlap so that their union covers the entire box.

2.2. Partition function zeros. Next we will discuss our assumptions and results concerning the zeros of the partition function. We assume that the functions $Z_L^{\text{per}}\colon \mathscr{O} \to \mathbb{C}$, playing the role of the partition function in a box of side L with periodic boundary conditions, are defined for each integer L, or, more generally, for any $L \in \mathbb{L}$, where $\mathbb{L} \subset \mathbb{N}$ is a fixed infinite set. Given any $m \in \mathcal{R}$ and $\epsilon > 0$, we use $\mathscr{S}_\epsilon(m)$ to denote the region where the phase m is "almost stable,"

$$\mathscr{S}_\epsilon(m) = \bigl\{z \in \mathscr{O}\colon |\zeta_m(z)| > e^{-\epsilon}\zeta(z)\bigr\}. \tag{2.7}$$

For any $\mathcal{Q} \subset \mathcal{R}$, we also introduce the region where all phases from $\mathcal{Q}$ are "almost stable" while the remaining ones are not,

$$\mathscr{U}_\epsilon(\mathcal{Q}) = \bigcap_{m\in\mathcal{Q}} \mathscr{S}_\epsilon(m) \setminus \bigcup_{n\in\mathcal{Q}^{\mathrm{c}}} \overline{\mathscr{S}_{\epsilon/2}(n)}, \tag{2.8}$$

with the bar denoting the set closure. Notice that the function ζ_m is non-vanishing on $\mathscr{S}_\epsilon(m)$ and that $\bigcup_{\mathcal{Q}\subset\mathcal{R}} \mathscr{U}_\epsilon(\mathcal{Q}) = \mathscr{O}$, see Fig. 2. Note also that $\mathscr{U}_\epsilon(\emptyset) = \emptyset$, so we may assume that $\mathcal{Q} \neq \emptyset$ for the rest of this paper.

Assumption B. There exist constants $\kappa, \tau \in (0,\infty)$ and, for each $m \in \mathcal{R}$, a positive integer q_m and a function $\zeta_m^{(L)}\colon \mathscr{S}_{\kappa/L}(m) \to \mathbb{C}$ such that for any $L \in \mathbb{L}$ the following is true:

(1) The function Z_L^{per} is analytic in $\mathscr{O}$.

(2) Each $\zeta_m^{(L)}$ is non-vanishing and analytic in $\mathscr{S}_{\kappa/L}(m)$. Furthermore,

$$\left|\log \frac{\zeta_m^{(L)}(z)}{\zeta_m(z)}\right| \le e^{-\tau L} \tag{2.9}$$

and

$$\left|\partial_z \log \frac{\zeta_m^{(L)}(z)}{\zeta_m(z)}\right| + \left|\partial_{\bar z} \log \frac{\zeta_m^{(L)}(z)}{\zeta_m(z)}\right| \le e^{-\tau L} \tag{2.10}$$

for all $m \in \mathcal{R}$ and all $z \in \mathscr{S}_{\kappa/L}(m)$. (Here "log" denotes the principal branch of the complex logarithm.)

(3) There exist constants $\tilde\alpha > 0$, $M < \infty$ and $\tilde L_0 < \infty$ such that for any $L \ge \tilde L_0$ we have

$$\left|\frac{\partial_z^\ell \zeta_m^{(L)}(z)}{\zeta_m^{(L)}(z)}\right| \le M, \tag{2.11}$$

whenever $m \in \mathcal{R}$, $\ell = 1, \dots, r$, and $z \in \mathscr{S}_{\kappa/L}(m)$. In addition,

$$\left|\frac{\partial_z \zeta_m^{(L)}(z)}{\zeta_m^{(L)}(z)} - \frac{\partial_z \zeta_n^{(L)}(z)}{\zeta_n^{(L)}(z)}\right| \ge \tilde\alpha \tag{2.12}$$

whenever $m, n \in \mathcal{R}$ are distinct and $z \in \mathscr{S}_{\kappa/L}(m) \cap \mathscr{S}_{\kappa/L}(n)$.

(4) There exist constants $C_\ell < \infty$, $\ell = 0, 1, \dots, r+1$, such that for any $\mathcal{Q} \subset \mathcal{R}$, the difference

$$\varXi_{\mathcal{Q},L}(z) = Z_L^{\text{per}}(z) - \sum_{m\in\mathcal{Q}} q_m \big[\zeta_m^{(L)}(z)\big]^{L^d} \tag{2.13}$$

satisfies the bound

$$\left|\partial_z^\ell \varXi_{\mathcal{Q},L}(z)\right| \le C_\ell L^{d(\ell+1)} \zeta(z)^{L^d} \Bigg(\sum_{m\in\mathcal{R}} q_m\Bigg) e^{-\tau L}, \tag{2.14}$$

for all $\ell = 0, 1, \dots, r+1$, uniformly in $z \in \mathscr{U}_{\kappa/L}(\mathcal{Q})$.

Remark 2. In applications, q_m will represent the degeneracy of the phase m; thus we have taken it to be a positive integer. However, our arguments would go through even if we assumed only that all q_m's are real and positive. It is also worth noting that in many physical models the partition function is not directly of the form required by Assumption B; but it can be brought into this form by extracting a multiplicative "fudge" factor $F(z)^{L^d}$, where $F(z) \ne 0$ in the region of interest. For instance, in the Ising model with z related to the complex external field h by $z = e^h$ we will have to take $F(z) = z^{-1/2}$ to make the partition function analytic in the neighborhood of $z = 0$.

Our first theorem in this section states that the zeros of $Z_L^{\text{per}}(z)$ are concentrated in a narrow strip along the phase boundaries. In addition, their maximal degeneracy near the multiple points of the phase diagram can be evaluated. In accord with the standard terminology, we will call a point z_0 a *k-times degenerate root* of an analytic function $h(z)$ if $h(z) = g(z)(z - z_0)^k$ for some $g(z)$ that is finite and non-zero in a neighborhood of z_0. Recalling the definition (2.8) of the set $\mathscr{U}_\epsilon(\mathcal{Q})$, we introduce the shorthand

$$\mathscr{G}_\epsilon = \bigcup_{m\ne n} \Big(\overline{\mathscr{S}_{\epsilon/2}(n)} \cap \overline{\mathscr{S}_{\epsilon/2}(m)}\Big) = \mathscr{O} \setminus \bigcup_{m\in\mathcal{R}} \mathscr{U}_\epsilon\big(\{m\}\big). \tag{2.15}$$

An easy way to check the second equality in (2.15) is by noting that $\mathscr{O} \setminus \mathscr{U}_\epsilon(\{m\})$ can be written as the union $\bigcup_{n: n\ne m} \overline{\mathscr{S}_{\epsilon/2}(n)}$. Then we have the following result.

Theorem 2.2. *Suppose that Assumptions A1-3 and B hold and let $\kappa > 0$ be as in Assumption B. Let (ω_L) be a sequence of positive numbers such that $\omega_L \to \infty$. Then there exists a constant $L_0 < \infty$ such that for $L \ge L_0$ all roots of Z_L^{per} lie in $\mathscr{G}_{L^{-d}\omega_L}$ and are at most $|\mathcal{R}| - 1$ times degenerate. For each $\mathcal{Q} \subset \mathcal{R}$, the roots of Z_L^{per} in $\mathscr{U}_{\kappa/L}(\mathcal{Q})$ are at most $|\mathcal{Q}| - 1$ times degenerate.*

In other words, as $L \to \infty$, the zeros of Z_L^{per} asymptotically concentrate on the set of coexistence points $\mathscr{G}$. Notice that we explicitly do *not* require Assumption A4 to hold; see Sect. 2.4 for further discussion. Theorem 2.2 is proved in Sect. 4.1.

Our next theorem deals with the zeros of Z_L^{per} in the regions where at most two phases from $\mathcal{R}$ are "almost stable." It turns out that we have a much better control on the location of zeros in regions that are sufficiently far from multiple points. To quantify the meaning of "sufficiently far," we let γ_L be a sequence of positive numbers (to be specified below) and, for any $\mathcal{Q} \subset \mathcal{R}$ with $|\mathcal{Q}| = 2$ and any $L \ge 0$, let $\delta_L : \mathscr{U}_{\gamma_L}(\mathcal{Q}) \to (0, \infty)$ be a function defined by

$$\delta_L(z) = \begin{cases} e^{-\tau L}, & \text{if } z \in \mathscr{U}_{\gamma_L}(\mathcal{Q}) \cap \mathscr{U}_{2\kappa/L}(\mathcal{Q}), \\ L^d e^{-\frac{1}{2}\gamma_L L^d}, & \text{otherwise.} \end{cases} \tag{2.16}$$

(Clearly, $\delta_L(z)$ depends on the index set $\mathcal{Q}$. However, this set will always be clear from the context and so we will not make it notationally explicit.) Finally, given $\epsilon > 0$ and $z \in \mathscr{O}$, let $\mathbb{D}_\epsilon(z)$ denote the open disc of radius ϵ centered at z.

The exact control of the roots in two-phase regions is then as follows.

Theorem 2.3. *Suppose that Assumptions A and B hold and let $\Omega_L^\star$ be the set of all zeros of the function $Z_L^{\text{per}}(z)$ in $\mathscr{O}$, including multiplicity. If $m, n \in \mathcal{R}$ are distinct indices, let $\mathcal{Q} = \{m, n\}$, and let $\Omega_L(\mathcal{Q})$ be the set of the solutions of the system of equations*

$$q_m^{1/L^d} |\zeta_m(z)| = q_n^{1/L^d} |\zeta_n(z)|, \tag{2.17}$$

$$L^d \operatorname{Arg}\bigl(\zeta_m(z)/\zeta_n(z)\bigr) = \pi \bmod 2\pi. \tag{2.18}$$

Let γ_L be such that

$$\liminf_{L\to\infty} \frac{L^d \gamma_L}{\log L} > 4d \quad \text{and} \quad \limsup_{L\to\infty} L^{d-1}\gamma_L < 2\tau, \tag{2.19}$$

and let $\delta_L : \mathscr{U}_{\gamma_L}(\mathcal{Q}) \to (0, \infty)$ be as defined in (2.16). Then there exist finite positive constants B, C, D, and L_0 such that for any $\mathcal{Q} \subset \mathcal{R}$ with $|\mathcal{Q}| = 2$ and any $L \ge L_0$ we have:

(1) For all $z \in \mathscr{G} \cap \mathscr{U}_{\gamma_L}(\mathcal{Q})$ with $\mathbb{D}_{DL^{-d}}(z) \subset \mathscr{O}$, the disc $\mathbb{D}_{DL^{-d}}(z)$ contains at least one root from $\Omega_L^\star$.
(2) For all $z \in \Omega_L^\star \cap \mathscr{U}_{\gamma_L}(\mathcal{Q})$ with $\mathbb{D}_{C\delta_L(z)}(z) \subset \mathscr{O}$, the disc $\mathbb{D}_{C\delta_L(z)}(z)$ contains exactly one point from $\Omega_L(\mathcal{Q})$.
(3) For all $z \in \Omega_L(\mathcal{Q}) \cap \mathscr{U}_{\gamma_L}(\mathcal{Q})$ with $\mathbb{D}_{C\delta_L(z)}(z) \subset \mathscr{O}$, the disc $\mathbb{D}_{C\delta_L(z)}(z)$ contains exactly one root from $\Omega_L^\star$.
(4) Any two distinct roots of Z_L^{per} in the set $\{z \in \mathscr{U}_{\gamma_L}(\mathcal{Q}) \colon \mathbb{D}_{BL^{-d}}(z) \subset \mathscr{O}\}$ are at least BL^{-d} apart.

Note that the first limit in (2.19) ensures that $L^d\delta_L(z) \to 0$ as $L \to \infty$ throughout $\mathscr{U}_{\gamma_L}(\mathcal{Q})$ (for any $\mathcal{Q} \subset \mathcal{R}$ with $|\mathcal{Q}| = 2$). Thus $\delta_L(z)$ is much smaller than the distance of the "neighboring" roots of (2.17–2.18). Theorem 2.3 is proved in Sect. 4.2.

Theorem 2.3 allows us to describe the asymptotic density of the roots of Z_L^{per} along the arcs of the complex phase diagram. Let $m, n \in \mathcal{R}$ be distinct and let $\mathscr{G}(m,n)$ be as in (2.6). For each $\epsilon > 0$ and each $z \in \mathscr{G}(m,n)$, let $\rho_{m,n}^{(L,\epsilon)}(z)$ be defined by

$$\rho_{m,n}^{(L,\epsilon)}(z) = \frac{1}{2\epsilon L^d}\big|\Omega_L^\star \cap \mathbb{D}_\epsilon(z)\big|, \tag{2.20}$$

where $|\Omega_L^\star \cap \mathbb{D}_\epsilon(z)|$ is the number of roots of Z_L^{per} in $\mathbb{D}_\epsilon(z)$ including multiplicity. Since $\mathscr{G}(m,n)$ is a union of simple arcs and closed curves, and since the roots of (2.17-2.18) are spaced within $O(L^{-d})$ from each other, $\rho_{m,n}^{(L,\epsilon)}(z)$ has the natural interpretation of the approximate *line density of zeros* of Z_L^{per} along $\mathscr{G}(m,n)$. As can be expected from Theorem 2.3, the approximate density $\rho_{m,n}^{(L,\epsilon)}(z)$ tends to an explicitly computable limit.

Proposition 2.4. *Let $m, n \in \mathcal{R}$ be distinct and let $\rho_{m,n}^{(L,\epsilon)}(z)$ be as in (2.20). Then the limit*

$$\rho_{m,n}(z) = \lim_{\epsilon \downarrow 0}\lim_{L\to\infty} \rho_{m,n}^{(L,\epsilon)}(z) \tag{2.21}$$

exists for all $z \in \mathscr{G}(m,n)$ such that $|\mathcal{Q}(z)| = 2$, and

$$\rho_{m,n}(z) = \frac{1}{2\pi}\left|\frac{\partial_z\zeta_m(z)}{\zeta_m(z)} - \frac{\partial_z\zeta_n(z)}{\zeta_n(z)}\right|. \tag{2.22}$$

Remark 3. Note that, on the basis of Assumption A3, we have that $\rho_{m,n}(z) \geq \alpha/(2\pi)$. In particular, the density of zeros is always positive. This is directly related to the fact that all points $z \in \mathscr{G}$ will exhibit a first-order phase transition (defined in an appropriate sense, once $\operatorname{Im} z \neq 0$ or $\operatorname{Re} z < 0$)—hence the title of the paper. The observation that the (positive) density of zeros and the order of the transition are closely related goes back to [42].

In order to complete the description of the roots of Z_L^{per}, we also need to cover the regions with more than two "almost stable" phases. This is done in the following theorem.

Theorem 2.5. *Suppose that Assumptions A and B are satisfied. Let z_{M} be a multiple point and let $\mathcal{Q} = \mathcal{Q}(z_{\mathrm{M}})$ with $q = |\mathcal{Q}| \geq 3$. For each $m \in \mathcal{Q}$, let*

$$\phi_m(L) = L^d \operatorname{Arg}\zeta_m(z_{\mathrm{M}}) \pmod{2\pi} \quad \text{and} \quad v_m = \frac{\partial_z\zeta_m(z_{\mathrm{M}})}{\zeta_m(z_{\mathrm{M}})}. \tag{2.23}$$

Consider the set $\Omega_L(\mathcal{Q})$ of all solutions of the equation

$$\sum_{m\in\mathcal{Q}} q_m\, e^{\mathrm{i}\phi_m(L) + L^d(z-z_{\mathrm{M}})v_m} = 0, \tag{2.24}$$

including multiplicity, and let (ρ_L) be a sequence of positive numbers such that

$$\lim_{L\to\infty} L^d\rho_L = \infty \qquad \text{but} \qquad \lim_{L\to\infty} L^{d-d/(2q)}\rho_L = 0. \tag{2.25}$$

Define $\rho'_L = \rho_L + L^{-d(1+1/q)}$. Then there exists a constant $L_0 < \infty$ and, for any $L \ge L_0$, an open, connected and simply connected set $\mathscr{U}$ satisfying $\mathbb{D}_{\rho_L}(z_{\mathrm{M}}) \subset \mathscr{U} \subset \mathbb{D}_{\rho'_L}(z_{\mathrm{M}})$ such that the zeros in $\Omega \cap \mathscr{U}$ are in one-to-one correspondence with the solutions in $\Omega(\mathcal{Q}) \cap \mathscr{U}$ and the corresponding points are not farther apart than $L^{-d(1+1/q)}$.

Theorem 2.5 is proved in Sect. 4.4. Sect. 2.4 contains a discussion of the role of Assumption A4 in this theorem; some information will also be provided concerning the actual form of the solutions of (2.24).

To finish the exposition of our results, we will need to show that the results of Theorems 2.2, 2.3 and 2.5 can be patched together to provide complete control of the roots of Z_L^{per}, at least in any compact subset of $\mathscr{O}$. This is done in the following claim, the proof of which essentially relies only on Assumption A and compactness arguments:

Proposition 2.6. *Suppose that Assumption A holds and let ω_L, γ_L and ρ_L be sequences of positive numbers such that $\omega_L \le \gamma_L L^d$, $\gamma_L \to 0$, and $\rho_L \to 0$. For each compact set $\mathscr{D} \subset \mathscr{O}$, there exist constants $\chi = \chi(\mathscr{D}) > 0$ and $L_0 = L_0(\mathscr{D}) < \infty$ such that, if $\rho_L \ge \chi\gamma_L$, we have*

$$\mathscr{G}_{L^{-d}\omega_L} \cap \mathscr{D} \subset \bigcup_{\substack{\mathcal{Q}\subset\mathcal{R}\\ |\mathcal{Q}|=2}} \mathscr{U}_{\gamma_L}(\mathcal{Q}) \cup \bigcup_{\substack{z_{\mathrm{M}}\in\mathscr{D}\\ |\mathcal{Q}(z_{\mathrm{M}})|\ge 3}} \mathbb{D}_{\rho_L}(z_{\mathrm{M}}) \tag{2.26}$$

for any $L \ge L_0$.

Note that in (2.26) we consider only that portion of $\mathscr{D}$ in $\mathscr{G}_{L^{-d}\omega_L}$, since by Theorem 2.2 the roots of Z_L^{per} are contained in this set. Note also that the conditions we impose on the sequences ω_L, γ_L and ρ_L in Theorems 2.1, 2.3 and 2.5 and Proposition 2.6 are not very restrictive. In particular, it is very easy to verify the existence of these sequences. (For example, one can take both γ_L and ρ_L to be proportional to $L^{-d}\log L$ with suitable prefactors and then let $\omega_L = L^d\gamma_L$.)

2.3. Local Lee-Yang theorem. As our last result, we state a generalized version of the classic Lee-Yang Circle Theorem [25], the proof of which is based entirely on the exact symmetries of the model.

Theorem 2.7. *Suppose that Assumptions A and B hold. Let $+$ and $-$ be two selected indices from $\mathcal{R}$ and let $\mathscr{U}$ be an open set with compact closure $\mathscr{D} \subset \mathscr{O}$ such that $\mathscr{U} \cap \{z\colon |z| = 1\} \ne \emptyset$. Assume that $\mathscr{D}$ is invariant under circle inversion $z \mapsto 1/z^*$, and*

(1) $Z_L^{\mathrm{per}}(z) = Z_L^{\mathrm{per}}(1/z^*)^*$,
(2) $\zeta_+(z) = \zeta_-(1/z^*)^*$ and $q_+ = q_-$

hold for all $z \in \mathscr{D}$ and all $L \in \mathbb{L}$. Then there exists a constant L_0 such that the following holds for all $L \ge L_0$: If the intersection of $\mathscr{D}$ with the set of coexistence points $\mathscr{G}$ is connected and if $+$ and $-$ are the only stable phases in $\mathscr{D}$, then all zeros in $\mathscr{D}$ lie on the unit circle, and the number of zeros on any segment of $\mathscr{D} \cap \{z\colon |z| = 1\}$ is proportional to L^d as $L \to \infty$.

Condition (2) is the rigorous formulation of the statement that the $+$ and $-$ phases are related by $z \leftrightarrow 1/z^*$ (or $h \leftrightarrow -h$, when $z = e^h$) symmetry. Condition (1) then stipulates that this symmetry is actually respected by the remaining phases and, in particular, by Z_L^{per} itself.

Remark 4. As discussed in Remark 2, in order to satisfy Assumption B it may be necessary to extract a multiplicative "fudge" factor from the partition function, perform the analysis of partition function zeros in various restricted regions in $\mathbb{C}$ and patch the results appropriately. A similar manipulation may be required in order to apply Theorem 2.7.

Here are the main steps of the proof of Theorem 2.7: First we show that the phase diagram in $\mathscr{D}$ falls exactly on the unit circle, i.e.,

$$\mathscr{D} \cap \mathscr{G} = \{z \in \mathscr{D} \colon |z| = 1\}. \tag{2.27}$$

This fact is essentially an immediate consequence of the symmetry between "+" and "−." *A priori* one would then expect that the zeros are close to, but not necessarily on, the unit circle. However, the symmetry of Z_L^{per} combined with the fact that distinct zeros are at least BL^{-d} apart is not compatible with the existence of zeros away from the unit circle. Indeed, if z is a root of Z_L^{per}, it is bound to be within a distance $O(e^{-\tau L})$ of the unit circle. If, in addition, $|z| \neq 1$, then the $z \leftrightarrow 1/z^*$ symmetry implies that $1/z^*$ is also a root of Z_L^{per}, again within $O(e^{-\tau L})$ of the unit circle. But then the distance between z and $1/z^*$ is of the order $e^{-\tau L}$ which is forbidden by claim (4) of Theorem 2.3.

This argument is made precise in the following proof.

Proof of Theorem 2.7. We start with the proof of (2.27). Let us suppose that $\mathscr{D} \subset \mathscr{O}$ and $\mathcal{Q}(z) \subset \{+,-\}$ for all $z \in \mathscr{D}$. Invoking the continuity of $\zeta_\pm$ and condition (2) above, we have $\mathcal{Q}(z) = \{+,-\}$ for all $z \in \mathscr{D} \cap \{z \colon |z| = 1\}$ and thus $\mathscr{D} \cap \{z \colon |z| = 1\} \subset \mathscr{G}$. Assume now that $\mathscr{G} \cap \mathscr{D} \setminus \{z \colon |z| = 1\} \neq \emptyset$. By the fact that $\mathscr{G} \cap \mathscr{D}$ is connected and the assumption that $\mathscr{U} \cap \{z \colon |z| = 1\} \neq \emptyset$, we can find a path $z_t \in \mathscr{G} \cap \mathscr{D}$, $t \in [-1,1]$, such that $z_t \in \mathscr{D} \cap \{z \colon |z| = 1\}$ if $t \le 0$ and $z_t \in \mathscr{G} \cap \mathscr{D} \setminus \{z \colon |z| = 1\}$ if $t > 0$. Since $\mathcal{Q}(z_0) = \{+,-\}$, we know that there is a disc $\mathbb{D}_\epsilon(z_0) \subset \mathscr{O}$ that contains no multiple points. Applying Theorem 2.1 to this disc, we conclude that there is an open disc $\mathbb{D}$ with $z_0 \in \mathbb{D} \subset \mathbb{D}_\epsilon(z_0)$, such that $\mathscr{G} \cap \mathbb{D}$ is a simple curve which ends at $\partial\mathbb{D}$. However, using condition (2) above, we note that as with z_t, also the curve $t \mapsto 1/z_t^*$ lies in $\mathscr{G} \cap \mathscr{D}$, contradicting the fact that $\mathscr{G} \cap \mathbb{D}$ is a simple curve. This completes the proof of (2.27).

Next, we will show that for any $z_0 \in \mathscr{D} \cap \{z \colon |z| = 1\}$, and any $\delta > 0$, there exists an open disc $\mathbb{D}_\epsilon(z_0) \subset \mathscr{O}$ such that the set $\mathscr{G} \cap \mathbb{D}_\epsilon(z_0)$ is a smooth curve with the property that for any $z \in \mathbb{D}_\epsilon(z_0)$ with $|z| \neq 1$, the line connecting z and $1/z^*$ intersects the curve $\mathscr{G} \cap \mathbb{D}_\epsilon(z_0)$ exactly once, and at an angle that lies between $\pi/2 - \delta$ and $\pi/2 + \delta$. If z_0 lies in the interior of $\mathscr{D}$, this statement (with $\delta = 0$) follows trivially from (2.27). If z_0 is a boundary point of $\mathscr{D}$, we first choose a sufficiently small disc $\mathbb{D} \ni z_0$ so that $\mathbb{D} \subset \mathscr{O}$ and, for all points in $\mathbb{D}$, only the phases $+$ and $-$ are stable. Then we use Theorem 2.3 and (2.27) to infer that ϵ can be chosen small enough to guarantee the above statement about intersection angles.

Furthermore, we claim that given $z_0 \in \mathscr{D} \cap \{z \colon |z| = 1\}$ and $\epsilon > 0$ such that $\mathbb{D}_{3\epsilon}(z_0) \subset \mathscr{O}$ and $\mathcal{Q}(z) \subset \{+,-\}$ for all $z \in \mathbb{D}_{3\epsilon}(z_0)$, one can choose L sufficiently large so that

$$\mathbb{D}_{2\epsilon}(z_0) \cap \mathscr{G}_{L^{-d}\omega_L} \subset \mathscr{U}_{\gamma_L}(\{+,-\}) \cap \mathscr{U}_{2\kappa/L}(\{+,-\}). \tag{2.28}$$

To prove this, let us first note that, for $\gamma_L \le 2\kappa/L$, the right hand side can be rewritten as

$$\mathscr{U}_{\gamma_L}(\{+,-\}) \setminus \bigcup_{m \neq -,+} \overline{\mathscr{S}_{\kappa/L}(m)}. \tag{2.29}$$

Next, by the compactness of $\overline{\mathbb{D}_{2\epsilon}(z_0)}$ and the fact that no $m \in \mathcal{R}$ different from $\pm$ is stable anywhere in $\mathbb{D}_{3\epsilon}(z_0)$, we can choose L_0 so large that $\overline{\mathscr{S}_{\kappa/L}(m)} \cap \overline{\mathbb{D}_{2\epsilon}(z_0)} = \emptyset$ for all $L \geq L_0$ and all $m \neq \pm$. Using the closure of $\mathbb{D}_{2\epsilon}(z_0)$ in place of the set $\mathscr{D}$ in (2.26), we get (2.28).

We are now ready to prove that for any $z_0 \in \mathscr{D} \cap \{z\colon |z| = 1\}$, there exist constants $\epsilon > 0$ and L_0 such that all roots of Z_L^{per} in $\mathbb{D}_\epsilon(z_0) \cap \mathscr{D}$ lie on the unit circle. To this end, let us first assume that ϵ has been chosen small enough to guarantee that $(1-\epsilon)^{-1} < 1+2\epsilon$, $\mathbb{D}_{3\epsilon}(z_0) \subset \mathscr{O}$, $\mathcal{Q}(z) \subset \{+,-\}$ for all $z \in \mathbb{D}_{3\epsilon}(z_0)$, and $\mathscr{G} \cap \mathbb{D}_{3\epsilon}(z_0)$ is a smooth curve with the above property about the intersections angles, with, say, $\delta = \pi/4$. Assume further that L is chosen so that (2.28) holds and $\epsilon > \max(C\delta_L(z_0), BL^{-d})$, where C and B are the constants from Theorem 2.3.

Let $z \in \mathbb{D}_\epsilon(z_0) \cap \mathscr{D}$ be a root of Z_L^{per}. If L is so large that Theorem 2.2 applies, we have $z \in \mathscr{G}_{L^{-d}\omega_L}$ and thus $\delta_L(z) = e^{-\tau L}$ in view of (2.28). By Theorem 2.3, there exists a solution $\tilde{z}$ to (2.17–2.18) that lies in a $C\delta_L(z)$-neighborhood of z, implying that z has distance less than $C\delta_L(z)$ from $\mathbb{D}_{2\epsilon}(z_0) \cap \mathscr{G}$. (Here we need that $q_+ = q_-$ to conclude that $\tilde{z} \in \mathscr{G}$.) Suppose now that $|z| \neq 1$. Then the condition (1) above implies that $z' = (z^*)^{-1}$ is a *distinct* root of Z_L^{per} in $\mathscr{D}$. Moreover, if ϵ is so small that $(1-\epsilon)^{-1} < 1+2\epsilon$, then $z' \in \mathscr{G}_{L^{-d}\omega_L} \cap \mathbb{D}_{2\epsilon}(z_0)$ and $\delta_L(z')$ also equals $e^{-\tau L}$, implying that z' has distance less than $C\delta_L(z)$ from $\mathbb{D}_{3\epsilon}(z_0) \cap \mathscr{G}$. Since both z and z' have distance less than $C\delta_L(z)$ from $\mathbb{D}_{3\epsilon}(z_0) \cap \mathscr{G}$, and the curve $\mathbb{D}_{3\epsilon}(z_0) \cap \mathscr{G}$ intersects the line through z and z' in an angle that is near $\pi/2$, we conclude that $|z - z'| \leq 2\sqrt{2}Ce^{-\tau L}$ which for L sufficiently large contradicts the last claim of Theorem 2.3. Hence, z must have been on the unit circle after all.

The rest of the argument is based on compactness. The set $\mathscr{D} \cap \{z\colon |z| = 1\}$ is compact, and can thus be covered by a finite number of such discs. Picking one such cover, let $\mathscr{D}'$ be the complement of these disc in $\mathscr{D}$. Then the set $\mathscr{D}'$ is a finite distance away from $\mathscr{G}$ and thus $\mathscr{D}' \cap \mathscr{G}_{L^{-d}\omega_L} = \emptyset$ for L sufficiently large. From here it follows that for some finite $L_0 < \infty$ (which has to exceed the maximum of the corresponding quantity for the discs that constitute the covering of $\mathscr{D} \cap \{z\colon |z| = 1\}$), all roots of Z_L^{per} in $\mathscr{U}$ lie on the unit circle. □

2.4. Discussion. We finish with a brief discussion of the results stated in the previous three sections. We will also mention the role of (and possible exceptions to) our assumptions, as well as extensions to more general situations.

We begin with the results on the complex phase diagram. Theorem 2.1 describes the situation in the generic cases when Assumptions A1-A4 hold. We note that Assumption A3 is crucial for the fact that the set $\mathscr{G}$ is a collection of *curves*. A consequence of this is also that the zeros of Z_L^{per} asymptotically concentrate on curves—exceptions to this "rule" are known, see, e.g., [36]. Assumption A4 prevents the phase coexistence curves from merging in a tangential fashion and, as a result of that, guarantees that multiple points do not proliferate throughout $\mathscr{O}$. Unfortunately, in several models of interest (e.g., the Potts and Blume-Capel model) Assumption A4 happens to be violated at some $\tilde{z}$ for one or two "critical" values of the model parameters. In such cases, the region $\mathscr{O}$ has to be restricted to the complement of some neighborhood of $\tilde{z}$ and, inside the neighborhood, the claim has to be verified using a refined and often model-specific analysis. (It often suffices to show that the phase coexistence curves meeting at $\tilde{z}$ have different curvatures, which amounts to a statement about the second derivatives of the

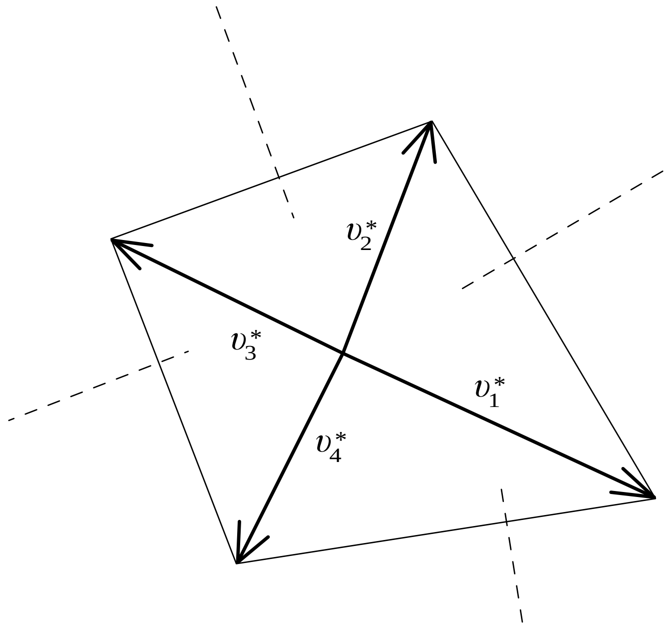

Fig. 3. An illustration of the situation around a quadruple point. Here $v_1^*, \dots, v_4^*$ are the complex conjugates of the quantities from (2.4) and $q_1 = q_2 = q_3 < q_4$. (The quadruple point lies at the common tail point of the vectors $v_1^*, \dots, v_4^*$.) The dashed lines indicate the asymptotes of the "strings" of zeros sufficiently far—on the scale L^{-d}—from the quadruple point. Note the lateral shift of these lines due to the fact that $q_4 > q_1, q_3$. The picture seems to suggest that, on the scale L^{-d}, the quadruple point splits into two triple points.

functions $\log \zeta_m(z)$.) Examples of such analysis have appeared in [2] for the Blume-Capel model and in [4] for the Potts model in a complex external field.

Next we will look at the results of Theorems 2.2 and 2.3. The fact that the roots of Z_L^{per} are only finitely degenerate is again independent of Assumption A4. (This is of some relevance in view of the aforementioned exceptions to this assumption.) The fact that, in the cases when all q_m's are the same, the zeros shift only by an exponentially small amount away from the two-phase coexistence lines is a direct consequence of our choice of the boundary conditions. Indeed, the factor $e^{-\tau L}$ in (2.16) can be traced to the similar factors in (2.9) and (2.14). For strong (e.g., fixed-spin) boundary conditions, we expect the corresponding terms in (2.9) and (2.14) to be replaced by $1/L$. In particular, in these cases, the lateral shift of the partition function zeros away from the phase-coexistence lines should be of the order $1/L$. See [44] for some results on this problem.

Finally, let us examine the situation around multiple points in some detail. Theorem 2.5 can be given the following geometrical interpretation: Let z_{M} be a multiple point. Introducing the parametrization $\mathfrak{z} = (z - z_{\text{M}})L^d$, we effectively zoom in on the scale L^{-d}, where the zeros of Z_L^{per} are well approximated by the roots of the linearized problem (2.24) with $\mathcal{Q} = \mathcal{Q}(z_{\text{M}})$. Let us plot the complex conjugates v_m^* of the logarithmic derivatives v_m (see (2.23)), $m \in \mathcal{Q}$, as vectors in $\mathbb{R}^2$. By Assumption A4, the vectors v_m^* are the endpoints of a convex set in $\mathbb{C} \simeq \mathbb{R}^2$. Let $v_1^*, \dots, v_q^*$ be the ordering of $\mathcal{Q}$ in the counterclockwise direction, see Fig. 3. Noting that the real part $\text{Re}(v_m \mathfrak{z})$ can be written in terms of the dot product, $\operatorname{Re}(\upsilon_m \mathfrak{z}) = \upsilon_m^* \cdot \mathfrak{z}$, (2.24) can be recast as

$$\sum_{m \in \mathcal{Q}(z_M)} q_m \, e^{i\phi'_m(L) + \upsilon_m^* \cdot \mathfrak{z}} = 0, \tag{2.30}$$

where $\phi'_m(L) = \phi_m(L) + \operatorname{Im}(\upsilon_m \mathfrak{z})$.

On the basis of (2.30), it is easy to verify the following facts: Let $\mathfrak{z} = |\mathfrak{z}|\hat{e}$, with $\hat{e}$ a unit vector in $\mathbb{C}$. An inspection of (2.30) shows that, for $|\mathfrak{z}| \gg 1$, the roots of (2.30) will concentrate along the "directions" for which the projection of $\hat{e}$ on at least two υ_n^*'s is the same. Invoking the convexity assumption (Assumption A4), this can only happen when $\upsilon_n^* \cdot \hat{e} = \upsilon_{n+1}^* \cdot \hat{e}$ for some n. In such cases, the contributions of the terms with indices $m \neq n, n+1$ in (2.30) are negligible—at least once $|\mathfrak{z}| \gg 1$—and the zeros will thus asymptotically lie along the half-lines given in the parametric form by

$$\mathfrak{z} = \mathfrak{z}(t) = \frac{\upsilon_n^* - \upsilon_{n+1}^*}{|\upsilon_n - \upsilon_{n+1}|^2} \log\Bigl(\frac{q_{n+1}}{q_n}\Bigr) + it(\upsilon_n^* - \upsilon_{n+1}^*), \quad t \in [0, \infty). \tag{2.31}$$

Clearly, the latter is a line perpendicular to the $(n, n+1)$-st side of the convex set with vertices $\upsilon_1^*, \dots, \upsilon_q^*$, which is shifted (away from the origin) along the corresponding side by a factor proportional to $\log(q_{n+1}/q_n)$, see Fig. 3.

Sufficiently far away from z_M (on the scale L^{-d}), the zeros resume the pattern established around the two-phase coexistence curves. In particular, the zeros are asymptotically equally spaced but their overall shift along the asymptote is determined by the factor $\phi_m(L)$—which we note depends very sensitively on L. Computer simulations show that, at least in generic cases, this pattern will persists all the way down to the multiple point. Thus, even on the "microscopic" level, the zeros seem to form a "phase diagram." However, due to the lateral shifts caused by $q_{m+1} \neq q_m$, a "macroscopic" quadruple point may resolve into two "microscopic" triple points, and similarly for higher-order multiple points.

3. Characterization of phase diagrams

The goal of this section is to give the proof of Theorem 2.1. We begin by proving a series of auxiliary lemmas whose purpose is to elevate the pointwise Assumptions A3-A4 into statements extending over a small neighborhood of each coexistence point.

3.1. Auxiliary claims. Recall the definitions of $\mathscr{S}_m$, $\mathcal{Q}(z)$ and $\upsilon_m(z)$, in (2.2), (2.5) and (2.23), respectively. The first lemma gives a limiting characterization of stability of phases around coexistence points.

Lemma 3.1. *Let Assumption A1–A2 hold and let $\bar{z} \in \mathscr{O}$ be such that $|\mathcal{Q}(\bar{z})| \geq 2$. Let (z_k) be a sequence of numbers $z_k \in \mathscr{O}$ such that $z_k \to \bar{z}$ but $z_k \neq \bar{z}$ for all k. Suppose that*

$$e^{i\theta} = \lim_{k\to\infty} \frac{z_k - \bar{z}}{|z_k - \bar{z}|} \tag{3.1}$$

exists and let $m \in \mathcal{Q}(\bar{z})$. If $z_k \in \mathscr{S}_m$ for infinitely many $k \geq 1$, then

$$\operatorname{Re}(e^{i\theta}\upsilon_m) \geq \operatorname{Re}(e^{i\theta}\upsilon_n) \quad \textit{for all } n \in \mathcal{Q}(\bar{z}), \tag{3.2}$$

where $v_n = v_n(\bar z)$. Conversely, if the inequality in (3.2) fails for at least one $n \in \mathcal{Q}(\bar z)$, then there is an $\epsilon > 0$ such that

$$\mathscr{W}_{\epsilon,\theta}(\bar z) = \Big\{z \in \mathscr{O}\colon |z - \bar z| < \epsilon,\ z \neq \bar z,\ \big|\tfrac{z-\bar z}{|z-\bar z|} - e^{\mathrm{i}\theta}\big| < \epsilon\Big\} \tag{3.3}$$

has empty intersection with $\mathscr{S}_m$, i.e., $\mathscr{S}_m \cap \mathscr{W}_{\epsilon,\theta}(\bar z) = \emptyset$. In particular, $z_k \notin \mathscr{S}_m$ for k large enough.

Remark 5. In the following, it will be useful to recall some simple facts about complex functions. Let f, g and h be functions $\mathbb{C} \to \mathbb{C}$ and let ∂_z and $\partial_{\bar z}$ be as in (2.1). If f satisfies $\partial_{\bar z} f(z_0) = 0$ (i.e., Cauchy-Riemann conditions), then all directional derivatives of f at $z_0 = x_0 + \mathrm{i}y_0$ can be expressed using one complex number $A = \partial_z f(x_0 + \mathrm{i}y_0)$, i.e., we have

$$f(x_0 + \epsilon\cos\varphi + \mathrm{i}y_0 + \mathrm{i}\epsilon\sin\varphi) - f(x_0 + \mathrm{i}y_0) = \epsilon A e^{\mathrm{i}\varphi} + o(\epsilon), \quad \epsilon \downarrow 0, \tag{3.4}$$

holds for every $\varphi \in [-\pi, \pi)$. Moreover, if g is differentiable with respect to x and y at $z_0 = x_0 + \mathrm{i}y_0$ and h satisfies $\partial_{\bar z} h(z') = 0$ at $z' = g(z_0)$, then the chain rule holds for $z \mapsto h(g(z))$ at $z = z_0$. In particular, $\partial_z h(g(z_0)) = (\partial_z h)(g(z_0))\partial_z g(z_0)$.

Proof of Lemma 3.1. Let $m \in \mathcal{Q}(\bar z)$ be fixed. Whenever $z_k \in \mathscr{S}_m$, we have

$$\log\big|\zeta_m(z_k)\big| - \log\big|\zeta_m(\bar z)\big| \ge \log\big|\zeta_n(z_k)\big| - \log\big|\zeta_n(\bar z)\big|, \qquad n \in \mathcal{Q}(\bar z), \tag{3.5}$$

because $|\zeta_m(\bar z)| = |\zeta_n(\bar z)|$, by our assumption that $m, n \in \mathcal{Q}(\bar z)$. Using the notation

$$F_{m,n}(z) = \frac{\zeta_m(z)}{\zeta_n(z)} \tag{3.6}$$

for $n \in \mathcal{Q}(\bar z)$ (which is well defined and non-zero in a neighborhood of $\bar z$), the inequality (3.5) becomes

$$\log\big|F_{m,n}(z_k)\big| - \log\big|F_{m,n}(\bar z)\big| \ge 0, \qquad n \in \mathcal{Q}(\bar z). \tag{3.7}$$

Note that the complex derivative $\partial_z F_{m,n}(\bar z)$ exists for all $n \in \mathcal{Q}(\bar z)$. Our task is then to prove that

$$\mathrm{Re}\Big(e^{\mathrm{i}\theta}\,\frac{\partial_{\bar z} F_{m,n}(\bar z)}{F_{m,n}(\bar z)}\Big) \ge 0, \qquad n \in \mathcal{Q}(\bar z). \tag{3.8}$$

Fix $n \in \mathcal{Q}(\bar z)$. Viewing $z \mapsto F_{m,n}(z)$ as a function of two real variables $x = \mathrm{Re} z$ and $y = \mathrm{Im} z$, we can expand $\log|F_{m,n}(z)|$ into a Taylor series around the point $\bar z$ to get

$$\log\big|F_{m,n}(z_k)\big| - \log\big|F_{m,n}(\bar z)\big| = \mathrm{Re}\bigg((z_k - \bar z)\frac{\partial_z F_{m,n}(\bar z)}{F_{m,n}(\bar z)}\bigg) + O(|z_k - \bar z|^2). \tag{3.9}$$

To derive (3.9) we recalled that $F_{m,n}$ is at least twice continuously differentiable (hence the error bound) and then applied the identity

$$\frac{\partial \log|F_{m,n}(\bar z)|}{\partial x}\,\Delta x_k + \frac{\partial \log|F_{m,n}(\bar z)|}{\partial y}\,\Delta y_k = \mathrm{Re}\bigg((z_k - \bar z)\frac{\partial_z F_{m,n}(\bar z)}{F_{m,n}(\bar z)}\bigg), \tag{3.10}$$

where $\Delta x_k = \mathrm{Re}(z_k - \bar z)$ and $\Delta y_k = \mathrm{Im}(z_k - \bar z)$. (To derive (3.10), we just have to apply the chain rule to the functions $z \mapsto \log F_{m,n}(z)$. See Remark 5 for a discussion of this point.) Using that $z_k \to \bar{z}$, the inequality (3.8) and hence also (3.2) now follows by combining (3.9) with (3.5), dividing by $|z_k - \bar{z}|$ and taking the limit $k \to \infty$.

If, on the contrary, the inequality (3.2) is violated for some $n \in \mathcal{Q}(\bar{z})$, then (3.8) fails to hold as well and hence (3.7) and (3.5), with z_k replaced by z, must be wrong for $z \in \mathscr{W}_{\epsilon,\theta}(\bar{z})$ whenever ϵ is small enough. But $m \in \mathcal{Q}(\bar{z})$ implies that $|\zeta_m(\bar{z})| = |\zeta_n(\bar{z})|$ and thus $|\zeta_m(z)| < |\zeta_n(z)|$ for all $z \in \mathscr{W}_{\epsilon,\theta}(\bar{z})$, proving that $\mathscr{S}_m \cap \mathscr{W}_{\epsilon,\theta}(\bar{z}) = \emptyset$. By (3.1) and the fact that $z_k \to \bar{z}$, we have $z_k \in \mathscr{W}_{\epsilon,\theta}(\bar{z})$ and hence $z_k \notin \mathscr{S}_m$ for all k large enough. $\square$

Lemma 3.1 directly implies the following corollary.

Corollary 3.2. *Let Assumption A1–A2 hold and let $m, n \in \mathcal{R}$ be distinct. Let (z_k) be a sequence of numbers $z_k \in \mathscr{S}_m \cap \mathscr{S}_n$ such that $z_k \to \bar{z} \in \mathscr{O}$ but $z_k \neq \bar{z}$ for all k. Suppose that the limit (3.1) exists and equals $e^{\mathrm{i}\theta}$. Then* $\mathrm{Re}(e^{\mathrm{i}\theta} \upsilon_m) = \mathrm{Re}(e^{\mathrm{i}\theta} \upsilon_n)$.

Proof. Follows immediately applying (3.2) twice. $\square$

The next lemma will ensure that multiple points do not cluster and that the coexistence lines always intersect at positive angles.

Lemma 3.3. *Suppose that Assumption A holds and let $\bar{z} \in \mathscr{O}$. Suppose there are two sequences (z_k) and (z'_k) of numbers from $\mathscr{O}$ such that $|z_k - \bar{z}| = |z'_k - \bar{z}| \neq 0$ for all k and $z_k, z'_k \to \bar{z}$ as $k \to \infty$. Let $a, b, c \in \mathcal{R}$ and suppose that $z_k \in \mathscr{S}_a \cap \mathscr{S}_b$ and $z'_k \in \mathscr{S}_a \cap \mathscr{S}_c$ for all k. Suppose the limit (3.1) exists for both sequences and let $e^{\mathrm{i}\theta}$ and $e^{\mathrm{i}\theta'}$ be the corresponding limiting values.*

(1) If a, b, c are distinct, then $e^{\mathrm{i}\theta} \neq e^{\mathrm{i}\theta'}$.
(2) If $a \neq b = c$ and $z_k \neq z'_k$ for infinitely many k, then $|\mathcal{Q}(\bar{z})| = 2$ and $e^{\mathrm{i}\theta} = -e^{\mathrm{i}\theta'}$.

Remark 6. The conclusions of part (2) have a very natural interpretation. Indeed, in this case, $\bar{z}$ is a point on a two-phase coexistence line (whose existence we have not established yet) and z_k and z'_k are the (eventually unique) intersections of this line with a circle of radius $|z_k - \bar{z}| = |z'_k - \bar{z}|$ around $\bar{z}$. As the radius of this circle decreases, the intersections z_k and z'_k approach $\bar{z}$ from “opposite” sides, which explains why we should expect to have $e^{\mathrm{i}\theta} = -e^{\mathrm{i}\theta'}$.

Proof of Lemma 3.3. Throughout the proof, we set $\upsilon_m = \upsilon_m(\bar{z})$. We begin by proving (1). Assume that $a, b, c \in \mathcal{R}$ are distinct and suppose that $e^{\mathrm{i}\theta} = e^{\mathrm{i}\theta'}$. Note that, since $\mathcal{Q}(\bar{z}) \supset \{a, b, c\}$, the point $\bar{z}$ is a multiple point. Corollary 3.2 then implies that

$$\mathrm{Re}(e^{\mathrm{i}\theta} \upsilon_a) = \mathrm{Re}(e^{\mathrm{i}\theta} \upsilon_b) = \mathrm{Re}(e^{\mathrm{i}\theta} \upsilon_c), \tag{3.11}$$

and hence υ_a, υ_b and υ_c lie on a straight line in $\mathbb{C}$. But then υ_a, υ_b and υ_c cannot simultaneously be vertices of a strictly convex polygon, in contradiction with Assumption A4.

In order to prove part (2), let $a \neq b = c$, suppose without loss of generality that $z_k \neq z'_k$ for all k. If $e^{\mathrm{i}\theta} \neq \pm e^{\mathrm{i}\theta'}$, then Corollary 3.2 implies that $\mathrm{Re}(e^{\mathrm{i}\theta}(\upsilon_a - \upsilon_b)) = 0 = \mathrm{Re}(e^{\mathrm{i}\theta'}(\upsilon_a - \upsilon_b))$ and hence $\upsilon_a = \upsilon_b$, in contradiction with Assumption A3. Next we will rule out the possibility that $e^{\mathrm{i}\theta} = e^{\mathrm{i}\theta'}$, regardless of how many phases are stable at $\bar{z}$. Let $G(z) = \zeta_a(z)/\zeta_b(z)$ and note that $|G(z_k)| = 1 = |G(z'_k)|$ for all k. Applying

Taylor's theorem (analogously to the derivation of (3.9)), dividing by $|z_k - z'_k|$ and passing to the limit $k \to \infty$, we derive

$$\lim_{k\to\infty} \mathrm{Re}\biggl(\frac{z_k - z'_k}{|z_k - z'_k|}\,\frac{\partial_z G(z_k)}{G(z_k)}\biggr) = 0. \tag{3.12}$$

The second ratio on the left-hand side tends to $\upsilon_a - \upsilon_b$. As for the first ratio, an easy computation reveals that, since $|z_k - \bar z| = |z'_k - \bar z| \neq 0$, we have

$$\frac{z_k - z'_k}{|z_k - z'_k|} = \mathrm{i}e^{\mathrm{i}\frac{1}{2}(\theta_k+\theta'_k)}\,\frac{\sin((\theta_k - \theta'_k)/2)}{|\sin((\theta_k - \theta'_k)/2)|}, \tag{3.13}$$

where

$$e^{\mathrm{i}\theta_k} = \frac{z_k - \bar z}{|z_k - \bar z|} \quad \text{and} \quad e^{\mathrm{i}\theta'_k} = \frac{z'_k - \bar z}{|z'_k - \bar z|}. \tag{3.14}$$

By our assumptions, we have $e^{\mathrm{i}\theta_k} \to e^{\mathrm{i}\theta}$ and $e^{\mathrm{i}\theta'_k} \to e^{\mathrm{i}\theta'}$ as $k \to \infty$. Suppose now that $e^{\mathrm{i}\theta} = e^{\mathrm{i}\theta'}$. Then, choosing a subsequence if necessary, the left-hand side of (3.13) tends to a definite sign times $\mathrm{i}e^{\mathrm{i}\theta}$. Inserting this into (3.12) and using Corollary 3.2, in addition to $\mathrm{Re}(e^{\mathrm{i}\theta}(\upsilon_a - \upsilon_b)) = 0$, we now get that also $\mathrm{Re}(\mathrm{i}e^{\mathrm{i}\theta}(\upsilon_a - \upsilon_b)) = \mathrm{Im}(e^{\mathrm{i}\theta}(\upsilon_a - \upsilon_b)) = 0$. Consequently, $\upsilon_a = \upsilon_b$, again contradicting Assumption A3.

To finish the proof of the claim (2), it remains to rule out the possibility that $e^{\mathrm{i}\theta'} = -e^{\mathrm{i}\theta}$ in the case when $\bar z$ is a multiple point. Let $n \in \mathcal{Q}(\bar z)$ be another phase stable at $\bar z$, i.e., $n \neq a, b$. By Lemma 3.1, we have

$$\mathrm{Re}\bigl(e^{\mathrm{i}\theta}(\upsilon_m - \upsilon_n)\bigr) \geq 0 \quad \text{and} \quad \mathrm{Re}\bigl(e^{\mathrm{i}\theta'}(\upsilon_m - \upsilon_n)\bigr) \geq 0, \qquad m = a, b. \tag{3.15}$$

But then $e^{\mathrm{i}\theta'} = -e^{\mathrm{i}\theta}$ would imply that $\mathrm{Re}(e^{\mathrm{i}\theta}\upsilon_a) = \mathrm{Re}(e^{\mathrm{i}\theta}\upsilon_n) = \mathrm{Re}(e^{\mathrm{i}\theta}\upsilon_b)$, in contradiction with Assumption A4. Therefore, $|\mathcal{Q}(\bar z)| < 3$, as claimed. □

Corollary 3.4. *Suppose that Assumption A holds and let $\bar z \in \mathscr{O}$ be a multiple point. Then there exists a constant $\delta > 0$ such that $|\mathcal{Q}(z)| \leq 2$ for all $z \in \{z' \in \mathscr{O} : 0 < |z' - \bar z| < \delta\}$. In particular, each multiple point in $\mathscr{O}$ is isolated.*

Proof. Suppose $\bar z \in \mathscr{O}$ is a non-isolated multiple point. Then there is a sequence $z_k \in \mathscr{O}$ such that $z_k \to \bar z$ and, without loss of generality, $\mathcal{Q}(z_k) = \mathcal{Q}_0$ with $|\mathcal{Q}_0| \geq 3$, $z_k \neq \bar z$ for all k, and such that the limit (3.1) exists. Taking for (z'_k) the identical sequence, $z'_k = z_k$, we get $e^{\mathrm{i}\theta} = e^{\mathrm{i}\theta'}$ in contradiction to Lemma 3.3(1). Therefore, every multiple point in $\mathscr{O}$ is isolated. □

Our last auxiliary claim concerns the connectivity of sets of θ such that (3.2) holds. As will be seen in the proof of Lemma 3.6, this will be crucial for characterizing the topology of the phase diagram in small neighborhoods of multiple points.

Lemma 3.5. *Suppose that Assumption A holds and let $\bar z \in \mathscr{O}$ be a multiple point. For $m \in \mathcal{Q}(\bar z)$, let $\upsilon_m = \upsilon_m(\bar z)$. Then, for each $m \in \mathcal{Q}(\bar z)$, the set*

$$I_m = \bigl\{e^{\mathrm{i}\theta} \colon \theta \in [0, 2\pi),\ \mathrm{Re}(e^{\mathrm{i}\theta}\upsilon_m) > \mathrm{Re}(e^{\mathrm{i}\theta}\upsilon_n),\ n \in \mathcal{Q}(\bar z) \setminus \{m\}\bigr\} \tag{3.16}$$

is connected and open as a subset of $\{z \in \mathscr{O} \colon |z| = 1\}$. In particular, if $e^{\mathrm{i}\theta}$ is such that

$$\mathrm{Re}(e^{\mathrm{i}\theta}\upsilon_m) = \max_{n\in\mathcal{Q}(\bar z)\smallsetminus\{m\}} \mathrm{Re}(e^{\mathrm{i}\theta}\upsilon_n), \tag{3.17}$$

then $e^{\mathrm{i}\theta}$ is one of the two boundary points of I_m.

Proof. By Assumption A4, the numbers v_m, $m \in \mathcal{Q}(\bar{z})$, are the vertices of a strictly convex polygon $\mathscr{P}$ in $\mathbb{C}$. Let $s = |\mathcal{Q}(\bar{z})|$ and let $(v_1, \dots, v_s)$ be an ordering of the vertices of $\mathscr{P}$ in the counterclockwise direction. For $m = 1, \dots, s$ define $\Delta v_m = v_m - v_{m-1}$, where we take $v_0 = v_s$. Note that, by strict convexity of $\mathscr{P}$, the arguments θ_m of Δv_m, i.e., numbers θ_m such that $\Delta v_m = |\Delta v_m| e^{\mathrm{i}\theta_m}$, are such that the vectors $e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_s}$ are ordered counterclockwise, with the angle between $e^{\mathrm{i}\theta_m}$ and $e^{\mathrm{i}\theta_{m+1}}$ lying strictly between 0 and π for all $m = 1, \dots s$ (again, we identify $m = 1$ and $m = s + 1$). In other words, for each m, the angles $\theta_1 \dots, \theta_s$ can be chosen in such a way that $\theta_m < \theta_{m+1} < \dots < \theta_{m+s}$, with $0 < \theta_{m+k} - \theta_{m+k-1} < \pi$, $k = 1, \dots, s$. (Again, we identified $m + k$ with $m + k - s$ whenever $m + k > s$).

Using J_m to denote the set $J_m = \big\{\mathrm{i}e^{-\mathrm{i}\vartheta} : \vartheta \in (\theta_m, \theta_{m+1})\big\}$, we claim that $I_m = J_m$ for all $m = 1, \dots, s$. First, let us show that $J_m \subset I_m$. Let thus $\vartheta \in (\theta_m, \theta_{m+1})$ and observe that

$$\mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta}\,\Delta v_m) = |\Delta v_m| \sin(\vartheta - \theta_m) > 0, \tag{3.18}$$

because $\theta_m < \vartheta < \theta_{m+1} < \theta_m + \pi$. Similarly,

$$\mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta}\,\Delta v_{m+1}) = |\Delta v_{m+1}| \sin(\vartheta - \theta_{m+1}) < 0, \tag{3.19}$$

because $\theta_{m+1} - \pi < \theta_m < \vartheta < \theta_{m+1}$. Consequently, $\mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta} v_m) > \mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta} v_n)$ holds for both $n = m + 1$ and $n = m - 1$.

It remains to show that $\mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta} v_m) > \mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta} v_n)$ is true also for all remaining $n \in \mathcal{Q}(\bar{z})$. Let $n \in \mathcal{Q}(\bar{z}) \setminus \{m, m \pm 1\}$. We will separately analyze the cases with $\theta_n - \theta_m \in (0, \pi]$ and $\theta_n - \theta_m \in (-\pi, 0)$. Suppose first that $\theta_n - \theta_m \in (0, \pi]$. This allows us to write $n = m + k$ for some $k \in \{2, \dots, s - 1\}$ and estimate

$$\begin{aligned}\mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta}(v_n - v_m)) &= \sum_{j=1}^{k} \mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta}\,\Delta v_{m+j}) \\ &= \sum_{j=1}^{k} |\Delta v_{m+j}| \sin(\vartheta - \theta_{m+j}) < 0.\end{aligned} \tag{3.20}$$

The inequality holds since, in light of $\vartheta < \theta_{m+1} < \dots < \theta_{m+k} \le \theta + \pi$, each sine is negative except perhaps for the last one which is allowed to be zero. On the other hand, if $\theta_n - \theta_m \in (-\pi, 0)$, we write $n = m - k$ instead, for some $k \in \{2, \dots, s - 1\}$, and estimate

$$\begin{aligned}\mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta}(v_m - v_n)) &= \sum_{j=-k+1}^{0} \mathrm{Re}(\mathrm{i}e^{-\mathrm{i}\vartheta}\,\Delta v_{m+j}) \\ &= \sum_{j=-k+1}^{0} |\Delta v_{m+j}| \sin(\vartheta - \theta_{m+j}) > 0.\end{aligned} \tag{3.21}$$

Here we invoked the inequalities $\vartheta - \pi < \theta_{m-k} < \dots < \theta_m < \vartheta$ to show that each sine on the right-hand side is strictly positive.

As a consequence of the previous estimates, we conclude that $J_m \subset I_m$ for all $m = 1, \dots, s$. However, the union of all J_m's covers the unit circle with the exception of s points and, since the sets I_m are open and disjoint, we must have $I_m = J_m$ for all $m \in \mathcal{Q}(\bar{z})$. Then, necessarily, I_m is connected and open. Now the left-hand side of

(3.17) is strictly greater than the right-hand side for $e^{\mathrm{i}\theta} \in I_m$, and strictly smaller than the right-hand side for $e^{\mathrm{i}\theta}$ in the interior of the complement of I_m. By continuity of both sides, (3.17) can hold only on the boundary of I_m. □

3.2. Proof of Theorem 2.1. Having all the necessary tools ready, we can start proving Theorem 2.1. First we will apply Lemma 3.5 to characterize the situation around multiple points.

Lemma 3.6. *Suppose that Assumption A holds and let $\bar z \in \mathscr{O}$ be a multiple point. For $\delta > 0$, let*

$$I_m^{(\delta)} = \bigl\{z \in \mathscr{O} \colon |z - \bar z| = \delta,\ Q(z) \ni m\bigr\}. \tag{3.22}$$

Then the following is true once δ is sufficiently small:

(1) For each $m \in \mathcal{Q}(\bar z)$, the set $I_m^{(\delta)}$ is connected and has a non-empty interior.
(2) $I_m^{(\delta)} = \emptyset$ whenever $m \notin \mathcal{Q}(\bar z)$.
(3) For distinct m and n, the sets $I_m^{(\delta)}$ and $I_n^{(\delta)}$ intersect in at most one point.

Proof. The fact that $I_m^{(\delta)} = \emptyset$ for $m \notin \mathcal{Q}(\bar z)$ once $\delta > 0$ is sufficiently small is a direct consequence of the continuity of the functions ζ_m and ζ. Indeed, if there were a sequence of points z_k tending to $\bar z$ such that a phase m were stable at each z_k, then m would be also stable at $\bar z$.

We will proceed by proving that, as $\delta \downarrow 0$, each set $I_m^{(\delta)}$, $m \in \mathcal{Q}(\bar z)$, will eventually have a non-empty interior. Let $m \in \mathcal{Q}(\bar z)$. Observe that, by Lemma 3.5, there is a value $e^{\mathrm{i}\theta}$ (namely, a number from I_m) such that $\mathrm{Re}(e^{\mathrm{i}\theta} v_m) > \mathrm{Re}(e^{\mathrm{i}\theta} v_n)$ for all $n \in \mathcal{Q}(\bar z) \setminus \{m\}$. But then the second part of Lemma 3.1 guarantees the existence of an $\epsilon > 0$ such that $\mathcal{Q}(z) = \{m\}$ for all $z \in \mathscr{W}_{\epsilon,\theta}(\bar z)$—see (3.3). In particular, the intersection $\mathscr{W}_{\epsilon,\theta}(\bar z) \cap \{z \colon |z - \bar z| = \delta\}$, which is non-empty and (relatively) open for $\delta < \epsilon$, is a subset of $I_m^{(\delta)}$. It follows that the set $I_m^{(\delta)}$ has a nonempty interior once δ is sufficiently small.

Next we will prove that each $I_m^{(\delta)}$, $m \in \mathcal{Q}(\bar z)$, is eventually connected. Suppose that there exist a phase $a \in \mathcal{Q}(\bar z)$ and a sequence $\delta_k \downarrow 0$ such that all sets $I_a^{(\delta_k)}$ are *not* connected. Then, using the fact that $I_a^{(\delta_k)}$ has nonempty interior and thus cannot consist of just two separated points, we conclude that the phase a coexists with some other phase at at least three distinct points on each circle $\{z \colon |z - \bar z| = \delta_k\}$. Explicitly, there exist (not necessarily distinct) indices $b_k^{(j)} \in \mathcal{Q}(\bar z) \setminus \{a\}$ and points $(z_k^{(j)})$, $j = 1, 2, 3$, with $|z_k^{(j)} - \bar z| = \delta_k$ and $z_k^{(j)} \neq z_k^{(\ell)}$ for $j \neq \ell$, such that $a, b_k^{(j)} \in \mathcal{Q}(z_k^{(j)})$. Moreover, (choosing subsequences if needed) we can assume that $b_k^{(j)} = b^{(j)}$ for some $b^{(j)} \in \mathcal{Q}(\bar z) \setminus \{a\}$ independent of k. Resorting again to subsequences, we also may assume that the limits in (3.1) exist for all three sequences.

Let us use $e^{\mathrm{i}\theta_j}$ to denote the corresponding limits for the three sequences. First we claim that the numbers $e^{\mathrm{i}\theta_j}$, $j = 1, 2, 3$, are necessarily all distinct. Indeed, suppose two of the $e^{\mathrm{i}\theta_j}$'s are the same and let b and c be the phases coexisting with a along the corresponding sequences. Then Lemma 3.3(1) forces $b = c$, which contradicts both conclusions of Lemma 3.3(2). Therefore, all three $e^{\mathrm{i}\theta_j}$ must be different. Applying now Corollary 3.2 and Lemma 3.1, we get $\mathrm{Re}(e^{\mathrm{i}\theta_j} v_a) = \max_{n \in \mathcal{Q}(\bar z) \setminus \{a\}} \mathrm{Re}(e^{\mathrm{i}\theta_j} v_n)$ for $j = 1, 2, 3$. According to Lemma 3.5, all three distinct numbers $e^{\mathrm{i}\theta_j}$, $j = 1, 2, 3$, are endpoints of I_a, which is not possible since I_a is a connected subset of the unit circle. Thus, we can conclude that $I_a^{(\delta)}$ must be connected once $\delta > 0$ is sufficiently small.

To finish the proof, we need to show that $I_a^{(\delta)} \cap I_b^{(\delta)}$ contains at most one point for any $a \neq b$. First note that we just ruled out the possibility that this intersection contains *three* distinct points for a sequence of δ's tending to zero. (Indeed, then a would coexist with b along three distinct sequences, which would in turn imply that a and b coexists along three distinct directions, in contradiction with Lemma 3.5.) Suppose now that $I_a^{(\delta)} \cap I_b^{(\delta)}$ contains two distinct points. Since both $I_a^{(\delta)}$ and $I_b^{(\delta)}$ are connected with open interior, this would mean that $I_a^{(\delta)}$ and $I_b^{(\delta)}$ cover the entire circle of radius δ. Once again, applying the fact that two $I_m^{(\delta)}$ have at most two points in common, we then must have $I_c^{(\delta)} = \emptyset$ for all $c \neq a, b$. But $\mathcal{Q}(\bar{z})$ contains at least three phases which necessitates that $I_m^{(\delta)} \neq \emptyset$ for at least three distinct m. Hence $I_a^{(\delta)} \cap I_b^{(\delta)}$ cannot contain more than one point. □

Next we will give a local characterization of two-phase coexistence lines.

Lemma 3.7. *Suppose that Assumption A holds and let $m, n \in \mathcal{R}$ be distinct. Let $z \in \mathscr{O}$ be such that $z \in \mathscr{S}_m \cap \mathscr{S}_n$ and $\mathcal{Q}(z') \subset \{m, n\}$ for $z' \in \mathbb{D}_\delta(z)$. Then there exist numbers $\delta' \in (0, \delta)$, $t_1 < 0$, $t_2 > 0$, and an twice continuously differentiable function $\gamma_z\colon (t_1, t_2) \to \mathbb{D}_{\delta'}(z)$ such that*

(1) $\gamma_z(0) = z$.
(2) $|\zeta_m(\gamma_z(t))| = |\zeta_n(\gamma_z(t))| = \zeta(\gamma_z(t)),\ t \in (t_1, t_2)$.
(3) $\lim_{t\downarrow t_1} \gamma_z(t),\ \lim_{t\uparrow t_2} \gamma_z(t) \in \partial\mathbb{D}_{\delta'}(z)$.

The curve $t \mapsto \gamma_z(t)$ is unique up to reparametrization. Moreover, the set $\mathbb{D}_{\delta'}(z) \setminus \gamma_z(t_1, t_2)$ has two connected components and m is the only stable phase in one of the components while n is the only stable phase in the other.

Proof. We begin by observing that by Assumption A3, the function

$$\phi_{m,n}(x, y) = \log|\zeta_m(x + \mathrm{i}y)| - \log|\zeta_n(x + \mathrm{i}y)| = \operatorname{Re}\log F_{m,n}(x + \mathrm{i}y), \qquad (3.23)$$

has at least one of the derivatives $\partial_x\phi_{m,n}, \partial_y\phi_{m,n}$ non-vanishing at $x + \mathrm{i}y = z$. By continuity, there exists a constant $\eta > 0$ such that one of the derivatives is uniformly bounded away from zero for all $z' = u + \mathrm{i}v \in \mathbb{D}_\eta(z)$. Since $z = x + \mathrm{i}y \in \mathscr{S}_m \cap \mathscr{S}_n$, we have $\phi_{m,n}(x, y) = 0$. By the implicit function theorem, there exist numbers t_0', t_1', x_0, x_1, y_0 and y_1 such that $t_0' < 0 < t_1'$, $x_0 < x < x_1$, $y_0 < y < y_1$ and $(x_0, x_1) \times (y_0, y_1) \subset \mathbb{D}_\eta(z)$, and twice continuously differentiable functions $u\colon (t_0', t_1') \to (x_0, x_1)$ and $v\colon (t_0', t_1') \to (y_0, y_1)$ such that

$$\phi_{m,n}\big(u(t), v(t)\big) = 0, \qquad t \in (t_0', t_1'), \qquad (3.24)$$

and

$$u(0) = x, \quad \text{and} \quad v(0) = y. \qquad (3.25)$$

Moreover, since the second derivatives of $\phi_{m,n}$ are continuous in $\mathscr{O}$ and therefore bounded in $\mathbb{D}_\eta(z)$, standard theorems on uniqueness of the solutions of ODEs guarantee that the solution to (3.24) and (3.25) is unique up to reparametrization. The construction of γ_z is now finished by picking δ' so small that $\mathbb{D}_{\delta'}(z) \subset (x_0, x_1) \times (y_0, y_1)$, and taking t_0 and t_1 to be the first backward and forward time, respectively, when $(u(t), v(t))$ leaves $\mathbb{D}_{\delta'}(z)$.

The fact that $\mathbb{D}_{\delta'}(z) \setminus \gamma_z(t_1, t_2)$ splits into two components is a consequence of the construction of γ_z. Moreover, γ_z is a (zero-)level curve of function $\phi_{m,n}$ which has a non-zero gradient. Hence, $\phi_{m,n} < 0$ on one component of $\mathbb{D}_{\delta'}(z) \setminus \gamma_z(t_1, t_2)$, while $\phi_{m,n} > 0$ on the other. Recalling the assumption that $\mathcal{Q}(z') \subset \{m, n\}$ for z' in a neighborhood of z, the claim follows. $\square$

Now we can finally give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\mathscr{M}$ denote the set of all multiple points in $\mathscr{O}$, i.e., let

$$\mathscr{M} = \bigl\{z \in \mathscr{O} \colon |\mathcal{Q}(z)| \geq 3\bigr\}. \tag{3.26}$$

By Corollary 3.4, we know that $\mathscr{M}$ is relatively closed in $\mathscr{O}$ and so the set $\mathscr{O}' = \mathscr{O} \setminus \mathscr{M}$ is open. Moreover, the set $\mathscr{G} \cap \mathscr{O}'$ consists solely of points where exactly two phases coexist. Lemma 3.7 then shows that for each $z \in \mathscr{G} \cap \mathscr{O}'$, there exists a disc $\mathbb{D}_{\delta'}(z)$ and a unique, smooth γ_z in $\mathbb{D}_{\delta'}(z)$ passing through z such that $\mathcal{Q}(z') = \mathcal{Q}(z)$ for all z' on the curve γ_z. Let $\tilde{\gamma}_z$ be a maximal extension of the curve γ_z in $\mathscr{O}'$. We claim that $\tilde{\gamma}_z$ is either a closed curve or an arc with both endpoints on $\partial\mathscr{O}'$. Indeed, if $\tilde{\gamma}_z$ were open with an end-point $\tilde{z} \in \mathscr{O}'$, then $\mathcal{Q}(\tilde{z}) \supset \mathcal{Q}(z)$, by continuity of functions ζ_m. But $\tilde{z} \in \mathscr{O}'$ and so $|\mathcal{Q}(\tilde{z})| \leq 2$, which implies that $\mathcal{Q}(\tilde{z}) = \mathcal{Q}(z)$. By Lemma 3.7, there exists a non-trivial curve $\gamma_{\tilde{z}}$ along which the two phases from $\mathcal{Q}(\tilde{z})$ coexist in a neighborhood of $\tilde{z}$. But then $\gamma_{\tilde{z}} \cup \tilde{\gamma}_z$ would be a non-trivial extension of $\tilde{\gamma}_z$, in contradiction with the maximality of $\tilde{\gamma}_z$. Thus we can conclude that $\tilde{z} \in \partial\mathscr{O}'$.

Let $\mathfrak{C}$ denote the set of maximal extensions of the curves $\{\gamma_z \colon z \in \mathscr{G} \cap \mathscr{O}'\}$. Let $\mathscr{D} \subset \mathscr{O}$ be a compact set and note that Corollary 3.4 implies that $\mathscr{D} \cap \mathscr{M}$ is finite. Let δ_0 be so small that, for each $z_{\mathrm{M}} \in \mathscr{M} \cap \mathscr{D}$, we have $\mathbb{D}_{\delta_0}(z_{\mathrm{M}}) \subset \mathscr{O}$, $\overline{\mathbb{D}_{\delta_0}(z_{\mathrm{M}})} \cap \mathscr{M} = \{z_{\mathrm{M}}\}$ and the statements in Lemma 3.6 hold true for $\delta \leq \delta_0$. Let $\delta \in (0, \delta_0]$. We claim that if a curve $\mathscr{C} \in \mathfrak{C}$ intersects the disc $\mathbb{D}_\delta(z_{\mathrm{M}})$ for a $z_{\mathrm{M}} \in \mathscr{M} \cap \mathscr{D}$, then the restriction $\mathscr{C} \cap \mathbb{D}_\delta(z_{\mathrm{M}})$ is a simple curve connecting z_{M} to $\partial\mathbb{D}_\delta(z_{\mathrm{M}})$. Indeed, each curve $\mathscr{C} \in \mathfrak{C}$ terminates either on $\partial\mathscr{O}$ or on $\mathscr{M}$. If $\mathscr{C}$ "enters" $\mathbb{D}_\delta(z_{\mathrm{M}})$ and does not hit z_{M}, our assumptions about δ_0 imply that $\mathscr{C}$ "leaves" $\mathbb{D}_\delta(z_{\mathrm{M}})$ through the boundary. But Lemma 3.7 ensures that one of the phases coexisting along $\mathscr{C}$ dominates in a small neighborhood on the "left" of $\mathscr{C}$, while the other dominates in a small neighborhood on the "right" of $\mathscr{C}$. The only way this can be made consistent with the connectivity of the sets $I_m^{(\delta)}$ in Lemma 3.6 is by assuming that $I_m^{(\delta)} \neq \emptyset$ only for the two m's coexisting along $\mathscr{C}$. But that still contradicts Lemma 3.6, by which $I_m^{(\delta)} \neq \emptyset$ for at least *three* distinct m. Thus, once a curve $\mathscr{C} \in \mathfrak{C}$ intersects $\mathbb{D}_\delta(z_{\mathrm{M}})$, it must terminate at z_{M}.

Let $\mathscr{D}_0 = \mathscr{D} \setminus \bigcup_{z \in \mathscr{M}} \mathbb{D}_{\delta_0}(z)$ and let $\Delta \colon \mathscr{D}_0 \to [0, \infty)$ be a function given by

$$\Delta(z) = \inf\bigl\{\delta' \in (0, \delta_0) \colon \mathbb{D}_{\delta'}(z) \subset \mathscr{O},\ \mathbb{D}_{\delta'}(z) \cap \textstyle\bigcup_{\mathscr{C} \in \mathfrak{C}} \mathscr{C} \text{ is disconnected}\bigr\}. \tag{3.27}$$

We claim that Δ is bounded from below by a positive constant. Indeed, Δ is clearly continuous and, since $\mathscr{D}_0$ is compact, Δ attains its minimum at some $z \in \mathscr{D}_0$. If $\Delta(z) = 0$, then z is a limit point of $\bigcup_{\mathscr{C} \in \mathfrak{C}} \mathscr{C}$ and thus $z \in \mathscr{C}$ for some $\mathscr{C} \in \mathfrak{C}$. Moreover, for infinitely many $\delta' \in (0, \delta_0)$, the circle $\partial\mathbb{D}_{\delta'}(z)$ intersects the set $\bigcup_{\mathscr{C} \in \mathfrak{C}} \mathscr{C}$ in at least three different points. Indeed, the curve $\mathscr{C} \ni z$ provides two intersections; the third intersection is obtained by adjusting the radius δ' so that $\mathbb{D}_{\delta'}(z) \cap \bigcup_{\mathscr{C} \in \mathfrak{C}}$ is disconnected. Thus, we are (again) able to construct three sequences (z_k), (z_k') and (z_k'') such that, without loss of generality, $z_k, z_k', z_k'' \in \mathscr{S}_a \cap \mathscr{S}_b$ for some distinct $a, b \in \mathcal{R}$ (only two phases can exist in sufficiently small neighborhoods of points in $\mathscr{D}_0$), $|z_k - \bar{z}| =$

$|z'_k - \bar{z}| = |z''_k - \bar{z}| \to 0$, but $z_k \neq z'_k \neq z''_k \neq z_k$ for all k. However, this contradicts Lemma 3.3, because its part (2) cannot hold simultaneously for all three pairs of sequences (z_k, z'_k), (z'_k, z''_k) and (z_k, z''_k).

Now we are ready to define the set of points $z_1, \dots, z_\ell$. Let ϵ be the minimum of the function Δ in $\mathscr{D}_0$ and let $\delta = \min(\delta_0, \epsilon)$. Consider the following collections of open finite discs:

$$\begin{aligned} \mathcal{S}_1 &= \big\{\mathbb{D}_\delta(z) \colon z \in \mathscr{M} \cap \mathscr{D}\big\}, \\ \mathcal{S}_2 &= \big\{\mathbb{D}_\delta(z) \colon z \in \mathscr{D} \cap \textstyle\bigcup_{\mathscr{C} \in \mathfrak{C}} \mathscr{C},\ \operatorname{dist}(z, \bigcup_{\mathbb{D} \in \mathcal{S}_1} \mathbb{D}) > \tfrac{2}{3}\delta\big\}, \\ \mathcal{S}_3 &= \big\{\mathbb{D}_\delta(z) \colon z \in \mathscr{D},\ \operatorname{dist}(z, \textstyle\bigcup_{\mathbb{D} \in \mathcal{S}_1 \cup \mathcal{S}_2} \mathbb{D}) > \tfrac{2}{3}\delta\big\}. \end{aligned} \tag{3.28}$$

It is easy to check that the union of these discs covers $\mathscr{D}$. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. By compactness of $\mathscr{D}$, we can choose a finite collection $\mathcal{S}' \subset \mathcal{S}$ still covering $\mathscr{D}$. It remains to show that the sets $\mathscr{A} = \mathscr{G} \cap \mathbb{D}$ for $\mathbb{D} \in \mathcal{S}'$ will have the desired properties. Let $\mathbb{D} \in \mathcal{S}'$ and let z be the center of $\mathbb{D}$. If $\mathbb{D} \in \mathcal{S}_3$, then $\mathscr{G} \cap \mathbb{D} = \emptyset$. Indeed, if z' is a coexistence point, then $\mathbb{D}_\delta(z') \in \mathcal{S}_1 \cup \mathcal{S}_2$ and thus $\operatorname{dist}(z, z') > \delta + \frac{2}{3}\delta$ and hence $z' \notin \mathbb{D}$. Next, if $\mathbb{D} \in \mathcal{S}_2$, then $z \in \mathscr{G}$ and, by the definition of δ_0 and ϵ, the disc $\mathbb{D}$ contains no multiple point and intersects $\mathscr{G}$ only in one component. This component is necessarily part of one of the curves $\mathscr{C} \in \mathfrak{C}$. Finally, if $\mathbb{D} \in \mathcal{S}_1$, then z is a multiple point and, relying on our previous reasoning, several curves $\mathscr{C} \in \mathfrak{C}$ connect z to the boundary of $\mathbb{D}$. Since Lemma 3.6 implies the existence of exactly $|\mathcal{Q}(z)|$ coexistence points on $\partial\mathbb{D}$, there are exactly $|\mathcal{Q}(z)|$ such curves. The proof is finished by noting that every multiple point appears as the center of some disc $\mathbb{D} \in \mathcal{S}'$, because that is how the collections (3.28) were constructed. $\square$

4. Partition function zeros

The goal of this section is to prove Theorems 2.2-2.5. The principal tool which enables us to control the distance between the roots of Z_L^{per} and the solutions of equations (2.17–2.18) or (2.24) is Rouché's Theorem (see e.g. [16]). For reader's convenience, we transcribe the corresponding statement here:

Theorem 4.1 (Rouché's Theorem). *Let $\mathscr{D} \subset \mathbb{C}$ be a bounded domain with piecewise smooth boundary $\partial\mathscr{D}$. Let f and g be analytic on $\mathscr{D} \cup \partial\mathscr{D}$. If $|g(z)| < |f(z)|$ for all $z \in \partial\mathscr{D}$, then f and $f + g$ have the same number of zeros in $\mathscr{D}$, counting multiplicities.*

More details on the use of this theorem and the corresponding bounds are stated in Sect. 4.2 for the case of two-phase coexistence and in Sect. 4.4 for the case of multiple phase coexistence.

Root degeneracy will be controlled using a link between the non-degeneracy conditions from Assumption B and certain Vandermonde determinants; cf Sect. 4.1. Throughout this section, we will use the shorthand

$$\mathscr{S}_\epsilon(\mathcal{Q}) = \bigcap_{m \in \mathcal{Q}} \mathscr{S}_\epsilon(m) \tag{4.1}$$

to denote the set of points $z \in \mathscr{O}$ where all phases from a non-empty $\mathcal{Q} \subset \mathcal{R}$ are "almost stable" (as quantified by $\epsilon > 0$).

4.1. Root degeneracy. In this section we will prove Theorem 2.2. We begin with a claim about the Vandermonde matrix defined in terms of the functions

$$b_m(z) = \frac{\partial_z \zeta_m^{(L)}(z)}{\zeta_m^{(L)}(z)}, \qquad z \in \mathscr{S}_{\kappa/L}(m), \tag{4.2}$$

where the dependence of b_m on L has been suppressed in the notation. Let us fix a non-empty $\mathcal{Q} \subset \mathcal{R}$ and let $q = |\mathcal{Q}|$. For each $z \in \mathscr{S}_{\kappa/L}(\mathcal{Q})$, we introduce the $q \times q$ Vandermonde matrix $\mathbb{M}(z)$ with elements

$$\mathbb{M}_{\ell,m}(z) = b_m(z)^\ell, \qquad m \in \mathcal{Q}, \ \ell = 0, 1, \dots, q-1. \tag{4.3}$$

Let $\|\mathbb{M}\|$ denote the $\ell^2(\mathcal{Q})$-norm of $\mathbb{M}$ (again without making the $\mathcal{Q}$-dependence of this norm notationally explicit). Explicitly, $\|\mathbb{M}\|^2$ is defined by the supremum

$$\|\mathbb{M}\|^2 = \sup\biggl\{\sum_{\ell=0}^{q-1}\Bigl|\sum_{m\in\mathcal{Q}} \mathbb{M}_{\ell,m}\hat{\mathrm{w}}_m\Bigr|^2 : \sum_{m\in\mathcal{Q}} |\hat{\mathrm{w}}_m|^2 = 1\biggr\}, \tag{4.4}$$

where $(\hat{\mathrm{w}}_m)$ is a $|\mathcal{Q}|$-dimensional complex vector.

Throughout the rest of this section, the symbol $\|\cdot\|$ will refer to the (vector or matrix) ℓ^2-norm as specified above. The only exceptions are the ℓ^p-norms $\|\mathbf{q}\|_1$, $\|\mathbf{q}\|_2$ and $\|\mathbf{q}\|_\infty$ of the r-tuple $(q_m)_{m\in\mathcal{R}}$, which are defined in the usual way.

Lemma 4.2. *Suppose that Assumption B3 holds and let $\tilde{L}_0$ be as in Assumption B3. For each $\mathcal{Q} \subset \mathcal{R}$, there exists a constant $K = K(\mathcal{Q}) < \infty$ such that*

$$\bigl\|\mathbb{M}^{-1}(z)\bigr\| \le K, \ \textit{for all } z \in \mathscr{S}_{\kappa/L}(\mathcal{Q}) \textit{ and } L \ge \tilde{L}_0. \tag{4.5}$$

In particular, $\mathbb{M}(z)$ is invertible for all $z \in \mathscr{S}_{\kappa/L}(\mathcal{Q})$ and $L \ge \tilde{L}_0$.

Proof. Let $\mathcal{Q} \subset \mathcal{R}$ and $q = |\mathcal{Q}|$. Let us choose a point $z \in \mathscr{S}_{\kappa/L}(\mathcal{Q})$ and let $\mathbb{M}$ and b_m, $m \in \mathcal{Q}$, be the quantities $\mathbb{M}(z)$ and $b_m(z)$, $m \in \mathcal{Q}$. First we note that, since $\mathbb{M}$ is a Vandermonde matrix, its determinant can be explicitly computed: $\det\mathbb{M} = \prod_{m<n}(b_n - b_m)$, where "$<$" denotes a complete order on $\mathcal{Q}$. In particular, Assumption B3 implies that $|\det\mathbb{M}| \ge \tilde{\alpha}^{q(q-1)/2} > 0$ once $L \ge \tilde{L}_0$.

To estimate the matrix norm of $\mathbb{M}^{-1}$, let $\lambda_1, \dots, \lambda_q$ be the eigenvalues of the Hermitian matrix $\mathbb{M}\mathbb{M}^+$ and note that $\lambda_\ell > 0$ for all $\ell = 1, \dots, q$ by our lower bound on $|\det\mathbb{M}|$. Now, $\|\mathbb{M}^+\|^2$ is equal to the spectral radius of the operator $\mathbb{M}\mathbb{M}^+$, and $\|\mathbb{M}^{-1}\|^2$ is equal to the spectral radius of the operator $(\mathbb{M}\mathbb{M}^+)^{-1}$. By the well-known properties of the norm we thus have

$$\|\mathbb{M}\|^2 = \|\mathbb{M}^+\|^2 = \max_{1\le\ell\le q} \lambda_\ell, \tag{4.6}$$

while

$$\|\mathbb{M}^{-1}\|^2 = \max_{1\le\ell\le q} \lambda_\ell^{-1}. \tag{4.7}$$

Now $|\det\mathbb{M}|^2 = \det\mathbb{M}\mathbb{M}^+ = \lambda_1 \dots \lambda_q$ and a simple algebraic argument gives us that

$$\|\mathbb{M}^{-1}\| \le \frac{\|\mathbb{M}\|^{q-1}}{|\det\mathbb{M}|}. \tag{4.8}$$

Using the lower bound on $|\det \mathbb{M}|$, this implies that $\|\mathbb{M}^{-1}\| \le \tilde{\alpha}^{-q\frac{q-1}{2}} \|\mathbb{M}\|^{q-1}$. The claim then follows by invoking the uniform boundedness of the matrix elements of $\mathbb{M}$ (see the upper bound from Assumption B3), which implies that $\|\mathbb{M}\|$ and hence also $\|\mathbb{M}^{-1}\|$ is uniformly bounded from above throughout $\mathscr{S}_{\kappa/L}(\mathcal{Q})$. □

Now we are ready to prove Theorem 2.2. To make the reading easier, let us note that for $\mathcal{Q} = \{m\}$, the expression (2.8) defining $\mathscr{U}_\epsilon(\mathcal{Q})$ can be simplified to

$$\mathscr{U}_\epsilon(\{m\}) = \bigl\{z \in \mathscr{O} \colon |\zeta_n(z)| < e^{-\epsilon/2}|\zeta(z)| \text{ for all } n \neq m\bigr\}, \tag{4.9}$$

a fact already mentioned right after (2.8).

Proof of Theorem 2.2. Let $m \in \mathcal{R}$. Since the sets $\mathscr{U}_{\kappa/L}(\mathcal{Q})$, $\mathcal{Q} \subset \mathcal{R}$, cover $\mathscr{O}$, it suffices to prove that $Z_L^{\text{per}} \neq 0$ in $\mathscr{U}_{L^{-d}\omega_L}(\{m\}) \cap \mathscr{U}_{\kappa/L}(\mathcal{Q})$ for each $\mathcal{Q} \subset \mathcal{R}$. In fact, since $z \in \mathscr{U}_{L^{-d}\omega_L}(\{m\})$ implies that m is stable, $|\zeta_m(z)| = \zeta(z)$, we may assume without loss of generality that $m \in \mathcal{Q}$, because otherwise $\mathscr{U}_{L^{-d}\omega_L}(\{m\}) \cap \mathscr{U}_{\kappa/L}(\mathcal{Q}) = \emptyset$. Thus, let $m \in \mathcal{Q} \subset \mathcal{R}$ and fix a point $z \in \mathscr{U}_{L^{-d}\omega_L}(\{m\}) \cap \mathscr{U}_{\kappa/L}(\mathcal{Q})$. By Assumption B4, we have the bound

$$\begin{aligned} \bigl|Z_L^{\text{per}}(z)\bigr| \ge \zeta(z)^{L^d} \Biggl(q_m \Bigl|\frac{\zeta_m^{(L)}(z)}{\zeta(z)}\Bigr|^{L^d} \\ - \sum_{n \in \mathcal{Q} \smallsetminus \{m\}} q_n \Bigl|\frac{\zeta_n^{(L)}(z)}{\zeta(z)}\Bigr|^{L^d} - C_0 L^d \|\mathbf{q}\|_1 e^{-\tau L} \Biggr). \end{aligned} \tag{4.10}$$

Since $z \in \mathscr{U}_{L^{-d}\omega_L}(\{m\})$, we have $|\zeta_n(z)| < \zeta(z) e^{-\frac{1}{2}L^{-d}\omega_L}$ for $n \neq m$. In conjunction with Assumption B2, this implies

$$\Bigl|\frac{\zeta_n^{(L)}(z)}{\zeta(z)}\Bigr|^{L^d} \le e^{L^d e^{-\tau L}} e^{-\frac{1}{2}\omega_L}, \qquad n \neq m. \tag{4.11}$$

On the other hand, we also have

$$\Bigl|\frac{\zeta_m^{(L)}(z)}{\zeta(z)}\Bigr|^{L^d} \ge e^{-L^d e^{-\tau L}}, \tag{4.12}$$

where we used that $|\zeta_m(z)| = \zeta(z)$. Since $\omega_L \to \infty$, (4.11–4.12) show that the right-hand side (4.10) is dominated by the term with index m, which is bounded away from zero uniformly in L. Consequently, $Z_L^{\text{per}} \neq 0$ throughout $\mathscr{U}_{L^{-d}\omega_L}(\{m\}) \cap \mathscr{U}_{\kappa/L}(\mathcal{Q})$, provided L is sufficiently large.

Next we will prove the claim about the degeneracy of the roots. Let us fix $\mathcal{Q} \subset \mathcal{R}$ and let, as before, $q = |\mathcal{Q}|$. Suppose that $L \ge \tilde{L}_0$ and let $z \in \mathscr{U}_{\kappa/L}(\mathcal{Q})$ be a root of Z_L^{per} that is at least q-times degenerate. Since Z_L^{per} is analytic in a neighborhood of z, we have

$$\partial_z^\ell Z_L^{\text{per}}(z) = 0, \qquad \ell = 0, 1, \dots, q-1. \tag{4.13}$$

It will be convenient to introduce q-dimensional vectors $\mathbf{x} = \mathbf{x}(z)$ and $\mathbf{y} = \mathbf{y}(z)$ such that (4.13) can be expressed as

$$\mathbb{M}(z)\mathbf{x} = \mathbf{y}, \tag{4.14}$$

with $\mathbb{M}(z)$ given by (4.2) and (4.3). Indeed, let $\mathbf{x} = \mathbf{x}(z)$ be the vector with components

$$x_m = q_m \Bigl(\frac{\zeta_m^{(L)}(z)}{\zeta(z)}\Bigr)^{L^d}, \qquad m \in \mathcal{Q}. \tag{4.15}$$

Similarly, let $\mathbf{y} = \mathbf{y}(z)$ be the vector with components $y_0, \dots, y_{q-1}$, where

$$\begin{aligned} y_\ell = {} & L^{-d\ell} \zeta(z)^{-L^d} \partial_z^\ell \varXi_{\mathcal{Q},L}(z) \\ & - \sum_{m \in \mathcal{Q}} q_m \, \zeta(z)^{-L^d} \Bigl\{ L^{-d\ell} \partial_z^\ell \bigl[\zeta_m^{(L)}(z)\bigr]^{L^d} - b_m(z)^\ell \bigl[\zeta_m^{(L)}(z)\bigr]^{L^d} \Bigr\}. \end{aligned} \tag{4.16}$$

Recalling the definition $\varXi_{\mathcal{Q},L}(z)$ from (2.13), it is easily seen that (4.14) is equivalent to (4.13).

We will now produce appropriate bounds on the $\ell^2(\mathcal{Q})$-norms $\|\mathbf{y}\|$ and $\|\mathbf{x}\|$ which hold uniformly in $z \in \mathscr{U}_{\kappa/L}(\mathcal{Q})$, and show that (4.14) contradicts Lemma 4.2. To estimate $\|\mathbf{y}\|$, we first note that there is a constant $A < \infty$, independent of L, such that, for all $\ell = 0, \dots, q-1$ and all $z \in \mathscr{U}_{\kappa/L}(\mathcal{Q})$,

$$\Bigl| L^{-d\ell} \partial_z^\ell \bigl[\zeta_m^{(L)}(z)\bigr]^{L^d} - b_m(z)^\ell \bigl[\zeta_m^{(L)}(z)\bigr]^{L^d} \Bigr| \le A L^{-d} \zeta(z)^{L^d}. \tag{4.17}$$

Here the leading order term from $L^{-d\ell}\partial_z^\ell[\zeta_m^{(L)}(z)]^{L^d}$ is exactly canceled by the term $b_m(z)^\ell[\zeta_m^{(L)}(z)]^{L^d}$, and the remaining terms can be bounded using (2.11). Invoking (4.17) in (4.16) and applying (2.14), we get

$$\|\mathbf{y}\| \le A \|\mathbf{q}\|_1 \sqrt{q} L^{-d} + \bigl(\max_{0 \le \ell \le q-1} C_\ell\bigr) \|\mathbf{q}\|_1 \sqrt{q} L^d e^{-\tau L}, \tag{4.18}$$

where the factor $\sqrt{q}$ comes from the conversion of ℓ^∞-type bounds (4.17) into a bound on the ℓ^2-norm $\|\mathbf{y}\|$. On the other hand, by (2.9) and $q_m \ge 1$ we immediately have

$$\|\mathbf{x}\| \ge e^{-e^{-\tau L}}. \tag{4.19}$$

But $\|\mathbf{x}\| \le \|\mathbb{M}^{-1}(z)\| \, \|\mathbf{y}\|$, so once L is sufficiently large, this contradicts the upper bound $\|\mathbb{M}^{-1}(z)\| \le K$ implied by Lemma 4.2. Therefore, the root at z cannot be more than $(q-1)$-times degenerate after all. $\square$

4.2. Two-phase coexistence. Here we will prove Theorem 2.3 on the location of partition function zeros in the range of parameter z where only two phases from $\mathcal{R}$ prevail. Throughout this section we will assume that Assumptions A and B are satisfied and use κ and τ to denote the constants from Assumption B. We will also use $\delta_L(z)$ for the function defined in (2.16).

The proof of Theorem 2.3 is based directly on three technical lemmas, namely, Lemma 4.3–4.5 below, whose proofs are deferred to Sect. 5.2. The general strategy is as follows: First, by Lemma 4.3, we will know that the solutions to (2.17–2.18) are within an $O(e^{-\tau L})$-neighborhood from the solutions of similar equations, where the functions ζ_m get replaced by their analytic counterparts $\zeta_m^{(L)}$. Focusing on specific indices m and n, we will write these analytic versions of (2.17–2.18) as $f(z) = 0$, where f is the function defined by

$$f(z) = q_m \zeta_m^{(L)}(z)^{L^d} + q_n \zeta_n^{(L)}(z)^{L^d}, \qquad z \in \mathscr{S}_{\kappa/L}(\{m,n\}). \tag{4.20}$$

The crux of the proof of Theorem 2.3 is then to show that the solutions of $f(z) = 0$ are located within an appropriate distance from the zeros of $Z_L^{\mathrm{per}}(z)$. This will be achieved by invoking Rouché's Theorem for the functions f and $f + g$, where g is defined by

$$g(z) = Z_L^{\mathrm{per}}(z) - f(z), \qquad z \in \mathscr{S}_{\kappa/L}(\{m,n\}). \tag{4.21}$$

To apply Rouché's Theorem, we will need that $|g(z)| < |f(z)|$ on boundaries of certain discs in $\mathscr{S}_{\kappa/L}(\{m,n\})$; this assumption will be verified by combining Lemma 4.4 (a lower bound on $|f(z)|$) with Lemma 4.5 (an upper bound on $|g(z)|$). The argument is then finished by applying Lemma 4.3 once again to conclude that any two distinct solutions of the equations (2.17–2.18), and thus also any two distinct roots of Z_L^{per}, are farther than a uniformly-positive constant times L^{-d}. The actual proof follows a slightly different path than indicated here in order to address certain technical details.

We begin by stating the aforementioned technical lemmas. The first lemma provides the necessary control over the distance between the solutions of (2.17–2.18) and those of the equation $f(z) = 0$. The function f is analytic and it thus makes sense to consider the multiplicity of the solutions. For that reason we will prefer to talk about the roots of the function f.

Lemma 4.3. *There exist finite, positive constants B_1, B_2, $\tilde{C}_1$ and L_1, satisfying the bounds $B_1 < B_2$ and $\tilde{C}_1 e^{-\tau L} < B_1 L^{-d}$ whenever $L \geq L_1$, such that for all $L \geq L_1$, all $s \leq (B_1 + B_2)L^{-d}$ and all $z_0 \in \mathscr{S}_{\kappa/(2L)}(\{m,n\})$ with $\mathbb{D}_s(z_0) \subset \mathscr{O}$, the disc $\mathbb{D}_s(z_0)$ is a subset of $\mathscr{S}_{\kappa/L}(\{m,n\})$ and the following statements hold:*

(1) If $s \leq B_1 L^{-d}$, then disc $\mathbb{D}_s(z_0)$ contains at most one solution of the equations (2.17–2.18) and at most one root of function f, which is therefore non-degenerate.
(2) If $s \geq \tilde{C}_1 e^{-\tau L}$ and if z_0 is a solution of the equations (2.17–2.18), then $\mathbb{D}_s(z_0)$ contains at least one root of f.
(3) If $s \geq \tilde{C}_1 e^{-\tau L}$ and if z_0 is a root of the function f, then $\mathbb{D}_s(z_0)$ contains at least one solution of the equations (2.17–2.18).
(4) If $s = B_2 L^{-d}$ and if both m and n are stable at z_0, then $\mathbb{D}_s(z_0)$ contains at least one solution of the equations (2.17–2.18).

The next two lemmas state bounds on $|f(z)|$ and $|g(z)|$ that will be needed to apply Rouché's Theorem. First we state a lower bound on $|f(z)|$:

Lemma 4.4. *There exist finite, positive constants $\tilde{c}_2$ and $\tilde{C}_2$ obeying $\tilde{c}_2 \leq \tilde{C}_2$ and, for any $\tilde{C} \geq \tilde{C}_2$ and any sequence (ϵ_L) of positive numbers satisfying*

$$\lim_{L\to\infty} L^d \epsilon_L = 0, \tag{4.22}$$

there exists a constant $L_2 < \infty$ such that for all $L \geq L_2$ the following is true: If z_0 is a point in $\mathscr{S}_{\kappa/(4L)}(\{m,n\}) \cap (\mathscr{S}_m \cup \mathscr{S}_n)$ and $\mathbb{D}_{\tilde{C}\epsilon_L}(z_0) \subset \mathscr{O}$, then there exists a number $s(z_0) \in \{\tilde{c}_2\epsilon_L, \tilde{C}_2\epsilon_L\}$ such that $\mathbb{D}_{s(z_0)}(z_0) \subset \mathscr{S}_{\kappa/(2L)}(\{m,n\})$ and

$$\liminf_{s\uparrow s(z_0)} \inf_{z:\, |z-z_0|=s} |f(z)| > \epsilon_L L^d \zeta(z_0)^{L^d}. \tag{4.23}$$

Moreover, if f has a root in $\mathbb{D}_{\tilde{c}_2\epsilon_L}(z_0)$, then $s(z_0)$ can be chosen as $s(z_0) = \tilde{C}\epsilon_L$.

The reasons why we write a limit in (4.23) will be seen in the proof of Theorem 2.3. At this point let us just say that we need to use Lemma 4.4 for the maximal choice $s(z_0) = \tilde{C}\epsilon_L$ in the cases when we know that $\mathbb{D}_{\tilde{C}\epsilon_L}(z_0) \subset \mathscr{O}$ but do not know the same about the closure of $\mathbb{D}_{\tilde{C}\epsilon_L}(z_0)$. In light of continuity of $z \mapsto |f(z)|$, once $s(z_0) < \tilde{C}\epsilon_L$, the limit is totally superfluous.

Now we proceed to state a corresponding upper bound on $|g(z)|$:

Lemma 4.5. *There exists a constant* $A_3 \in (0,\infty)$ *and, for each* $C \in (0,\infty)$ *and any sequence* γ_L *obeying the assumptions (2.19), there exists a number* $L_3 < \infty$ *such that*

$$\sup_{z\colon |z-z_0|<C\delta_L(z_0)} |g(z)| \le A_3\delta_L(z_0)L^d\zeta(z_0)^{L^d} \tag{4.24}$$

holds for any $L \ge L_3$ *and any* $z_0 \in \mathscr{U}_{\gamma_L}$ *with* $\mathbb{D}_{C\delta_L(z_0)}(z_0) \subset \mathscr{O}$.

With Lemmas 4.4–4.5 in hand, the proof of Theorem 2.3 is rather straightforward.

Proof of Theorem 2.3. Let m and n be distinct indices from $\mathcal{R}$ and let us abbreviate $\mathscr{U}_{\gamma_L} = \mathscr{U}_{\gamma_L}(\{m,n\})$ and $\mathscr{S}_\epsilon = \mathscr{S}_\epsilon(\{m,n\})$. Let $f(z)$ and $g(z)$ be the functions from (4.20–4.21). Let B_1, B_2, $\tilde{C}_1$, $\tilde{c}_2$, $\tilde{C}_2$ and A_3 be the constants whose existence is guaranteed by Lemmas 4.3-4.5 and let L_1 be as in Lemma 4.3. Since A_3 appears on the right-hand side of an upper bound, without loss of generality we can assume that

$$\tilde{c}_2A_3 \ge \tilde{C}_1. \tag{4.25}$$

Further, let us choose the constants C and D such that

$$C = \tilde{C}_1 + \tilde{C}_2A_3, \quad \text{and} \quad D = B_1 + B_2. \tag{4.26}$$

Next, let L_2 be the constant for which Lemma 4.4 holds for both $\tilde{C} = \tilde{C}_2$ and $\tilde{C} = C/A_3$ and for both $\epsilon_L = A_3e^{-\tau L}$ and $\epsilon_L = A_3L^de^{-\frac{1}{2}\gamma_L L^d}$. Finally, let L_3 be the constant for which Lemma 4.5 holds with C as defined above.

The statement of Theorem 2.3 involves two additional constants chosen as follows: First, a constant B for which we pick a number from $(0, \frac{2}{\sqrt{3}}B_1)$ (e.g, $B_1/3$ will do). Second, a constant L_0 which we choose such that $L_0 \ge \max\{L_1, L_2, L_3\}$ and that the bounds

$$\gamma_L \le \frac{\kappa}{4L}, \quad e^{-\tau L} \le L^de^{-\frac{1}{2}\gamma_L L^d}, \quad CL^de^{-\frac{1}{2}\gamma_L L^d} + \tilde{C}_1e^{-\tau L} \le \frac{\sqrt{3}-1}{2}BL^{-d} \tag{4.27}$$

hold true for all $L \ge L_0$. Fix $L \ge L_0$ and consider the set

$$\mathscr{U} = \bigl\{z_0 \in \mathscr{U}_{\gamma_L} : \mathbb{D}_{C\delta_L(z_0)}(z_0) \subset \mathscr{O}\bigr\}. \tag{4.28}$$

Notice that our choice of L_0 guarantees that $\mathscr{U} \subset \mathscr{U}_{\gamma_L} \subset \mathscr{S}_{\kappa/(4L)} \cap (\mathscr{S}_m \cup \mathscr{S}_n)$, while the fact that $\tilde{C} \le C/A_3$ for both choices of $\tilde{C}$ above ensures that for any $z_0 \in \mathscr{U}$, the disc $\mathbb{D}_{\tilde{C}A_3\delta(z_0)}(z_0)$ is contained in $\mathscr{O}$. These observations verify the assumptions of Lemma 4.4—with $\epsilon_L = A_3\delta_L(z_0)$ and $\tilde{C}$ equal to either $\tilde{C}_2$ or C/A_3—as well as of Lemma 4.5, for any $z_0 \in \mathscr{U}$.

First, we will attend to the proof of claim (2). Let $z_0 \in \Omega_L^\star \cap \mathscr{U}$ be a root of $Z_L^{\text{per}} = f + g$. Lemma 4.4 with $\tilde{C} = \tilde{C}_2$ and $\epsilon_L = A_3\delta_L(z_0)$ and Lemma 4.5 then imply the existence of a radius $s(z_0)$ with $s(z_0) \le \tilde{C}_2\epsilon_L = \tilde{C}_2 A_3 \delta_L(z_0) < C\delta_L(z_0)$ such that

$$\big|f(z)\big| > \big|g(z)\big|, \qquad z \in \partial\mathbb{D}_s(z_0) \tag{4.29}$$

holds for $s = s(z_0)$. (Note that here the limit in (4.23) can be omitted.) Hence, by Rouché's Theorem, f and $f+g$ have an equal number of roots in $\mathbb{D}_{s(z_0)}(z_0)$, including multiplicity. In particular, the function f has a root z_1 in $\mathbb{D}_{s(z_0)}(z_0)$ which by Lemma 4.4 lies also in $\mathscr{S}_{\kappa/(2L)}$. Since $s(z_0) + \tilde{C}_1 e^{-\tau L} \le C\delta_L(z_0)$ by the definition of C and the second bound in (4.27), we may use Lemma 4.3(3) to infer that the equations (2.17–2.18) have a solution $z \in \mathbb{D}_{\tilde{C}_1 e^{-\tau L}}(z_1) \subset \mathbb{D}_{C\delta_L(z_0)}(z_0)$. Moreover, (4.27) implies that $C\delta_L(z_0) \le B_1 L^{-d}$ so by Lemma 4.3(1) there is only one such solution in the entire disc $\mathbb{D}_{C\delta_L(z_0)}(z_0)$.

Next, we will prove claim (3). Let $z_0 \in \boldsymbol{\Omega}_L(\mathcal{Q}) \cap \mathscr{U}$ be a solution to the equations (2.17–2.18). By Lemma 4.3(2), there exists a root $z_1 \in \mathbb{D}_{\tilde{C}_1 e^{-\tau L}}(z_0) \subset \mathbb{D}_{C\delta_L(z_0)}(z_0)$ of the function f. Lemma 4.3(1) then shows that z_1 is in fact the only root of f in $\mathbb{D}_{C\delta_L(z_0)}(z_0)$. Applying Lemma 4.4 for the point z_0 and the choices $\epsilon_L = A_3\delta_L(z_0)$ and $\tilde{C} = C/A_3$ in conjunction with Lemma 4.5, there exists a radius $s(z_0)$ such that (4.29) holds true for any $s < s(z_0)$ sufficiently near $s(z_0)$. Moreover, by the bound (4.25) we know that $z_1 \in \mathbb{D}_{\tilde{C}_1 e^{-\tau L}}(z_0) \subset \mathbb{D}_{\tilde{c}_2\epsilon_L}(z_0)$ is a root of f within distance $\tilde{c}_2\epsilon_L$ from z_0, and so the last clause of Lemma 4.4 allows us to choose $s(z_0) = C\delta_L(z_0)$. Let $s_0 < s(z_0)$ be such that (4.29) holds for $s \in (s_0, s(z_0))$ and pick an $s \in (s_0, s(z_0))$. Rouché's Theorem for the discs $\mathbb{D}_s(z_0)$ and the fact that f has only one root in $\mathbb{D}_{C\delta_L(z_0)}(z_0)$ imply the existence of a unique zero z of $f(z) + g(z) = Z_L^{\text{per}}(z)$ in $\mathbb{D}_s(z_0)$. The proof is finished by taking the limit $s \uparrow C\delta_L(z_0)$.

Further, we will pass to claim (4). Let z_1 and z_2 be two distinct roots of Z_L^{per} in $\mathscr{U}_{\gamma_L}$ such that both $\mathbb{D}_{BL^{-d}}(z_1) \subset \mathscr{O}$ and $\mathbb{D}_{BL^{-d}}(z_2) \subset \mathscr{O}$ are satisfied. We will suppose that $|z_1 - z_2| < BL^{-d}$ and derive a contradiction. Let $z = \frac{1}{2}(z_1 + z_2)$ be the middle point of the segment between z_1 and z_2. Since $|z_1 - z_2| < BL^{-d}$, a simple geometrical argument shows that the disc of radius $s = \frac{\sqrt{3}}{2} BL^{-d}$ centered at z is entirely contained in $\mathbb{D}_{BL^{-d}}(z_1) \cup \mathbb{D}_{BL^{-d}}(z_2) \subset \mathscr{O}$. Next, by Lemmas 4.4-4.5, there exist two roots z_1' and z_2' of f such that $z_1' \in \mathbb{D}_{C\delta(z_1)}(z_1)$ and $z_2' \in \mathbb{D}_{C\delta(z_2)}(z_2)$. (We may have that $z_1 = z_2$, in which case $z_1 = z_2$ would be a degenerate root of f.) Now our assumptions on B and L_0 imply that

$$\frac{\sqrt{3}}{2} BL^{-d} \ge \frac{B}{2} L^{-d} + C\delta_L(z_1) \ge |z - z_1| + |z_1 - z_1'| \ge |z - z_1'|, \tag{4.30}$$

and similarly for z_2'. Consequently, both z_1' and z_2' lie in $\mathbb{D}_s(z)$. But this contradicts Lemma 4.3 and the bound $\frac{\sqrt{3}}{2} B < B_1$, implying that $\mathbb{D}_s(z_0)$ contains at most one non-degenerate root of f.

Finally, we will prove claim (1). Let $z_0 \in \mathscr{G} \cap \mathscr{U}_{\gamma_L}(\mathcal{Q})$ with $\mathbb{D}_{DL^{-d}}(z) \subset \mathscr{O}$. According to Lemma 4.3(4), the disc $\mathbb{D}_{B_2 L^{-d}}(z)$ contains at least one one solution z_1 of the equations (2.17–2.18). Checking that $B_2 L^{-d} + C\delta_L(z_1) \le (B_2 + B_1)L^{-d}$ in view of (4.27) and the definition of B, we know that $\mathbb{D}_{C\delta(z_1)}(z_1) \subset \mathscr{O}$ and we can use already proven claim (3) to get the existence of a root of Z_L^{per} in $\mathbb{D}_{C\delta_L(z_1)}(z_1)$, which is a subset of $\mathbb{D}_{DL^{-d}}(z_0)$. $\square$

This concludes the proof of Theorem 2.3 subject to the validity of Lemmas 4.3-4.5. The proofs of these lemmas have been deferred to Sect. 5.2.

4.3. Proof of Proposition 2.4. Fix distinct indices $m, n \in \mathcal{R}$. Our strategy is to first prove the claim for the density of the solutions of the equations (2.17–2.18),

$$\tilde{\rho}^{(L,\epsilon)}_{m,n}(z) = \frac{1}{2\epsilon L^d}\big|\Omega_L(\{m,n\}) \cap \mathbb{D}_\epsilon(z)\big|, \tag{4.31}$$

and then to argue that the density $\rho^{(L,\epsilon)}_{m,n}$ yields the same limit.

Let $z_0 \in \mathscr{G}(\{m,n\}) \setminus \mathscr{M}$, where $\mathscr{M}$ is the set of all multiple points. By Theorem 2.1 and Assumptions A1-A2, there exists an $\epsilon > 0$ such that, throughout the disc $\mathbb{D}_\epsilon = \mathbb{D}_\epsilon(z_0) \subset \mathscr{O}$, we have $Q(z) \subset \{m,n\}$ and the the function $F_{m,n}(z) = \zeta_m(z)/\zeta_n(z)$ is twice continuously differentiable and nonvanishing. Clearly, all solutions of the equations (2.17–2.18) in $\mathbb{D}_\epsilon$ must lie in the set

$$\mathscr{G}^{(L)} = \big\{z \in \mathbb{D}_\epsilon : |F_{m,n}(z)| = (q_n/q_m)^{1/L^d}\big\}. \tag{4.32}$$

Denoting the set $\mathscr{G}(\{m,n\} \cap \mathbb{D}_\epsilon$ by $\mathscr{G}^{(\infty)}$, we now claim that for sufficiently small ϵ, the sets $\mathscr{G}^{(\infty)}$ and $\mathscr{G}^{(L)}$ can be viewed as differentiable parametric curves $\gamma\colon (t_-, t_+) \to \mathbb{D}_\epsilon$ and $\gamma^{(L)}\colon (t_-^{(L)}, t_+^{(L)}) \to \mathbb{D}_\epsilon$ for which

(1) $t_-^{(L)} \to t_-$ and $t_+^{(L)} \to t_+$
(2) $\gamma^{(L)} \to \gamma$ uniformly on $\in (t_-, t_+)$
(3) $\hat{\mathrm{v}}_L \to \hat{\mathrm{v}}$ uniformly on (t_-, t_+)

hold true as $L \to \infty$. Here $\hat{\mathrm{v}}_L(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma^{(L)}(t)$ and $\hat{\mathrm{v}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$ denote the tangent vectors to $\gamma^{(L)}$ and γ, respectively.

We will construct both curves as solutions to the differential equation

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \mathrm{i}\frac{\partial_z\phi_{m,n}(z(t))}{|\partial_z\phi_{m,n}(z(t))|} \tag{4.33}$$

with $\phi_{m,n}(z) = \log|F_{m,n}(z)|$ (note that for ϵ small enough, the right hand side is a well defined, continuously differentiable function of $z(t) \in \mathbb{D}_\epsilon$ by Assumptions A1-A2 and the fact that $|\partial_z\phi_{m,n}(z_0)| \ge \alpha/2$ according to Assumption A3). In order to define the curves $\gamma^{(L)}(\cdot)$ and $\gamma(\cdot)$ we will choose a suitable starting point at $t = 0$. For $\gamma(\cdot)$, this will just be the point z_0, while for $\gamma^{(L)}(\cdot)$ we will choose a point $z_0^{(L)} \in \mathbb{D}_\epsilon$ which obeys the conditions $\phi_{m,n}(z_0^{(L)}) = \eta_L$ and $|z_0 - z_0^{(L)}| \le 3\alpha^{-1}\eta_L$, where $\eta_L = L^{-d}\log(q_n/q_m)$. To construct the point $z_0^{(L)} \in \mathbb{D}_\epsilon$, we use again the smoothness of $\phi_{m,n}$. Namely, by Assumption A1-2, the function $\phi_{m,n}(x+\mathrm{i}y) = \log|F_{m,n}(x+\mathrm{i}y)|$ is twice continuously differentiable on $\mathbb{D}_\epsilon$ if ϵ is sufficiently small, and by Assumption A3 we either have $|\partial\phi_{m,n}(x+\mathrm{i}y)/\partial x| \ge \alpha/3$, or $|\partial\phi_{m,n}(x+\mathrm{i}y)/\partial y| \ge \alpha/3$. Assuming, without loss of generality, that $|\partial\phi_{m,n}(x+\mathrm{i}y)/\partial y| \ge \alpha/3$ on all of $\mathbb{D}_\epsilon$, we then define $z_0^{(L)}$ to be the unique point for which $\mathrm{Re} z_0^{(L)} = \mathrm{Re} z_0$ and $\phi_{m,n}(z_0^{(L)}) = \eta_L$. By the assumption $|\partial\phi_{m,n}(x+\mathrm{i}y)/\partial y| \ge \alpha/3$, we then have $|z_0 - z_0^{(L)}| \le 3\alpha^{-1}\eta_L$, as desired.

Having chosen $z_0^{(L)}$, the desired curves $\gamma^{(L)}\colon (t_-^{(L)}, t_+^{(L)}) \to \mathbb{D}_\epsilon$ and $\gamma\colon (t_-, t_+) \to \mathbb{D}_\epsilon$ are obtained as the solutions of the equation (4.33) with initial condition $\gamma^{(L)}(0) = z_0^{(L)}$ and $\gamma(0) = z_0$, respectively. Here $t_-^{(L)}$, $t_+^{(L)}$, t_-, and t_+ are determined by the condition that $t_-^{(L)}$ and t_- are the largest values $t < 0$ for which $\gamma^{(L)}(t) \in \partial\mathbb{D}_\epsilon$ and $\gamma(t) \in \partial\mathbb{D}_\epsilon$, respectively, and $t_+^{(L)}$ and t_+ are the smallest values $t > 0$ for which

$\gamma^{(L)}(t) \in \partial\mathbb{D}_\epsilon$ and $\gamma(t) \in \partial\mathbb{D}_\epsilon$, respectively. Since the right-hand side of (4.33) has modulus one, both curves are parametrized by the arc-length. Moreover, decreasing ϵ if necessary, the functions $\gamma^{(L)}$ can be extended to all $t \in (t_-, t_+)$. To see that the limits in (1-3) above hold, we just refer to the Lipschitz continuity of the right hand side of (4.33) and the fact that, by definition, $|\gamma^{(L)}(0) - \gamma(0)| = O(L^{-d})$. Let K be the Lipschitz constant of the right-hand side of (4.33) in a neighborhood containing $\gamma^{(L)}(t)$ for all $t \in (t_-, t_+)$. Choosing ϵ so small that both $t_+ - t_-$ and $t_+^{(L)} - t_-^{(L)}$ are less than, say, $1/(2K)$, integrating (4.33) and invoking the Lipschitz continuity, we get

$$\sup_{t_-<t<t_+} |\gamma^{(L)}(t) - \gamma(t)| \le |\gamma^{(L)}(0) - \gamma(0)| + \tfrac{1}{2} \sup_{t_-<t<t_+} |\gamma^{(L)}(t) - \gamma(t)|. \tag{4.34}$$

This shows that $\gamma^{(L)}(t) \to \gamma(t)$ uniformly in $t \in (t_-, t_+)$. Using Lipschitz continuity once more, we get a similar bound on the derivatives. But then also the arc-lengths corresponding to $\gamma^{(L)}$ must converge to the arc-length of γ, which shows that also $t_+^{(L)} \to t_+$ and $t_-^{(L)} \to t_-$.

Consider now the curve $\gamma(t)$. Given that $|F_{m,n}(z)|$ is constant along γ, we have

$$\frac{\mathrm{d}\,\mathrm{Arg}\, F_{m,n}(\gamma(t))}{\mathrm{d}t} = \frac{1}{\mathrm{i}} \frac{\mathrm{d} \log F_{m,n}(\gamma(t))}{\mathrm{d}t} = -\mathrm{i}\partial_z \log F_{m,n}(z)\big|_{z=\gamma(t)} \hat{\mathrm{v}}(t). \tag{4.35}$$

Referring to Assumption A3 and the fact that $|\hat{\mathrm{v}}(t)| = 1$, we find that the modulus of the left-hand side is bounded below by α. Using continuity of the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Arg}\, F_{m,n}$ in $\mathbb{D}_\epsilon$, we observe that one of the two alternatives occurs on all the interval $(t_-^{(L)}, t_+^{(L)})$:

$$\text{either} \qquad \frac{\mathrm{d}\,\mathrm{Arg}\, F_{m,n}(\gamma^{(L)}(t))}{\mathrm{d}t} \ge \frac{\alpha}{2} \qquad \text{or} \qquad \frac{\mathrm{d}\,\mathrm{Arg}\, F_{m,n}(\gamma^{(L)}(t))}{\mathrm{d}t} \le -\frac{\alpha}{2}, \tag{4.36}$$

provided ϵ is sufficiently small. By Lemma 4.3, the disc $\mathbb{D}_\epsilon$ contains a finite number $k = 2\epsilon L^d \tilde{\rho}_{m,n}^{(L,\epsilon)}(z_0)$ of solutions of the equations (2.17) and (2.18) which in the present notation read

$$|F_{m,n}(z)| = \Bigl(\frac{q_n}{q_m}\Bigr)^{1/L^d}, \tag{4.37}$$

$$L^d\, \mathrm{Arg}\, F_{m,n}(z) = \pi \bmod 2\pi. \tag{4.38}$$

Assuming, without loss of generality, that the former alternative in (4.36) takes place, and ordering all the solutions consecutively along the curve $\gamma^{(L)}$, i.e., letting $z_1 = \gamma^{(L)}(t_1),\dots, z_k = \gamma^{(L)}(t_k)$ where $t_-^{(L)} \le t_1 < \cdots < t_k \le t_+^{(L)}$, we have

$$\mathrm{Arg}\, F_{m,n}(z_{j+1}) - \mathrm{Arg}\, F_{m,n}(z_j) = 2\pi L^{-d} \tag{4.39}$$

for any $j = 1, \dots, k-1$, as well as

$$\mathrm{Arg}\, F_{m,n}(z_1) - \mathrm{Arg}\, F_{m,n}(z_-) \le 2\pi L^{-d} \tag{4.40}$$

and

$$\mathrm{Arg}\, F_{m,n}(z_+) - \mathrm{Arg}\, F_{m,n}(z_k) \le 2\pi L^{-d}. \tag{4.41}$$

In view of the first equality in (4.35) rephrased for $\gamma^{(L)}$, the left hand side of (4.39) can be rewritten as

$$\operatorname{Arg} F_{m,n}(z_{j+1}) - \operatorname{Arg} F_{m,n}(z_j) = \int_{t_j}^{t_{j+1}} \Bigl|\frac{\mathrm{d} \log F_{m,n}(\gamma^{(L)}(t))}{\mathrm{d}t}\Bigr| \mathrm{d}t \tag{4.42}$$

and thus

$$\Bigl|\int_{t_-^{(L)}}^{t_+^{(L)}} \Bigl|\frac{\mathrm{d} \log F_{m,n}(\gamma^{(L)}(t))}{\mathrm{d}t}\Bigr| \mathrm{d}t - 2k\pi L^{-d}\Bigr| \le 2\pi L^{-d}. \tag{4.43}$$

Let us divide the whole expression by L^d and take the limit $L \to \infty$. Now $\gamma^{(L)}$ converge to γ along with their first derivatives, uniformly in $t \in (t_-, t_+)$, and the limits $t_\pm^{(L)}$ converge to $t_\pm$. The Bounded Convergence Theorem then shows that the integral in (4.43) converges to a corresponding integral over γ. Recalling that $\tilde{\rho}_{m,n}^{(L,\epsilon)}(z_0) = k/(2\epsilon L^d)$, we thus get

$$\begin{aligned} \lim_{L\to\infty} \tilde{\rho}_{m,n}^{(L,\epsilon)}(z_0) &= \frac{1}{4\pi\epsilon} \int_{t_-}^{t_+} \Bigl|\frac{\mathrm{d} \log F_{m,n}(\gamma_0(t))}{\mathrm{d}t}\Bigr| \mathrm{d}t \\ &= \frac{1}{4\pi\epsilon} \int_{\gamma_0} \bigl|\partial_z \log F_{m,n}(z)\bigr| |dz| \end{aligned} \tag{4.44}$$

where the last integral denotes the integration with respect to the arc length. Taking into account the Lipschitz continuity of $|\partial_z \log F_{m,n}(z)|$, the last integral in (4.44) can be approximated by $\bigl(\bigl|\partial_z \log F_{m,n}(z_0)\bigr| + O(\epsilon)\bigr)|\gamma|$. By the smoothness of the curve γ, we estimate its length by $|\gamma| = 2\epsilon\bigl(1 + O(\epsilon)\bigr)$, so that

$$\lim_{\epsilon\downarrow 0} \lim_{L\to\infty} \tilde{\rho}_{m,n}^{(L,\epsilon)}(z_0) = \frac{1}{2\pi}\bigl|\partial_z \log F_{m,n}(z_0)\bigr| = \frac{1}{2\pi}\Bigl|\frac{\partial_z \zeta_m(z_0)}{\zeta_m(z_0)} - \frac{\partial_z \zeta_n(z_0)}{\zeta_n(z_0)}\Bigr|. \tag{4.45}$$

To finish the proof, we need to show that $\rho_{m,n}^{(L,\epsilon)}(z_0)$ will converge to the same limit. According to Theorem 2.3, we have

$$\bigl||\Omega_L^* \cap \mathbb{D}_\epsilon(z)| - |\Omega_L(\{m,n\}) \cap \mathbb{D}_\epsilon(z)|\bigr| \le 2 \tag{4.46}$$

for all $z \in \mathscr{G}(m,n)$ such that $|\mathcal{Q}(z)| = 2$ and ϵ sufficiently small. Hence

$$\bigl|\rho_{m,n}^{(L,\epsilon)}(z) - \tilde{\rho}_{m,n}^{(L,\epsilon)}(z)\bigr| \le \frac{1}{\epsilon L^d}, \tag{4.47}$$

and the claim of the proposition follows by (4.45). □

4.4. Multiple phase coexistence. In this section we will prove Theorem 2.5, which deals with the zeros of Z_L^{per} in the vicinity of multiple points. Let $z_{\mathrm{M}} \in \mathscr{O}$ be a multiple point and let $\mathcal{Q} = \mathcal{Q}(z_{\mathrm{M}})$. For each $m \in \mathcal{Q}$, let $\phi_m(L)$ and v_m be as in (2.23). Define the functions

$$\tilde{f}(z) = \sum_{m\in\mathcal{Q}} q_m\, e^{\mathrm{i}\phi_m(L) + v_m(z - z_{\mathrm{M}})L^d}, \tag{4.48}$$

$$\tilde{g}(z) = Z_L^{\text{per}}(z)\zeta(z_{\mathrm{M}})^{-L^d} - f(z), \tag{4.49}$$

and

$$\xi(z) = \exp\Bigl\{\max_{m\in\mathcal{Q}} \operatorname{Re}(\upsilon_m(z - z_{\mathrm{M}}))\Bigr\}. \tag{4.50}$$

As in the case of two-phase coexistence, the proof uses Rouché's Theorem for the functions $\tilde{f}$ and $\tilde{f} + \tilde{g}$. For this we will need a lower bound on $|\tilde{f}|$ and an upper bound on $|\tilde{g}|$.

Lemma 4.6. *Suppose Assumptions A and B hold. Given $\mathcal{Q} \subset \mathcal{R}$ with $|\mathcal{Q}| \geq 3$ and abbreviating $q = |\mathcal{Q}|$ and $R_L = L^{-d(1+1/q)}$, let (ϵ_L) be a sequence of positive numbers such that*

$$\lim_{L\to\infty} L^{2d}\epsilon_L = \infty \qquad \textit{but} \qquad \lim_{L\to\infty} L^{2d-d/q}\epsilon_L = 0. \tag{4.51}$$

Then there is a constant $L_5 < \infty$ such that for any $z_0 \in \mathbb{C}$ and any $L \geq L_5$ there exists $s(z_0) \in [R_L/q, R_L]$ for which the bound

$$\inf_{z\colon |z-z_0|=s(z_0)} \bigl|f(z)\bigr| > L^d \epsilon_L\, \xi(z_0)^{L^d} \tag{4.52}$$

holds.

Lemma 4.7. *Let $z_{\mathrm{M}} \in \mathscr{O}$ be a multiple point, let $\mathcal{Q} = \mathcal{Q}(z_{\mathrm{M}})$, $q = |\mathcal{Q}|$, and $R_L = L^{-d(1+1/q)}$. There exists a constant $A_6 \in (0,\infty)$ and, for each sequence (ρ_L) of positive numbers obeying (2.25), a number $L_6 < \infty$ such that if $L \geq L_6$ then $\mathbb{D}_{\rho_L'}(z_{\mathrm{M}}) \subset \mathscr{U}_{\kappa/L}(\mathcal{Q})$, where $\rho_L' = \rho_L + R_L$. Furthermore, we have*

$$\sup_{z\colon |z-z_0|\leq R_L} \bigl|\tilde{g}(z)\bigr| \leq A_6 \rho_L^2 L^d \xi(z_0)^{L^d} \tag{4.53}$$

whenever $z_0 \in \mathbb{D}_{\rho_L}(z_{\mathrm{M}})$.

With these two lemmas we can proceed directly to the proof of Theorem 2.5.

Proof of Theorem 2.5. The proof is close in spirit to the proof of Theorem 2.3. Let z_{M} be a multiple point and let $\mathcal{Q} = \mathcal{Q}(z_{\mathrm{M}})$. Consider a sequence (ρ_L) of positive numbers such that (2.25) holds. Choosing $\epsilon_L = A_6\rho_L^2$, where A_6 is the constant from Lemma 4.7, we note that the conditions (4.51) are satisfied due to our conditions on ρ_L from (2.25). We will then prove Theorem 2.5 with $L_0 = \max\{L_5, L_6\}$, where L_5 and L_6 are the constants from Lemma 4.6 and 4.7, respectively. The proof again boils down to a straightforward application of Rouché's Theorem.

Indeed, let $L \geq L_0$ and note that by Lemmas 4.6 and 4.7, for each $z_0 \in \mathbb{D}_{\rho_L}(z_{\mathrm{M}})$ there is an $s(z_0) \in [R_L/q, R_L]$ such that on $\mathbb{D}_{s(z_0)}(z_0)$, we have

$$\bigl|\tilde{f}(z)\bigr| > \bigl|\tilde{g}(z)\bigr|. \tag{4.54}$$

Consider the set of these discs $\mathbb{D}_{s(z_0)}(z_0)$—one for every $z_0 \in \mathbb{D}_{\rho_L}(z_{\mathrm{M}})$. These discs cover the closure of $\mathbb{D}_{\rho_L}(z_{\mathrm{M}})$, so we can choose a finite subcover $\mathcal{S}$. Next we note that (4.54) implies that neither $\tilde{f}$ nor $\tilde{f} + \tilde{g}$ have more than finitely many zeros in $\mathbb{D}_{\rho_L}(z_{\mathrm{M}})$ (otherwise, one of these functions would be identically zero). Without loss of generality, we can thus assume that the discs centered at the zeros of $\tilde{f}$ and $\tilde{f} + \tilde{g}$ in $\mathbb{D}_{\rho_L}(z_{\mathrm{M}})$ are included in $\mathcal{S}$. Defining $\mathscr{U} = \bigcup_{\mathbb{D}\in\mathcal{S}} \mathbb{D}$, we clearly have $\mathbb{D}_{\rho_L}(z_{\mathrm{M}}) \subset \mathscr{U} \subset \mathbb{D}_{\rho_L'}(z_{\mathrm{M}})$.

Let now $\mathcal{K}$ be the set of all components of $\mathscr{U} \setminus \bigcup_{\mathbb{D}\in\mathcal{S}} \partial\mathbb{D}$. Let $\mathscr{K} \in \mathcal{K}$ be one such component. By (4.54) we know that $|\tilde{f}(z)| > |\tilde{g}(z)|$ on the boundary of $\mathscr{K}$ and

Rouché's Theorem then guarantees that $\tilde f$ has as many zeros in $\mathscr{K}$ as $\tilde f + \tilde g$, provided we count multiplicity correctly. Moreover, both functions $\tilde f$ or $\tilde f + \tilde g$ have no zeros on $\bigcup_{\mathbb{D}\in\mathcal{S}} \partial\mathbb{D}$. Since $\tilde f(z)+\tilde g(z) = Z_L^{\mathrm{per}}(z)\zeta(z_{\mathrm{M}})^{-L^d}$ and $\zeta(z_{\mathrm{M}})^{-L^d} > 0$, the zeros of $\tilde f+\tilde g$ are exactly those of Z_L^{per}. The above construction of $\mathscr{U}$ and $\mathcal{S}$ then directly implies the desired correspondence of the zeros. Namely, in each $\mathscr{K} \in \mathcal{K}$, both $\tilde f$ and Z_L^{per} have the same (finite) number of zeros, which can therefore be assigned to each other. Now $\tilde f$ and Z_L^{per} have no zeros in $\mathscr{U} \setminus \bigcup_{\mathscr{K}\in\mathcal{K}} \mathscr{K}$, so choosing one such assignment in each $\mathscr{K} \in \mathcal{K}$ extends into a one-to-one assignment of $\boldsymbol{\Omega}_L^\star \cap \mathscr{U}$ and $\boldsymbol{\Omega}_L(\mathcal{Q}) \cap \mathscr{U}$. Moreover, if $z \in \boldsymbol{\Omega}_L^\star \cap \mathscr{K}$ and $\tilde z \in \boldsymbol{\Omega}_L(\mathcal{Q}) \cap \mathscr{K}$ for some $\mathscr{K} \in \mathcal{K}$ (which is required if z and $\tilde z$ are the corresponding roots), then z belongs to the disc $\widetilde{\mathbb{D}} \in \mathcal{S}$ centered at $\tilde z$ and $\tilde z$ belongs to the disc $\mathbb{D} \in \mathcal{S}$ centered at z. Consequently, z and $\tilde z$ are not farther apart than $R_L = L^{-d(1+1/q)}$. This completes the proof. □

4.5. Proof of Proposition 2.6. Assuming that $L^{-d}\omega_L \le \gamma_L$, it clearly suffices to ascertain that

$$\bigcup_{\mathcal{Q}\colon |\mathcal{Q}|\ge 3} \mathscr{S}_{\gamma_L}(\mathcal{Q}) \cap \mathscr{D} \subset \bigcup_{z_{\mathrm{M}}\in\mathscr{D}\cap\mathscr{M}} \mathbb{D}_{\rho_L}(z_{\mathrm{M}}). \tag{4.55}$$

To this end let us first observe that continuity of the functions ζ_m implies

$$\lim_{L\to\infty} \mathscr{S}_{\gamma_L}(\mathcal{Q}) = \bigcap_{m\in\mathcal{Q}} \mathscr{S}_m \tag{4.56}$$

since $\gamma_L \to 0$. The set $\mathscr{D} \cap \mathscr{M}$ is finite according to Theorem 2.1. Hence, there exists a constant $\delta_0 > 0$ and, for each $\delta \in (0, \delta_0]$, a constant $L_0 = L_0(\delta)$, such that the discs $\mathbb{D}_\delta(z_{\mathrm{M}})$, $z_{\mathrm{M}} \in \mathscr{D} \cap \mathscr{M}$, are mutually disjoint,

$$\mathcal{Q}(z) \subset \mathcal{Q}(z_{\mathrm{M}}) \qquad \text{whenever} \qquad z \in \mathbb{D}_\delta(z_{\mathrm{M}}), \tag{4.57}$$

and

$$\bigcup_{\mathcal{Q}\colon |\mathcal{Q}|\ge 3} \mathscr{S}_{\gamma_L}(\mathcal{Q}) \cap \mathscr{D} \subset \bigcup_{z_{\mathrm{M}}\in\mathscr{D}\cap\mathscr{M}} \mathbb{D}_{\delta}(z_{\mathrm{M}}) \tag{4.58}$$

whenever $0 < \delta \le \delta_0$ and $L \ge L_0(\delta)$. It is therefore enough to show that there exist constants $\chi > 0$ and $\delta \in (0, \delta_0)$ such that for any multiple point $z_{\mathrm{M}} \in \mathscr{D}$, we have

$$\mathbb{D}_\delta(z_{\mathrm{M}}) \cap \mathscr{S}_{\gamma_L}(\mathcal{Q}(z_{\mathrm{M}})) \subset \mathbb{D}_{\rho_L}(z_{\mathrm{M}}) \tag{4.59}$$

once $\rho_L \ge \chi\gamma_L$ and $L \ge L_0(\delta)$.

We will prove (4.59) in two steps: First we will show that there is a constant $\chi > 0$ such that for any multiple point z_{M}, any $z \ne z_{\mathrm{M}}$, and any $n \in \mathcal{Q}(z_{\mathrm{M}})$, there exists $m \in \mathcal{Q}(z_{\mathrm{M}})$ for which

$$\mathrm{Re}\big[(z - z_{\mathrm{M}})(v_n(z_{\mathrm{M}}) - v_m(z_{\mathrm{M}}))\big] \ge 2\chi|z - z_{\mathrm{M}}|, \tag{4.60}$$

and then we will show that (4.60) implies (4.59). To prove (4.60), we first refer to the fact that we are dealing with a finite number of strictly convex polygons with vertices $\{v_k(z_{\mathrm{M}})\colon k \in \mathcal{Q}(z_{\mathrm{M}})\}$ according to Assumption A4 and thus, given z and n, the label m can be always chosen so that the angle between the complex numbers $z - z_{\mathrm{M}}$ and $v_n(z_{\mathrm{M}}) - v_m(z_{\mathrm{M}})$ is not smaller than a given fixed value. Combining this fact with the lower bound from Assumption A3, we get (4.60).

We are left with the proof of (4.59). Let us thus consider a multiple point $z_{\mathrm{M}} \in \mathscr{D}$ with $\mathcal{Q}(z_{\mathrm{M}}) = \mathcal{Q}$, and a point $z \in \mathbb{D}_\delta(z_{\mathrm{M}}) \setminus \mathbb{D}_{\rho_L}(z_{\mathrm{M}})$. We will have to show that there exists an $m \in \mathcal{Q}$ with $z \notin \mathscr{S}_{\gamma_L}(m)$. Recalling that $\mathcal{Q}(z') \subset \mathcal{Q}$ for all $z' \in \mathbb{D}_\delta(z_{\mathrm{M}})$, let $n \in \mathcal{Q}$ be such that $|\zeta_n(z)| = \zeta(z)$. Choosing $m \in \mathcal{Q}(z_{\mathrm{M}})$ so that (4.60) is satisfied and using, as in the proof of Lemma 3.1, $F_{n,m}(z)$ to denote the function $F_{n,m}(z) = \zeta_n(z)/\zeta_m(z)$, we apply, as in (3.9), the Taylor expansion to $\log|F_{n,m}(z)|$ to get

$$\begin{aligned} \log|F_{n,m}(z)| &= \mathrm{Re}\big[(z - z_{\mathrm{M}})(v_n(z_{\mathrm{M}}) - v_m(z_{\mathrm{M}}))\big] + O(|z - z_{\mathrm{M}}|^2) \\ &\geq \chi|z - z_{\mathrm{M}}| \geq \chi\rho_L. \end{aligned} \tag{4.61}$$

Here we also used that $|F_{n,m}(z_{\mathrm{M}})| = 1$ and assumed that δ was chosen small enough to guarantee that the error term is smaller than $\chi|z - z_{\mathrm{M}}|$. As a result, we get

$$|\zeta_m(z)| \leq e^{-\chi\rho_L}\zeta(z) \leq e^{-\gamma_L}\zeta(z) \tag{4.62}$$

implying that $z \notin \mathscr{S}_{\gamma_L}(m)$. Thus, the inclusion (4.59) is verified and (4.55) follows. □

5. Technical lemmas

The goal of this section is to provide the proofs of Lemmas 4.3-4.7. We will begin with some preparatory statements concerning Lipschitz continuity of the ζ_m and ζ.

5.1. Lipschitz properties of the functions $\log|\zeta_m|$ *and* $\log\zeta$. In this section, we prove two auxiliary lemmas needed for the proofs of our main theorems. For any $z_1, z_2 \in \mathbb{C}$, we will use $[z_1, z_2]$ to denote the closed segment

$$[z_1, z_2] = \big\{tz_1 + (1-t)z_2 \colon t \in [0,1]\big\}. \tag{5.1}$$

The following Lipschitz bounds are (more or less) a direct consequence of formulas (2.9) and (2.11) in Assumption B.

Lemma 5.1. *Suppose Assumptions A and B hold and let κ, τ, and M be as in Assumption B. Let $m \in \mathcal{R}$, and let $z_1, z_2 \in \mathscr{S}_{\kappa/L}(m)$ be such that $[z_1, z_2] \subset \mathscr{S}_{\kappa/L}(m)$. Then*

$$\left|\frac{\zeta_m(z_1)}{\zeta_m(z_2)}\right| \leq e^{2e^{-\tau L} + M|z_1 - z_2|}. \tag{5.2}$$

Moreover, for all $z_1, z_2 \in \mathscr{O}$ such that $[z_1, z_2] \subset \mathscr{O}$, we have

$$\frac{\zeta(z_1)}{\zeta(z_2)} \leq e^{M|z_1 - z_2|}. \tag{5.3}$$

Remark 7. Since $z \mapsto |\zeta_m(z)|$ are all twice continuously differentiable and hence Lipschitz throughout $\mathscr{O}$, so is their maximum $z \mapsto \zeta(z)$. The reason why we provide a (rather demanding) proof of (5.3) is that we need this bound to hold uniformly throughout $\mathscr{O}$ and the constant M from Assumption B(3) to appear explicitly on the right-hand side. The first part of the lemma underlines what is hard about the second part: On the basis of Assumption B, the uniform Lipchitz bound in (5.2) can be guaranteed only in the region where m is "almost stable."

Proof of Lemma 5.1. Let $[z_1, z_2] \subset \mathscr{S}_{\kappa/L}(m)$. The bound (5.2) is directly proved by combining (2.9) with the estimate

$$\bigl|\log|\zeta_m^{(L)}(z_1)| - \log|\zeta_m^{(L)}(z_2)|\bigr| \le M|z_1 - z_2|, \tag{5.4}$$

implied by (2.11). Indeed, introducing $\varphi(t) = \zeta_m^{(L)}(z_1 + t(z_2 - z_1))$, we have

$$\Bigl|\frac{\mathrm{d}}{\mathrm{d}t}\log|\varphi(t)|\Bigr| = \Bigl|\frac{1}{\varphi(t)}\frac{\mathrm{d}|\varphi(t)|}{\mathrm{d}t}\Bigr| \le \Bigl|\frac{1}{\varphi(t)}\Bigr|\Bigl|\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\Bigr| \le M|z_2 - z_1| \tag{5.5}$$

implying (5.4). By passing to the limit $L \to \infty$, we conclude that

$$\bigl|\log\zeta(z_1) - \log\zeta(z_2)\bigr| \le M|z_1 - z_2| \tag{5.6}$$

holds provided $[z_1, z_2] \subset \mathscr{S}_m$.

To prove (5.3), let $z_1, z_2 \in \mathscr{O}$ with $[z_1, z_2] \subset \mathscr{O}$. If the segment $[z_1, z_2]$ intersects the coexistence set $\mathscr{G}$ only in a finite number of points, then (5.3) is an easy consequence of (5.6). However, this may not always be the case and hence we need a more general argument. Note that continuity of both sides requires us to prove (5.3) only for a dense set of points z_1 and z_2. This and the fact that each compact subset of $\mathscr{O}$ contains only a finite number of multiple points from $\mathscr{M} = \{z \in \mathscr{O} \colon |\mathcal{Q}(z)| \ge 3\}$ permit us to assume that $z_1, z_2 \notin \mathscr{G}$ and that the segment $[z_1, z_2]$ does not contain a multiple point, i.e., $[z_1, z_2] \cap \mathscr{M} = \emptyset$.

Suppose now that the bound (5.3) fails. We claim that then there exist a point $\bar{x} \in [z_1, z_2]$, with $\bar{x} \neq z_1, z_2$, and two sequences (x_n) and (y_n) of points from $[z_1, \bar{x}] \cap \mathscr{G}$ and $[\bar{x}, z_2] \cap \mathscr{G}$, respectively, such that the following holds:

(1) $x_n \neq y_n$ for all n and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \bar{x}$.
(2) There exists a number $M' > M$ such that

$$\Bigl|\log\frac{\zeta(x_n)}{\zeta(y_n)}\Bigr| > M'|x_n - y_n| \tag{5.7}$$

for all n.

The proof of these facts will be simplified by introducing the *Lipschitz ratio,* which for any pair of distinct numbers $x, y \in [z_1, z_2]$ is defined by the formula

$$R(x, y) = \frac{|\log\zeta(x) - \log\zeta(y)|}{|x - y|}. \tag{5.8}$$

The significance of this quantity stems from its behavior under subdivisions of the interval. Namely, if x and y are distinct points and $z \in (x, y)$, then we have

$$R(x, y) \le \max\bigl\{R(x, z), R(z, y)\bigr\}, \tag{5.9}$$

with the inequality being strict unless $R(x, z) = R(z, y)$.

To prove the existence of sequences satisfying (1) and (2) above, we need a few observations: First, we note that $M' = R(z_1, z_2) > M$ from our assumption that (5.3) fails. Second, whenever $x, y \in [z_1, z_2]$ are such that $R(x, y) > M$, then (5.6) implies the existence of $x', y' \in [x, y]$ such that $x', y' \in \mathscr{G}$ and $R(x', y') \ge R(x, y)$. Indeed, we choose x' to be the nearest point to x from the closed set $[x, y] \cap \mathscr{G}$, and similarly for y'. The fact that the Lipschitz ratio increases in the process is a direct consequence of (5.9). Finally, if distinct $x, y \in [z_1, z_2] \cap \mathscr{G}$ satisfy $R(x, y) > M$, then there exists a pair of distinct points $x', y' \in [x, y] \cap \mathscr{G}$ such that $|x' - y'| \leq \frac{1}{2}|x - y|$ and $R(x', y') \geq R(x, y)$. To prove this we use (5.9) with $z = \frac{1}{2}(x + y)$ to choose the one of the segments $[x, z]$ or $[z, y]$ that has the Lipschitz ratio not smaller than $R(x, y)$ and then use the preceding observation on the chosen segment.

Equipped with these observations, we are ready to prove the existence of the desired sequences. Starting with the second observation above applied for $x = z_1$ and $y = z_2$, we get $x_1, x_2 \in [z_1, z_2] \cap \mathscr{G}$ such that $R(x_1, x_2) > M'$. Notice that $x_1 \neq z_1$ and $x_2 \neq z_2$ since $z_1, z_2 \notin \mathscr{G}$. Next, whenever the pair x_n, y_n is chosen, we use the third observation to construct the pair $x_{n+1}, y_{n+1} \in [x_n, y_n] \cap \mathscr{G}$ of points such that $|x_{n+1} - y_{n+1}| \leq \frac{1}{2}|x_n - y_n|$ and $R(x_{n+1}, y_{n+1}) \geq R(x_n, y_n) \geq M'$. Clearly, the sequences (x_n) and (y_n) converge to a common limit $\bar{x} \in [x_1, y_1]$, which is distinct from z_1 and z_2.

We will now show that (5.7) still leads to a contradiction with (5.3). First we note that the point $\bar{x}$, being a limit of points from $\mathscr{G} \setminus \mathscr{M}$, is a two-phase coexistence point and so Theorem 2.1(2) applies in a disc $\mathbb{D}_\epsilon(\bar{x})$ for $\epsilon > 0$ sufficiently small. Hence, there is a unique smooth coexistence curve $\mathscr{C}$ connecting $\bar{x}$ to the boundary of $\mathbb{D}_\epsilon(\bar{x})$ and, since (x_n) and (y_n) eventually lie on $\mathscr{C}$, its tangent vector at $\bar{x}$ is colinear with the segment $[z_1, z_2]$. Since in $\mathbb{D}_\epsilon(\bar{x})$, the coexistence curve is at least twice continuously differentiable, the tangent vector to $\mathscr{C}$ has a bounded derivative throughout $\mathbb{D}_\epsilon(\bar{x})$. As a consequence, in the disc $\mathbb{D}_\delta(\bar{x})$ with $\delta \leq \epsilon$, the curve $\mathscr{C}$ will not divert from the segment $[z_1, z_2]$ by more than $C\delta^2$, where $C = C(\epsilon) < \infty$.

Now we are ready to derive the anticipated contradiction: Fix n and let δ_n be the maximum of $|x_n - \bar{x}|$ and $|y_n - \bar{x}|$. Let $\hat{e}$ be a unit vector orthogonal to the segment $[z_1, z_2]$ and consider the shifted points $x'_n = x_n + 2C\delta_n^2\hat{e}$ and $y'_n = y_n + 2C\delta_n^2\hat{e}$. Then we can write

$$\frac{\zeta(x_n)}{\zeta(y_n)} = \frac{\zeta(x_n)}{\zeta(x'_n)}\frac{\zeta(x'_n)}{\zeta(y'_n)}\frac{\zeta(y'_n)}{\zeta(y_n)}. \tag{5.10}$$

Assuming that n is sufficiently large to ensure that $\delta_n\sqrt{1 + 4C^2\delta_n^2} \leq \epsilon$, the segment $[x'_n, y'_n]$ lies in $\mathbb{D}_\epsilon(\bar{x})$ entirely on one "side" of $\mathscr{C}$ and is thus contained in $\mathscr{S}_m$ for some $m \in \mathcal{R}$. On the other hand, given the bounded derivative of the tangent vector to $\mathscr{C}$, each segment $[x_n, x'_n]$ and $[y_n, y'_n]$ intersects the curve $\mathscr{C}$ exactly once, which in light of $x_n, y_n \in \mathscr{G}$ happens at the endpoint. This means that also $[x_n, x'_n] \subset \mathscr{S}_m$ and $[y_n, y'_n] \subset \mathscr{S}_m$ for the same m. Consequently, all three ratios can be estimated using (5.3), yielding

$$R(x_n, y_n) \leq M\,\frac{|x_n - x'_n| + |x'_n - y'_n| + |y'_n - y_n|}{|x_n - y_n|} \leq M + 4MC\delta_n, \tag{5.11}$$

where we used that $|x'_n - y'_n| = |x_n - y_n|$ and $|x_n - y_n| \geq \delta_n$. But $\delta_n \to 0$ with $n \to \infty$ and thus the ratio $R(x_n, y_n)$ is eventually strictly less than M', in contradiction with (5.7). Hence, (5.3) must have been true after all. □

The previous lemma will be particularly useful in terms of the following corollary.

Corollary 5.2. *Suppose that Assumptions A and B hold and let* $0 < \tilde{\kappa} \leq \kappa$*, where* κ *is the constant from Assumption B. Then there exist constants* $c < \infty$ *and* $L_4 < \infty$ *such that the following is true for all* $L \geq L_4$ *and all* $s \leq c/L$*:*

(1) For $m \in \mathcal{R}$ *and* $z \in \mathscr{S}_{\tilde{\kappa}/(2L)}(m)$ *with* $\mathbb{D}_s(z) \subset \mathscr{O}$*, we have*

$$\mathbb{D}_s(z) \subset \mathscr{S}_{\tilde{\kappa}/L}(m). \tag{5.12}$$

(2) For $z \in \mathscr{O}$ with $\mathbb{D}_s(z) \subset \mathscr{O}$, the set

$$\mathcal{Q}' = \bigl\{m \in \mathcal{R} \colon \mathbb{D}_s(z) \subset \mathscr{S}_{\tilde\kappa/L}(m)\bigr\} \tag{5.13}$$

in non-empty and

$$\mathbb{D}_s(z) \subset \mathscr{U}_{\tilde\kappa/L}(\mathcal{Q}'). \tag{5.14}$$

(3) For $\gamma_L \le \tilde\kappa/(2L)$, $\mathcal{Q} \subset \mathcal{R}$ and $z \in \mathscr{U}_{\gamma_L}(\mathcal{Q}) \cap \mathscr{U}_{2\tilde\kappa/L}(\mathcal{Q})$ with $\mathbb{D}_s(z) \subset \mathscr{O}$, we have

$$\mathbb{D}_s(z) \subset \mathscr{U}_{\tilde\kappa/L}(\mathcal{Q}). \tag{5.15}$$

Proof. Let M be as in Assumption B. We then choose $c > 0$ sufficiently small and $L_4 < \infty$ sufficiently large to ensure that for $L \ge L_4$ we have

$$\frac{\tilde\kappa}{8M} - \frac{1}{M} L e^{-\tau L} \ge 2c. \tag{5.16}$$

First, we will show that the claims (1), (2), and (3) above reduce to the following statement valid for each $m \in \mathcal{R}$: If z, z' are complex numbers such that the bound $|z - z'| \le 2c/L$, the inclusion $[z, z') \subset \mathscr{O}$, and $z \in \mathscr{O} \setminus \mathscr{S}_{\tilde\kappa/L}(m)$ hold, then also

$$[z, z') \subset \mathscr{O} \setminus \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)}. \tag{5.17}$$

We proceed with the proof of (1-3) given this claim; the inclusion (5.17) will be established at the end of this proof.

Ad (1): Let $z \in \mathscr{S}_{\tilde\kappa/(2L)}$ with $\mathbb{D}_s(z) \subset \mathscr{O}$ and assume that (5.12) fails. Then there exist some $z' \in \mathscr{O} \setminus \mathscr{S}_{\tilde\kappa/L}(m)$ with $|z - z'| < s$ and $[z, z'] \subset \mathscr{O}$. But by (5.17), this implies $[z', z) \cap \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)} = \emptyset$, which means that $[z', z] \cap \mathscr{S}_{\tilde\kappa/(2L)}(m) = \emptyset$. This contradicts the fact that $z \in \mathscr{S}_{\tilde\kappa/(2L)}(m)$.

Ad (2): Let $z \in \mathscr{O}$ with $\mathbb{D}_s(z) \subset \mathscr{O}$. By the definition of stable phases, there is at least one $m \in \mathcal{R}$ such that $z \in \mathscr{S}_m \subset \mathscr{S}_{\tilde\kappa/(2L)}(m)$. Combined with (5.12), this proves that the set $\mathcal{Q}'$ is non-empty. To prove (5.14), it remains to show that $\mathbb{D}_s(z) \subset \mathscr{O} \setminus \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)}$ whenever $m \notin \mathcal{Q}'$. By the definition of $\mathcal{Q}'$, $m \notin \mathcal{Q}'$ implies that there exists a $z' \in \mathbb{D}_s(z)$ such that $z' \in \mathscr{O} \setminus \mathscr{S}_{\tilde\kappa/L}(m)$. Consider an arbitrary $z'' \in \partial\mathbb{D}_s(z)$. For such a z'', we have that $|z' - z''| \le 2c/L$ and $[z', z'') \subset \mathscr{O}$, so by (5.17), we conclude that $[z', z'') \subset \mathscr{O} \setminus \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)}$. Since this is true for all $z'' \in \partial\mathbb{D}_s(z)$, we get the desired statement $\mathbb{D}_s(z) \subset \mathscr{O} \setminus \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)}$.

Ad (3): Let $\mathcal{Q} \subset \mathcal{R}$, $z \in \mathscr{U}_{\gamma_L}(\mathcal{Q}) \cap \mathscr{U}_{2\tilde\kappa/L}(\mathcal{Q})$ and $\mathbb{D}_s(z) \subset \mathscr{O}$. If $m \in \mathcal{Q}$, then $z \in \mathscr{S}_{\gamma_L}(m) \subset \mathscr{S}_{\tilde\kappa/(2L)}(m)$ by the definition of $\mathscr{U}_{\gamma_L}(\mathcal{Q})$ and the condition that $\gamma_L \le \tilde\kappa/(2L)$. With the help of (5.12), this implies that $\mathbb{D}_s(z) \subset \mathscr{S}_{\tilde\kappa/L}(m)$ for all $m \in \mathcal{Q}$. Recalling the definition of $\mathscr{U}_{\tilde\kappa/L}(\mathcal{Q})$, we are left with the proof that $\mathbb{D}_s(z) \subset \mathscr{O} \setminus \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)}$ whenever $m \notin \mathcal{Q}$. But if $m \notin \mathcal{Q}$, then $z \in \mathscr{O} \setminus \mathscr{S}_{\tilde\kappa/L}(m)$ because we assumed that $z \in \mathscr{U}_{2\tilde\kappa/L}(\mathcal{Q})$. By (5.17) we conclude that $[z, z') \subset \mathscr{O} \setminus \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)}$ whenever $z' \in \partial\mathbb{D}_s(z)$, which proves $\mathbb{D}_s(z) \subset \mathscr{O} \setminus \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)}$.

We are left with the proof of (5.17), which will be done by contradiction. Assume thus that $m \in \mathcal{R}$ and let z, z' be two points such that $|z - z'| \le 2c/L$, $[z, z') \subset \mathscr{O}$ and $z \in \mathscr{O} \setminus \mathscr{S}_{\tilde\kappa/L}(m)$ hold, while (5.17) fails to hold, so that $[z, z') \cap \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)} \ne \emptyset$. Let $z_1 \in [z, z') \cap \overline{\mathscr{S}_{\tilde\kappa/(2L)}(m)}$. Since $[z, z') \subset \mathscr{O}$, we have in particular that $[z_1, z] \subset \mathscr{O}$. Let z_2 be defined as the nearest point to z_1 on the linear segment $[z_1, z]$ such that

$z_2 \notin \mathscr{S}_{3\tilde{\kappa}/(4L)}(m)$. By continuity of the functions ζ_k, we have $[z_1, z_2] \subset \mathscr{S}_{\tilde{\kappa}/L}(m) \subset \mathscr{S}_{\kappa/L}(m)$ so that the bounds in Lemma 5.1 are at our disposal. Putting (5.2–5.3) together, we have

$$\left|\frac{\zeta_m(z_1)}{\zeta(z_1)}\right|\left|\frac{\zeta(z_2)}{\zeta_m(z_2)}\right| \le e^{2e^{-\tau L}+2M|z_1-z_2|}. \tag{5.18}$$

Now, since $z_1 \in \mathscr{S}_{\tilde{\kappa}/(2L)}(m)$ and $z_2 \notin \mathscr{S}_{3\tilde{\kappa}/(4L)}(m)$, we can infer that the left-hand side is larger than $e^{\tilde{\kappa}/(4L)}$. Hence, we must have

$$|z_1 - z_2| \ge \frac{\kappa}{8ML} - \frac{1}{M}e^{-\tau L} \ge \frac{2c}{L}, \tag{5.19}$$

where the last inequality is a consequence of (5.16). Now $z_1, z_2 \in [z, z')$ implies $|z_1 - z_2| < |z - z'|$, which contradicts the assumption that $|z - z'| \le 2c/L$ and thus proves (5.17). $\square$

5.2. Proofs of Lemmas 4.3-4.5. Here we will establish the three technical lemmas on which the proof of Theorem 2.3 was based. Throughout this section, we fix distinct $m, n \in \mathcal{R}$ and introduce the abbreviations $\mathscr{S}_\epsilon = \mathscr{S}_\epsilon(\{m, n\})$ and $\mathscr{U}_\epsilon = \mathscr{U}_\epsilon(\{m, n\})$. We will also let f and g be the functions defined in (4.20–4.21).

First we will need to establish a few standard facts concerning the local inversion of analytic maps and its behavior under perturbations by continuous functions. The proof is based on Brouwer's Fixed Point Theorem, see e.g. [30, Chapter 2].

Lemma 5.3. *Let $z_0 \in \mathbb{C}$, $\epsilon > 0$, and let $\phi\colon \mathbb{D}_\epsilon(z_0) \to \mathbb{C}$ be an analytic map for which*

$$|\phi'(z_0)|^{-1}\bigl|\phi'(z) - \phi'(z_0)\bigr| \le \frac{1}{2} \tag{5.20}$$

holds for all $z \in \mathbb{D}_\epsilon(z_0)$. Let $\delta \le \epsilon|\phi'(z_0)|/2$. Then, for every $w \in \mathbb{D}_\delta(\phi(z_0))$, there exists a unique point $z \in \mathbb{D}_\epsilon(z_0)$ such that $\phi(z) = w$.

In addition, let $\eta \in [0, \delta/2)$ and let $\theta\colon \mathbb{D}_\epsilon(z_0) \to \mathbb{C}$ be a continuous map satisfying

$$|\theta(z)| \le \eta, \qquad z \in \mathbb{D}_\epsilon(z_0). \tag{5.21}$$

Then for each $z \in \mathbb{D}_\epsilon(z_0)$ with $\phi(z) \in \mathbb{D}_\eta(\phi(z_0))$ there exists a point $z' \in \mathbb{D}_\epsilon(z_0)$ such that

$$\phi(z') + \theta(z') = \phi(z). \tag{5.22}$$

Moreover, $|z' - z| \le 2\eta|\phi'(z_0)|^{-1}$.

Proof. Following standard proofs of the theorem about local inversion of differentiable maps (see, e.g., [13], Sect. 3.1.1), we seek the inverse of w as a fixed point of the (analytic) function $z \mapsto \psi(z) = z + \phi'(z_0)^{-1}(w - \phi(z))$. The condition (5.20) guarantees that $z \mapsto \psi(z)$ is a contraction on $\mathbb{D}_\epsilon(z_0)$. Indeed, for every $z \in \mathbb{D}_\epsilon(z_0)$ we have

$$|\psi'(z)| = \bigl|1 - \phi'(z_0)^{-1}\phi'(z)\bigr| \le |\phi'(z_0)|^{-1}\bigl|\phi'(z) - \phi'(z_0)\bigr| \le \tfrac{1}{2}, \tag{5.23}$$

which implies that $|\psi(z) - \psi(z')| \le \frac{1}{2}|z - z'|$ for all $z, z' \in \mathbb{D}_\epsilon(z_0)$. The actual solution to $\phi(z) = w$ is obtained as the limit $z = \lim_{n\to\infty} z_n$ of iterations $z_{n+1} = \psi(z_n)$ starting at z_0. In view of the above estimates, we have $|z_{n+1} - z_n| \le \frac{1}{2}|z_n - z_{n-1}|$ and, summing over n, we get $|z_n - z_0| \le 2|z_1 - z_0| \le 2|\phi'(z_0)|^{-1}|w - \phi(z_0)|$. Since $|w - \phi(z_0)| < \delta$, we have that z_n as well as its limit belongs to $\mathbb{D}_\epsilon(z_0)$.

Next we shall attend to the second part of the claim. The above argument allows us to define the left inverse of ϕ as the function $\phi^{-1}\colon \mathbb{D}_\delta(\phi(z_0)) \to \mathbb{D}_\epsilon(z_0)$ such that $\phi^{-1}(w)$ is the unique value $z \in \mathbb{D}_\epsilon(z_0)$ for which $\phi(z) = w$. Let $\eta \in [0, \delta/2)$ and let $z \in \mathbb{D}_\epsilon(z_0)$ be such that $\phi(z) \in \mathbb{D}_\eta(\phi(z_0))$. Consider the function $\Psi\colon \mathbb{D}_\delta(\phi(z_0)) \to \mathbb{C}$ defined by

$$\Psi(w) = \phi(z) - \theta(\phi^{-1}(w)). \tag{5.24}$$

By our choice of z and (5.21), we have $|\Psi(w)| \le 2\eta$ for any $w \in \mathbb{D}_\delta(\phi(z_0))$. Thus, Ψ maps the closed disc $\overline{\mathbb{D}_{2\eta}(\phi(z_0))}$ into itself and, in light of continuity of Ψ, Brouwer's Theorem implies that Ψ has a fixed point w' in $\overline{\mathbb{D}_{2\eta}(\phi(z_0))}$. From the relation $\Psi(w') = w'$ we then easily show that (5.22) holds for $z' = \phi^{-1}(w')$. To control the distance between z and z', we just note that the above Lipschitz bound on ψ allows us to conclude that $|z' - z| \le 2|\phi'(z_0)|^{-1}|\phi(z') - \phi(z)|$. Applying (5.22) and (5.21), the right-hand side is bounded by $2\eta|\phi(z_0)|^{-1}$. $\square$

Now we are ready to start proving Lemmas 4.3-4.5. The first claim to prove concerns the relation of the solutions of (2.17–2.18) and the roots of the function f defined in (4.20).

Proof of Lemma 4.3. Let $\tilde\alpha$, M and τ be the constants from Assumption B. Let c and L_4 be the constants from Corollary 5.2 with $\tilde\kappa = \kappa$. The proof will be carried out for the constants B_1, $\tilde C_1$ and L_1 chosen as follows: We let

$$B_1 = \frac{1}{4M}, \quad B_2 = \frac{16 + 4|\log(q_n/q_m)|}{\tilde\alpha} \quad \text{and} \quad \tilde C_1 = \frac{10}{\tilde\alpha}, \tag{5.25}$$

and assume that L_1 is so large that $L_1 \ge L_4$ and for all $L \ge L_1$, we have $\tilde C_1 e^{-\tau L} < B_1 L^{-d}$ and the bounds:

$$(B_1 + B_2)L^{-d} \le \frac{c}{L} \le \frac{1}{4M}, \quad 2e^{-\tau L} + \frac{\kappa}{L} \le \frac{1}{4}, \tag{5.26}$$

$$\frac{2}{\tilde\alpha}(M + M^2)(B_1 + B_2)L^{-d} \le \frac{1}{2}, \tag{5.27}$$

and also

$$2e^{-\tau L} + 2MB_1L^{-d} \le L^{-d}, \quad \tilde\alpha > 2\sqrt{2}e^{-\tau L}, \tag{5.28}$$

$$\pi L^{-d} + 2e^{-\tau L} < 4L^{-d} \quad \text{and} \quad \tilde C_1 e^{-\tau L} \le \tfrac{1}{2} B_2 L^{-d}. \tag{5.29}$$

Let us fix a value $L \ge L_1$ and choose a point $z_0 \in \mathscr{S}_{\kappa/(2L)}$ and a number $s \le (B_1 + B_2)L^{-d}$ such that $\mathbb{D}_s(z_0) \subset \mathscr{O}$. Corollary 5.2(1) combined with the first bound in (5.26) implies that $\mathbb{D}_s(z_0) \subset \mathscr{S}_{\kappa/L}$.

We will apply Lemma 5.3 for suitable choices of ϕ and θ defined in terms of the functions $F_{m,n}\colon \mathbb{D}_s(z_0) \to \mathbb{C}$ and $F^{(L)}_{m,n}\colon \mathbb{D}_s(z_0) \to \mathbb{C}$ defined by

$$F_{m,n}(z) = \frac{\zeta_m(z)}{\zeta_n(z)} \quad \text{and} \quad F^{(L)}_{m,n}(z) = \frac{\zeta^{(L)}_m(z)}{\zeta^{(L)}_n(z)}. \tag{5.30}$$

We will want to define $\phi(z)$ as the logarithm of $F^{(L)}_{m,n}(z)$, and $\theta(z)$ as the logarithm of the ratio $F^{(L)}_{m,n}(z)/F_{m,n}(z)$, but in order to do so, we will have to specify the branch of the complex logarithm we are using. To this end, we will first analyze the image of the functions $F_{m,n}^{(L)}(z)$ and $F_{m,n}^{(L)}(z)/F_{m,n}(z)$.

According to Assumption B2, for any $z \in \mathbb{D}_s(z_0) \subset \mathscr{S}_{\kappa/L}$, we have $|F_{m,n}^{(L)}(z)| \in (2/3, 3/2)$ in view of the second bound in (5.26) with the observation that $\frac{1}{4} < \log\frac{3}{2}$. A simple calculation and the bound (2.11) show that $\operatorname{Arg} F_{m,n}^{(L)}(z)$ and $\operatorname{Arg} F_{m,n}^{(L)}(z_0)$ differ by less than $2M(B_1+B_2)L^{-d} \le \frac{1}{2}$. Indeed, the difference $\operatorname{Arg} F_{m,n}^{(L)}(z) - \operatorname{Arg} F_{m,n}^{(L)}(z_0)$ is expressed in terms of the integral of $\partial_z F_{m,n}^{(L)}/F_{m,n}^{(L)}$ along any path in $\mathbb{D}_s(z_0)$ connecting z_0 and z. The latter logarithmic derivative is bounded uniformly by $2M$ throughout $\mathbb{D}_s(z_0)$. Consequently, $z \mapsto F_{m,n}^{(L)}(z)$ maps $\mathbb{D}_s(z_0)$ into the open set of complex numbers $\{\rho e^{\mathrm{i}\omega}\colon \rho \in (\frac{2}{3}, \frac{3}{2}), |\omega - \omega_0| < \frac{1}{2}\}$, where $\omega_0 = \operatorname{Arg} F_{m,n}(z_0)$. The function $z \mapsto F_{m,n}^{(L)}(z)/F_{m,n}(z)$, on the other hand, maps $\mathbb{D}_s(z_0)$ into the open set of complex numbers $\{\rho e^{\mathrm{i}\omega}\colon \rho \in (\frac{2}{3}, \frac{3}{2}), |\omega| < \frac{1}{4}\}$, as can be easily inferred from Assumption B2 and the second bound in (5.26). Given these observations, we choose the branch of the complex logarithm with cut along the ray $\{re^{-\mathrm{i}\omega_0/2}\colon r > 0\}$, and define

$$\phi(z) = \log F_{m,n}^{(L)}(z) \tag{5.31}$$

and

$$\theta(z) = \log \frac{F_{m,n}^{(L)}(z)}{F_{m,n}(z)}. \tag{5.32}$$

Having defined the functions ϕ and θ, we note that, by Assumptions A and B, ϕ is analytic while θ is twice continuously differentiable throughout $\mathbb{D}_s(z_0)$. Moreover, these functions are directly related to the equations $f(z) = 0$ and (2.17–2.18). Indeed, $f(z) = 0$ holds for some $z \in \mathbb{D}_s(z_0)$ if and only if $F_{m,n}^{(L)}(z)$ is an L^d-th root of $-(q_n/q_m)$, i.e., $\phi(z) = (\log(q_n/q_m) + \mathrm{i}\pi(2k+1))L^{-d}$ for some integer k. Similarly, $z \in \mathbb{D}_s(z_0)$ is a solution of (2.17–2.18) if and only if $\phi(z) + \theta(z)$ is of the form $(\log(q_n/q_m) + \mathrm{i}\pi(2k+1))L^{-d}$ for some integer k. Furthermore, these functions obey the bounds

$$\tilde{\alpha} \le |\phi'(z)| \le 2M, \qquad |\phi'(z) - \phi'(z_0)| \le 2(M+M^2)(B_1+B_2)L^{-d}, \tag{5.33}$$

and

$$|\theta(z)| \le 2e^{-\tau L}, \qquad |\theta(z) - \theta(z')| \le 2\sqrt{2}e^{-\tau L}|z - z'| \tag{5.34}$$

for all $z, z' \in \mathbb{D}_s(z_0)$. Here the first three bounds are obvious consequences of Assumption B, while the third follows from Assumption B by observing that the derivative matrix $D\theta(z)$ is bounded in norm by $2\sqrt{2}$ times the right hand side of (2.10). Note that, in light of (5.26–5.27), these bounds directly verify the assumptions (5.20) and (5.21) of Lemma 5.3 for $\eta = 2e^{-\tau L}$ and any $\epsilon \le s$. We proceed by applying Lemma 5.3 with different choices of ϵ to give the proof of (2-4) of Lemma 4.3, while part (1) turns out to be a direct consequence of the bounds (5.33–5.34).

Indeed, let us first show that for $s \le B_1 L^{-d}$ the disc $\mathbb{D}_s(z_0)$ contains at most one solution to (2.17–2.18) and at most one root of the equation $f(z) = 0$. We will prove both statements by contradiction. Starting with the solutions to (2.17–2.18), let us thus assume that $z_1, z_2 \in \mathbb{D}_s(z_0)$ are two distinct solutions to the equations (2.17–2.18). Setting $w_1 = \phi(z_1) + \theta(z_1)$ and $w_2 = \phi(z_2) + \theta(z_2)$ this means that $w_1 - w_2$ is an integer multiple of $2\pi\mathrm{i}L^{-d}$. However, the bounds (5.33) and (5.34) combined with the first bound in (5.28) guarantee that $|w_1 - w_2| \le 4e^{-\tau L} + 4MB_1L^{-d} \le 2L^{-d}$

and thus $w_1 = w_2$. But then the bound $|\phi(z_1) - \phi(z_2)| \geq \tilde{\alpha}|z_1 - z_2|$ implies that $|\theta(z_1) - \theta(z_2)| \geq \tilde{\alpha}|z_1 - z_2|$, which, in view of the second bound in (5.28), contradicts the second bound in (5.34). Hence, we must have had $z_1 = z_2$ in the first place. Turning to the equation $f(z) = 0$, let us now assume that z_1 and z_2 are two different roots of this equation. Setting $w_1 = \phi(z_1)$ and $w_2 = \phi(z_2)$, we again have $w_1 = w_2$, this time by the first bound in (5.33) and the very definition of B_1, which implies that $4MB_1 = 1$. But once we have $w_1 = w_2$, we must have $z_1 = z_2$ since $|\phi(z_1) - \phi(z_2)| \geq \tilde{\alpha}|z_1 - z_2|$ by our lower bound on $\phi'(z)$, implying that there exists at most one $z \in \mathbb{D}_s(z_0)$ that solves the equation $f(z) = 0$. If such a solution z exists, Assumption B immediately implies that $f'(z) \neq 0$, and so z is a non-degenerate root of f.

Next, we will show that within a $\tilde{C}_1 e^{-\tau L}$-neighborhood of each solution z_0 of the equations (2.17–2.18) there is a root of f. Indeed, let $\epsilon = \tilde{C}_1 e^{-\tau L}$ and $\delta = 5e^{-\tau L}$. By the first bound in (5.33) and our choice of $\tilde{C}_1$, we then have $\delta \leq \epsilon|\phi'(z_0)|/2$, so the first part of Lemma 5.3 is at our disposal. Since z_0 is assumed to be a solution to (2.17–2.18), we have that $\phi(z_0) + \theta(z_0)$ is of the form $(\log(q_n/q_m) + \mathrm{i}\pi(2k+1))L^{-d}$, where k is an integer. In light of the bound $|\theta(z_0)| \leq 2e^{-\tau L}$, the disc $\mathbb{D}_\delta(\phi(z_0))$ contains the point $w = \phi(z_0) + \theta(z_0)$. By the first part of Lemma 5.3, there exists a point $z \in \mathbb{D}_\epsilon(z_0)$ such that $\phi(z) = w$, implying that z is a root of f.

As a third step we will prove that if z_0 is a root of f, then there exists a solution to (2.17–2.18) in $\mathbb{D}_{\tilde{C}_1 e^{-\tau L}}(z_0)$. By the relation between f and ϕ we now know that $\phi(z_0)$ is of the form $(\log(q_n/q_m) + \mathrm{i}\pi(2k+1))L^{-d}$ for some integer k. We again set $\epsilon = \tilde{C}_1 e^{-\tau L}$ and $\delta = 5e^{-\tau L}$. Choosing $\eta = 2e^{-\tau L}$ and noting that $2\eta < \delta$, we apply the second part of Lemma 5.3 to conclude that there must be a point $z' \in \mathbb{D}_\epsilon(z_0)$ such that $\phi(z') + \theta(z') = \phi(z_0) = (\log(q_n/q_m) + \mathrm{i}\pi(2k+1))L^{-d}$, which means that z' is a solution to (2.17–2.18).

Finally, we will show that if $z_0 \in \mathscr{S}_m \cap \mathscr{S}_n$, then there exists a solution to (2.17–2.18) in the disc $\mathbb{D}_{B_2 L^{-d}}(z_0)$. To this end, we first note that $z_0 \in \mathscr{S}_m \cap \mathscr{S}_n$ implies that $\phi(z_0) + \theta(z_0)$ is purely imaginary. Combined with the first bound in (5.34) we conclude that within a distance of at most $(|\log(q_m/q_n)| + \pi)L^{-d} + 2e^{-\tau L}$ from $\phi(z_0)$, there exists a point of the form $w = (\log(q_n/q_m) + \mathrm{i}\pi(2k+1))L^{-d}$ for some integer k. We now set $\epsilon = B_2 L^{-d}/2$ and $\delta = (|\log(q_m/q_n)| + 4)L^{-d}$. By the first condition in (5.29), we then have $|\phi(z_0) - w| < \delta$, while the first bound in (5.33) together with the definition of B_2 implies that $\delta \leq \epsilon|\phi'(z_0)|/2$. We therefore can use the first part of Lemma 5.3 to conclude that there must be a point $z' \in \mathbb{D}_\epsilon(z_0)$ such that $\phi(z') = w$, implying that z' is a root of $f(z') = 0$. Finally, by the already proven statement (3) of the lemma, there must be a solution of the equations (2.17–2.18) within a distance strictly less than $\tilde{C}_1 e^{-\tau}$ from z'. Since $\epsilon + \tilde{C}_1 e^{-\tau} \leq B_2 L^{-d}$ by the second condition in (5.29), this gives the desired solution of the equations (2.17–2.18) in the disc $\mathbb{D}_{B_2 L^{-d}}(z_0)$. □

Next we will prove Lemma 4.4 which provides a lower bound on $f(z)$ on the boundary of certain discs.

Proof of Lemma 4.4. Let $\tilde{\alpha}$ and M be as in Assumption B3, let $\tilde{\kappa} = \kappa/2$, and let c and L_4 be the constants from Corollary 5.2. We will prove the claim with

$$\tilde{c}_2 = (2eM\|\mathbf{q}\|_\infty)^{-1} \qquad \text{and} \qquad \tilde{C}_2 = \max\{\tilde{c}_2, 22e\tilde{\alpha}^{-1}\} \tag{5.35}$$

and, given $\tilde{C} \geq \tilde{C}_2$, with L_2 defined by the condition that $L_2 \geq L_4$ and

$$\tilde{C}\epsilon_L \leq c/L, \quad L^d e^{-\tau L} \leq 1, \quad e^{\tilde{C}ML^d\epsilon_L} \leq 2 \tag{5.36}$$

and

$$2e(M+M^2)\|\mathbf{q}\|_\infty \tilde{C}^2 L^d \epsilon_L \le 1 \tag{5.37}$$

hold whenever $L \ge L_2$.

Fix $L \ge L_2$ and choose a point $z_0 \in \mathscr{S}_{\kappa/(4L)} \cap (\mathscr{S}_m \cup \mathscr{S}_n)$ with $\mathbb{D}_{\tilde{C}\epsilon_L}(z_0) \subset \mathscr{O}$. Let $s < \tilde{C}\epsilon_L$ and note that by (5.36) we have $s < c/L$. Applying Corollary 5.2(1) to the disc $\mathbb{D}_s(z_0)$ we find that $\mathbb{D}_s(z_0) \subset \mathscr{S}_{\kappa/(2L)} \subset \mathscr{S}_{\kappa/L}$. In particular, the bounds of Assumption B are at our disposal whenever $z \in \mathbb{D}_{\tilde{C}\epsilon_L}(z_0)$. The proof will proceed by considering two separate cases depending (roughly) on whether $|f(z_0)|$ is "small" or "large." We will first address the latter situations. Let us therefore suppose that $|f(z_0)| > 4L^d\epsilon_L\,\zeta(z_0)^{L^d}$. In this case, we will show that (4.23) holds with $s(z_0) = \tilde{c}_2\epsilon_L$. (Note that $s(z_0) \le \tilde{C}_2\epsilon_L \le \tilde{C}\epsilon_L$ by our definition of $\tilde{C}_2$.) A crucial part of the proof consists of the derivation of an appropriate estimate on the derivative of f. Let $s < \tilde{C}\epsilon_L$ and let z be such that $|z-z_0| \le s$. Recalling the definition (4.2) of $b_m(z)$ and using Assumptions B2-B3, the second and third bound in (5.36) and the fact that one of the values $|\zeta_m(z_0)|$ and $|\zeta_n(z_0)|$ must be equal to $\zeta(z_0)$, we have

$$\begin{aligned} |f'(z)| &= L^d\Big|q_m b_m(z)\zeta_m^{(L)}(z)^{L^d} + q_n b_n(z)\zeta_n^{(L)}(z)^{L^d}\Big| \\ &\le L^d\Big[q_m M|\zeta_m(z_0)|^{L^d} + q_n M|\zeta_n(z_0)|^{L^d}\Big]e^{M|z-z_0|L^d + L^d e^{-\tau L}} \\ &\le 4eM\|\mathbf{q}\|_\infty L^d\zeta(z_0)^{L^d} \end{aligned} \tag{5.38}$$

whenever $z \in \mathscr{S}_{\kappa/L}$. As argued above, $z \in \mathbb{D}_{\tilde{C}\epsilon_L}(z_0)$ implies that $[z_0, z] \subset \mathscr{S}_{\kappa/L}$, so by the Fundamental Theorem of Calculus we have

$$\big|f(z)\big| \ge \big|f(z_0)\big| - 4eM\|\mathbf{q}\|_\infty L^d\zeta(z_0)^{L^d} s \ge 4L^d\,\zeta(z)^{L^d}\Big(\epsilon_L - \frac{s}{2\tilde{c}_2}\Big) \tag{5.39}$$

for all $z \in \mathbb{D}_s(z_0)$. The bound (4.23) now follows by letting $s \uparrow \tilde{c}_2\epsilon_L$.

Next we will address the cases with $|f(z_0)| \le 4L^d\epsilon_L\zeta(z_0)^{L^d}$. Let $s < \tilde{C}\epsilon_L$ and pick z such that $|z-z_0| = s$. This point belongs to the disc $\mathbb{D}_{\tilde{C}\epsilon_L}(z_0)$ which we recall is a subset of $\mathscr{S}_{\kappa/L}$. The second-order expansion formula

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + (z-z_0)^2\int_0^1 \mathrm{d}t\int_0^t \mathrm{d}\tilde{t}\; f''\big(\tilde{t}z + (1-\tilde{t})z_0\big) \tag{5.40}$$

then yields the estimate

$$\big|f(z)\big| \ge \big|f(z_0) + (z-z_0)f'(z_0)\big| - \widetilde{K}\big(\tilde{C}\epsilon_L\big)^2 L^{2d}\zeta(z_0)^{L^d} \tag{5.41}$$

where

$$\widetilde{K} = \frac{1}{2}\zeta(z_0)^{-L^d}L^{-2d}\sup\big\{|f''(z)|\colon z \in \mathscr{U},\ |z-z_0| < \tilde{C}\epsilon_L\big\}. \tag{5.42}$$

Proceeding as in the bound (5.38), we easily get

$$\widetilde{K} \le 2e\|\mathbf{q}\|_\infty\big[M^2(1-L^{-d}) + ML^{-d}\big], \tag{5.43}$$

which implies that $\widetilde{K} \le 2e\|\mathbf{q}\|_\infty[M^2+M]$.

It remains to estimate the absolute value on the right-hand side of (5.41). Abbreviating $b_m = b_m(z_0)$ and $b_n = b_n(z_0)$, we can write

$$\begin{aligned} f'(z_0) &= L^d\bigl(b_m q_m \zeta_m^{(L)}(z_0)^{L^d} + b_n q_n \zeta_n^{(L)}(z_0)^{L^d}\bigr) \\ &= L^d(b_m - b_n) q_m \zeta_m^{(L)}(z_0)^{L^d} + b_n L^d f(z_0). \end{aligned} \tag{5.44}$$

Without loss of generality, let us suppose that $|\zeta_m(z_0)| \ge |\zeta_n(z_0)|$ and, consequently, $|\zeta_m(z_0)| = \zeta(z_0)$, because $z_0 \in \mathscr{S}_m \cup \mathscr{S}_n$. Applying Assumption B3 together with the assumed upper bound on $|f(z_0)|$, we get

$$\bigl|(z - z_0) f'(z_0) + f(z_0)\bigr| \ge \bigl(\tilde{\alpha} q_m s e^{-L^d e^{-\tau L}} - 4\epsilon_L\,(1 + sL^d M)\bigr) L^d \zeta(z_0)^{L^d}, \tag{5.45}$$

where we recalled that $|z - z_0| = s$. Since $s \le \tilde{C}\epsilon_L$, the third inequality in (5.36) gives that $sL^d M \le \tilde{C} M L^d \epsilon_L < 1$. Let now s be so large that $s \ge \frac{1}{2}\tilde{C}\epsilon_L$. Using this bound in the first term in (5.45) and using the second inequality in (5.36) we thus get

$$\bigl|(z - z_0) f'(z_0) + f(z_0)\bigr| \ge \bigl(\tfrac{1}{2}\tilde{\alpha}\tilde{C}_2 e^{-1} - 8\bigr) L^d \epsilon_L \zeta(z_0)^{L^d} \ge 3L^d \epsilon_L \zeta(z_0)^{L^d}. \tag{5.46}$$

Moreover, using the above bound on $\widetilde{K}$ and the inequality in (5.37), the last term on the right-hand side of (5.41) can be shown not to exceed $L^d \epsilon_L \zeta(z_0)^{L^d}$. Putting (5.41) and (5.46) together with these estimates, we have $|f(z)| \ge 2L^d \epsilon_L \zeta(z_0)^{L^d}$ for all $z \in \mathbb{D}_{\tilde{C}\epsilon_L}(z_0)$ such that $s = |z - z_0|$ satisfies $\frac{1}{2}\tilde{C}\epsilon_L \le s < \tilde{C}\epsilon_L$. The proof is finished by taking $s \uparrow \tilde{C}\epsilon_L$.

The last statement of the lemma is an immediate consequence of the fact that whenever the above procedure picks $s(z_0) = \tilde{c}_2\epsilon_L$ and $\tilde{c}_2 < \tilde{C}$, then the argument (5.38–5.39) implies the stronger bound

$$\inf_{z\colon |z - z_0| < s(z_0)} |f(z)| \ge 2L^d \epsilon_L \zeta(z_0)^{L^d}. \tag{5.47}$$

Now, if f has a root in $\mathbb{D}_{\tilde{c}_2\epsilon_L}(z_0)$, then this bound shows that we could not have chosen $s(z_0) = \tilde{c}_2\epsilon_L$. Therefore, $s(z_0)$ must be equal to the other possible value, i.e., we must have $s(z_0) = \tilde{C}\epsilon_L$. $\square$

Proof of Lemma 4.5. We will prove (4.24) with $A_3 = 2C_0\|\mathbf{q}\|_1$, where C_0 is as in (2.14) for $\ell = 0$. Let $\tilde{L}_0$ and M be as in Assumption B and let L_4 and c be as in Corollary 5.2. Let $C \in (0, \infty)$ and let us choose $L_3 \ge \max\{L_4, \tilde{L}_0\}$ in such a way that

$$\max\bigl\{Ce^{-\tau L}, CL^d e^{-\frac{1}{2}L^d \gamma_L}\bigr\} \le \frac{c}{L}, \qquad MCL^d e^{-\tau L} \le \log 2, \tag{5.48}$$

$$\frac{1}{2} L^d \gamma_L + MCL^{2d} e^{-\frac{1}{2}L^d \gamma_L} \le \tau L, \tag{5.49}$$

and

$$\gamma_L \le \frac{\kappa}{2L} \quad \text{and} \quad MCL^{2d} e^{-\frac{1}{2}L^d \gamma_L} + L^d e^{-\tau L} \le 2d \log L + \log C_0 \tag{5.50}$$

hold for all $L \ge L_3$.

We will treat separately the cases $z_0 \in \mathscr{U}_{\gamma_L} \cap \mathscr{U}_{2\kappa/L}(z_0)$ and $z_0 \in \mathscr{U}_{\gamma_L} \setminus \mathscr{U}_{2\kappa/L}(z_0)$. Let us first consider the former case, so that $\delta_L(z_0) = e^{-\tau L}$. The first condition in

(5.48), the fact that $\mathbb{D}_{C\delta_L(z_0)}(z_0) \subset \mathscr{O}$ and $\gamma_L \le \kappa/(2L)$ therefore allow us to use Corollary 5.2(3), from which we conclude that $\mathbb{D}_{C\delta_L(z_0)}(z_0) \subset \mathscr{U}_{\gamma_L}$. For $z \in \mathbb{D}_{C\delta_L(z_0)}(z_0)$ we may thus apply the $\ell = 0$ version of (2.14) to the function $g(z) = \varXi_{\{m,n\},L}(z)$. Combined with the bound (5.3), the second condition in (5.48) and our definition of A_3 this immediately gives the desired bound (4.24).

Next we will attend to the cases when z_0 lies in $\mathscr{U}_{\gamma_L} \setminus \mathscr{U}_{2\kappa/L}$, so that $\delta_L(z_0) = L^d e^{-\frac{1}{2}L^d\gamma_L}$. Let us define $\mathcal{Q}'$ as in (5.13) with $s = C\delta_L(z_0)$, i.e.,

$$\mathcal{Q}' = \big\{k \in \mathcal{R}\colon \mathbb{D}_{C\delta_L(z_0)} \subset \mathscr{S}_{\kappa/L}(k)\big\}. \tag{5.51}$$

By Corollary 5.2(2), the set $\mathcal{Q}'$ is non-empty and $\mathbb{D}_{C\delta_L(z_0)}(z_0) \subset \mathscr{U}_{\kappa/L}(\mathcal{Q}')$. Let $z \in \mathbb{D}_{C\delta_L(z_0)}(z_0)$ and let us estimate $g(z)$. We will proceed analogously to the preceding case; the only difference is that this time we have

$$g(z) = \varXi_{\mathcal{Q}',L}(z) + h(z), \tag{5.52}$$

where the extra term $h(z)$ is given by

$$h(z) = \sum_{k\in\mathcal{Q}'\setminus\{m,n\}} q_k\big[\zeta_k^{(L)}(z)\big]^{L^d}. \tag{5.53}$$

Now $|\varXi_{\mathcal{Q}',L}(z)|$ is estimated as before: Using that $z \in \mathscr{U}_{\kappa/L}(\mathcal{Q}')$, the bounds (2.14) and (5.3) immediately yield that $|\varXi_{\mathcal{Q}',L}(z)| \le C_0\|\mathbf{q}\|_1 L^d\delta_L(z_0)\zeta(z_0)^{L^d}$. (Here we used that the term $e^{ML^dC\delta_L(z_0)}e^{-\tau L}$ is bounded by $e^{-\frac{1}{2}L^d\gamma_L} \le \delta_L(z_0)$ as follows from (5.49).)

Therefore, we just need to produce an appropriate bound on $|h(z)|$. To that end, we note that, since $[z_0, z]\subset \mathscr{U}_{\kappa/L}(\mathcal{Q}')$ and $|z - z_0| \le C\delta_L(z_0)$, we have from (5.4) and Assumption B2 that

$$\big|\zeta_k^{(L)}(z)\big|^{L^d} \le \big|\zeta_k^{(L)}(z_0)\big|^{L^d} e^{MCL^d\delta_L(z_0)} \le \big|\zeta_k(z_0)\big|^{L^d} e^{MCL^d\delta_L(z_0)+L^de^{-\tau L}} \tag{5.54}$$

whenever $k \in \mathcal{Q}'$. Since $z_0 \in \mathscr{U}_{\gamma_L}$, which implies $|\zeta_k^{(L)}(z)| \le \zeta(z_0)e^{-\gamma_L/2}$ whenever $k \notin \{m, n\}$, we thus have

$$\big|\zeta_k^{(L)}(z)\big|^{L^d} \le e^{MCL^d\delta_L(z_0)+L^de^{-\tau L}} e^{-\frac{1}{2}\gamma_L L^d}\zeta(z_0)^{L^d} \tag{5.55}$$

for every $k \in \mathcal{Q}' \setminus \{m, n\}$. Using the last bound in (5.50), we conclude that $|h(z)|$ is bounded by $C_0\|\mathbf{q}\|_1 L^d\delta_L(z_0)\zeta(z_0)^{L^d}$. From here (4.24) follows. $\square$

5.3. Proof of Lemmas 4.6 and 4.7. Here we will establish the two technical lemmas on which the proof of Theorem 2.5 was based. Throughout this section we will assume that a multiple point $z_{\mathrm{M}} \in \mathscr{O}$ is fixed and that $\mathcal{Q} = \mathcal{Q}(z_{\mathrm{M}})$. We will also use $\tilde f$, $\tilde g$ and ξ to denote the functions defined in (4.48–4.50).

Lemma 4.6 is an analogue of Lemma 4.4 from Sect. 4.2 the corresponding proofs are also analogous. Namely, the proof of Lemma 4.4 was based on the observation that either $|f(z)|$ was itself large in a neighborhood of z_0, or it was small, in which case we knew that $|f'(z)|$ was large. In Lemma 4.6, the function $\tilde f(z)$ is more complicated; however, a convenient reformulation in terms of Vandermonde matrices allows us to conclude that at least one among its first $(q-1)$ derivatives is large. This is enough to push the argument through.

Proof of Lemma 4.6. Abbreviating $q=|\mathcal{Q}|$ and using $A(q)=2q^{3q(q+1)/2}q!\sqrt{q}$ and the constants $K=K(\mathcal{Q})$ and $\tilde{L}_0$ from Lemma 4.2 and M from Assumption B, let $\epsilon=1/(3K)$ and $L_5\ge\tilde{L}_0$ be such that

$$e^{ML^dR_L}\le 2,\quad 2\|q\|_1M^q\le L^{2d}\epsilon_L\quad\text{and}\quad A(q)L^{2d-d/q}\epsilon_L\le\epsilon/\sqrt{q} \tag{5.56}$$

for all $L\ge L_5$. A choice of L_5 yielding (5.56) is possible in view of (4.51).

Choosing $z_0\in\mathbb{C}$, we use $F(z)$ to denote the function $F(z)=\tilde{f}(z)\xi(z_0)^{-L^d}$. First, we claim that if (4.52) fails to hold for some $L\ge L_5$, then we have

$$\bigl|F^{(\ell)}(z_0)\bigr|\le\frac{\epsilon}{\sqrt{q}}L^{d\ell},\qquad \ell=0,\dots,q-1. \tag{5.57}$$

Indeed, let us observe that, if (4.52) fails to hold, then there must exist a collection of points z_k, with $k=1,\dots,q$, such that

$$|z_k-z_0|=\tfrac{k}{q}R_L\quad\text{and}\quad\bigl|F(z_k)\bigr|\le L^d\epsilon_L, \tag{5.58}$$

for all $k=1,\dots,q$. Further, notice that, for $|z-z_0|\le R_L$, we have the bound

$$\Bigl|e^{v_m(z-z_{\mathrm{M}})L^d}\xi(z_0)^{-L^d}\Bigr|\le e^{\mathrm{Re}(v_m(z-z_0))L^d}\le e^{ML^dR_L},\qquad m\in\mathcal{Q}, \tag{5.59}$$

implying $|F^{(q)}(z)|\le 2\sum_{m\in\mathcal{Q}}q_m\,|v_m|^qL^{dq}$ in view of the first condition in (5.56). In particular, we have $|F^{(q)}(z)|R_L^q\le 2\|\mathbf{q}\|_1M^qL^{-d}$ for all z in the R_L-neighborhood of z_0. With help of the second condition in (5.56), Taylor's theorem yields

$$\Bigl|\sum_{\ell=0}^{q-1}\frac{F^{(\ell)}(z_0)}{\ell!}(z_k-z_0)^\ell\Bigr|\le 2L^d\epsilon_L,\qquad k=1,\dots,q. \tag{5.60}$$

Now we will write (5.60) in vector notation and use our previous estimates on Vandermonde matrices to derive (5.57). Let $\mathbf{x}=(x_0,x_1,\dots,x_{q-1})$ be the vector with components

$$x_\ell=R_L^\ell\,\frac{F^{(\ell)}(z_0)}{\ell!}\Bigl(\frac{z_k-z_0}{|z_k-z_0|}\Bigr)^\ell,\qquad \ell=0,1,\dots,q-1, \tag{5.61}$$

and let $\mathbb{N}=(\mathbb{N}_{k,\ell})$ be the $q\times q$-matrix with elements $\mathbb{N}_{k,\ell}=|z_k-z_0|^\ell R_L^{-\ell}=(k/q)^\ell$. The bound (5.60) then implies that the vector $\mathbb{N}\mathbf{x}$ has each component bounded by $2L^d\epsilon_L$ and so $\|\mathbb{N}\mathbf{x}\|\le 2\sqrt{q}L^d\epsilon_L$. On the other hand, since $\mathbb{N}$ is a Vandermonde matrix, the norm of its inverse can be estimated as in (4.8). Namely, using the inequalities $|\det\mathbb{N}|\ge q^{-q(q-1)/2}$ and $\|\mathbb{N}\|\le q$, we get

$$\|\mathbb{N}^{-1}\|\le\frac{\|\mathbb{N}\|^{q-1}}{|\det\mathbb{N}|}\le q^{q(q-1)/2+q(q-1)}. \tag{5.62}$$

But then $\|\mathbf{x}\|\le\|\mathbb{N}^{-1}\|\,\|\mathbb{N}\mathbf{x}\|\le q^{3q(q-1)/2}2\sqrt{q}L^d\epsilon_L$ implying

$$L^{-d\ell}|F^{(\ell)}(z_0)|\le\ell!(L^dR_L)^{-\ell}\|\mathbf{x}\|\le A(q)L^{2d-d/q}\epsilon_L, \tag{5.63}$$

where we used that $L^d(L^dR_L)^{-\ell}$ is maximal for $\ell = q-1$, in which case it equals $L^{2d-d/q}$. With the help of the last condition in (5.56), the claim (5.57) follows for all $L \geq L_5$.

Having proved (5.57), we will now invoke the properties of Vandermonde matrices once again to show that (5.57) contradicts Lemma 4.2. Let $\mathbf{y}$ be the q-dimensional vector with components

$$y_m = q_m e^{\mathrm{i}\phi_m(L)+v_m(z-z_\mathrm{M})L^d}\xi(z_0)^{-L^d}, \qquad m \in \mathcal{Q}. \tag{5.64}$$

Let $\mathbb{O} = (\mathbb{O}_{\ell,m})$ be the $q \times q$ matrix with matrix elements $\mathbb{O}_{\ell,m} = v_m^\ell$. (Here ℓ takes values between 0 and $q-1$, while $m \in \mathcal{Q}$.) Recalling the definition of $F(z)$, the bound (5.57) can be rewritten as $|[\mathbb{O}\mathbf{y}]_\ell| \leq \epsilon/\sqrt{q}$. It therefore implies that

$$\|\mathbb{O}\mathbf{y}\| \leq \epsilon. \tag{5.65}$$

The matrix $\mathbb{O}$ corresponds to the $L \to \infty$ limit of the matrix $\mathbb{M}$ in (4.3) evaluated at z_M. In particular, since $z_\mathrm{M} \in \mathscr{S}_{\kappa/L}(m)$ for all L and all $m \in \mathcal{Q}(z_\mathrm{M})$ and in view of the second bound in Assumption B2, the bound (4.5) applies to $\mathbb{O}$ as well. Having thus $\|\mathbb{O}^{-1}\| \leq K$ with the constant K from Lemma 4.2, we can conclude that

$$\|\mathbf{y}\| \leq \|\mathbb{O}^{-1}\|\|\mathbb{O}\mathbf{y}\| \leq K\|\mathbb{O}\mathbf{y}\| \leq K\epsilon \leq \frac{1}{3} \tag{5.66}$$

using our choice $\epsilon = 1/(3K)$. On the other hand, let m be an index for which the maximum in the definition of $\xi(z_0)$ is attained. Then we have

$$\left|e^{v_m(z-z_\mathrm{M})L^d}\xi(z_0)^{-L^d}\right| = e^{\mathrm{Re}(v_m(z-z_0))L^d} \geq e^{-ML^dR_L} \geq \frac{1}{2}, \qquad m \in \mathcal{Q}, \tag{5.67}$$

according to the first condition in (5.56). Moreover, $q_m \geq 1$ and thus $\|y\| \geq \frac{1}{2}$ in contradiction to (5.66). Thus, (4.52) must hold for some $s(z_0) \in [R_L/q, R_L]$ once L exceeds L_5. $\square$

Lemma 4.7 is also quite similar to the corresponding statement (Lemma 4.5) from two-phase coexistence.

Proof of Lemma 4.7. We will prove the Lemma for $A_6 = 2e(C_0+3)(M+M^2)\|\mathbf{q}\|_1$, where M and C_0 are the constants from Assumption B.

Let c and L_4 be the constants from Corollary 5.2 for $\tilde{\kappa} = \kappa$. Since $z_\mathrm{M} \in \mathscr{O}$ is a multiple point with $\mathcal{Q}(z_\mathrm{M}) = \mathcal{Q}$, we clearly have that $z_\mathrm{M} \in \mathscr{U}_\epsilon(\mathcal{Q})$ whenever ϵ is small enough. Since $\mathscr{O}$ is open, we also have that $\mathbb{D}_s(z_\mathrm{M}) \subset \mathscr{O}$ whenever s is sufficiently small. As a consequence, there is a constant $\tilde{L}_6 = \tilde{L}_6(z_\mathrm{M})$ such that $z_\mathrm{M} \in \mathscr{U}_{2\kappa/L}(Q) \cap \mathscr{U}_{\kappa/2L}(Q)$ and $\mathbb{D}_{c/L}(z_\mathrm{M}) \subset \mathscr{O}$ whenever $L \geq \tilde{L}_6$. Using Corollary 5.2, we reach the conclusion that $\mathbb{D}_s(z_\mathrm{M}) \subset \mathscr{U}_{\kappa/L}(\mathcal{Q})$ whenever $L \geq \max\{\tilde{L}_6, L_4\}$ and $s \leq c/L$. We now choose $L_6 \geq \max\{\tilde{L}_6, L_4\}$ in such a way that

$$\begin{gathered}\rho_L' \leq c/L, \qquad \rho_L' \leq 2\rho_L, \qquad (1+2\rho_L)e^{-\tau L} \leq (M+M^2)\rho_L^2, \\ 4(M+M^2)\rho_L^2L^d \leq 1, \quad e^{ML^dR_L} \leq 2\end{gathered} \tag{5.68}$$

whenever $L \geq L_6$. By the above conclusion and the first condition in (5.68), we then have $\mathbb{D}_{\rho_L'}(z_\mathrm{M}) \subset \mathscr{U}_{\kappa/L}(\mathcal{Q})$ whenever $L \geq L_6$.

To prove (4.53), let us recall the definition of $\varXi_{\mathcal{Q},L}(z)$ in formula (2.13) from Assumption B4. Then we can write $\tilde{g}(z)$ as $\varXi_{\mathcal{Q},L}(z)\zeta(z_{\mathrm{M}})^{-L^d} + h(z)$, where

$$h(z) = \sum_{m\in\mathcal{Q}} q_m \left[\left(\frac{\zeta_m^{(L)}(z)}{\zeta(z_{\mathrm{M}})}\right)^{L^d} - e^{\mathrm{i}\phi_m(L)+\upsilon_m(z-z_{\mathrm{M}})L^d}\right]. \tag{5.69}$$

Our goal is to show that both $\varXi_{\mathcal{Q},L}(z)\zeta(z_{\mathrm{M}})^{-L^d}$ and $h(z)$ satisfy a bound of the type (4.53).

We will begin with the bound on $h(z)$. First we recall the definition of $\phi_m(L)$ to write

$$\left(\frac{\zeta_m^{(L)}(z)}{\zeta(z_{\mathrm{M}})}\right)^{L^d} = \left(\frac{\zeta_m^{(L)}(z)}{\zeta_m^{(L)}(z_{\mathrm{M}})}\right)^{L^d} \left(\frac{\zeta_m^{(L)}(z_{\mathrm{M}})}{\zeta_m(z_{\mathrm{M}})}\right)^{L^d} e^{\mathrm{i}\phi_m(L)}. \tag{5.70}$$

The first term on the right-hand side is to the leading order equal to $e^{b_m(z-z_{\mathrm{M}})L^d}$, which is approximately equal to $e^{\upsilon_m(z-z_{\mathrm{M}})L^d}$. To control the difference between these two terms, and to estimate the deviations from the leading order behavior, we combine the bound (2.10) with the second-order Taylor formula and (2.11) to show that, for all $z \in \mathbb{D}_{\rho'_L}(z_{\mathrm{M}})$ and all $m \in \mathcal{Q}$,

$$\left|\log\bigl(\zeta_m^{(L)}(z)/\zeta_m^{(L)}(z_{\mathrm{M}})\bigr) - \upsilon_m(z-z_{\mathrm{M}})\right| \le e^{-\tau L}\rho'_L + \frac{1}{2}(M+M^2)(\rho'_L)^2, \tag{5.71}$$

where we have chosen the principal branch of the complex logarithm. Combining this estimate with the second and third condition in (5.68) and the bound (2.9) from Assumption B2, we get

$$\left|L^d \log\bigl(\zeta_m^{(L)}(z)/\zeta^{(L)}(z_{\mathrm{M}})\bigr) - \upsilon_m(z-z_{\mathrm{M}})L^d - \mathrm{i}\phi_m(L)\right| \le 3(M+M^2)\rho_L^2 L^d. \tag{5.72}$$

Using the fourth condition in (5.68) and the fact that $|e^w - 1| \le e|w|$ whenever $|w| \le 1$, we get

$$\bigl|h(z)\bigr| \le 3e(M+M^2)\|\mathbf{q}\|_1\, L^d\rho_L^2\xi(z)^{L^d}. \tag{5.73}$$

Now $\xi(z)^{L^d} \le \xi(z_0)^{L^d}e^{ML^dR_L} \le 2\xi(z_0)^{L^d}$ by the fifth condition in (5.68), so we finally have the bound $|h(z)| \le A\xi(z_0)^{L^d}L^d\rho_L^2$, with A given by $A = 6e(M+M^2)\|\mathbf{q}\|_1$.

It remains to prove a corresponding bound for $\varXi_{\mathcal{Q},L}(z)\zeta(z_{\mathrm{M}})^{-L^d}$. First we recall our previous observation that $\mathbb{D}_{\rho'_L}(z_{\mathrm{M}}) \subset \mathscr{U}_{\kappa/L}(\mathcal{Q})$, so we have Assumption B4 at our disposal. Then (2.14) yields

$$\bigl|\varXi_{\mathcal{Q},L}(z)\zeta(z_{\mathrm{M}})^{-L^d}\bigr| \le C_0 L^d\|\mathbf{q}\|_1 e^{-\tau L}\left[\frac{\zeta(z)}{\zeta(z_{\mathrm{M}})}\right]^{L^d}, \qquad z \in \mathbb{D}_{\rho'_L}(z_{\mathrm{M}}). \tag{5.74}$$

Also, by the definition of $\mathscr{U}_{\kappa/L}(\mathcal{Q})$, we have that $\zeta(z) = \min_{m\in\mathcal{Q}}|\zeta_m(z)|$ whenever $z \in \mathbb{D}_{\rho'_L}(z_{\mathrm{M}})$. For $z \in \mathbb{D}_{\rho'_L}(z_{\mathrm{M}})$, we can therefore find a index $m \in \mathcal{Q}$ such that $|\zeta_m(z)| = \zeta(z)$. With the help of (5.3) and the bound (2.9) from Assumption B, we thus get

$$\left[\frac{\zeta(z)}{\zeta(z_{\mathrm{M}})}\right]^{L^d} \le \left|\frac{\zeta_m(z_0)}{\zeta(z_{\mathrm{M}})}\right|^{L^d}\left|\frac{\zeta_m(z)}{\zeta_m(z_0)}\right|^{L^d} \le \left|\frac{\zeta_m^{(L)}(z_0)}{\zeta(z_{\mathrm{M}})}\right|^{L^d} e^{MR_LL^d}e^{L^de^{-\tau}}. \tag{5.75}$$

Combined with the estimate (5.72) for $z = z_0$, and the last three conditions in (5.68), this gives

$$\left[\frac{\zeta(z)}{\zeta(z_\mathrm{M})}\right]^{L^d} \le e^{MR_L L^d} e^{L^d e^{-\tau}} e^{3(M+M^2)\rho_L^2 L^d} \xi(z_0)^{L^d} \le 2e\xi(z_0)^{L^d}. \tag{5.76}$$

Using the third condition in (5.68) one last time, we can bound the right-hand side (5.74) by $2eC_0\|\mathbf{q}\|_1(M+M^2)L^d\rho_L^2\xi(z_0)^{L^d}$. Combined with the above bound on $|h(z)|$, this finally proves (4.53). $\square$

Acknowledgement. The authors would like to thank A. Sokal for his interest in this work, R. Shrock for bringing the references [8, 9, 28, 29, 38, 39] to our attention, and A. van Enter for pointing out the reference [36]. M.B. and R.K. would like to acknowledge the hospitality of Microsoft Research in Redmond, where large parts of this work were carried out. The research of R.K. was partly supported by the grants GAČR 201/00/1149, 201/03/0478, and MSM 110000001. The authors also wish to thank anonymous referees for helpful suggestions concerning the presentation.

References

1. S. Beraha, J. Kahane and N.J. Weiss, *Limits of zeroes of recursively defined families of polynomials*, In G.-C. Rota (ed.), *Studies in Foundations and Combinatorics* (Advances in Mathematics Supplementary Studies, Vol. 1), pp. 213–232. Academic Press, New York, 1978.
2. M. Biskup, C. Borgs, J.T. Chayes, L.J. Kleinwaks, and R. Kotecký, *General theory of Lee-Yang zeros in models with first-order phase transitions*, Phys. Rev. Lett. **84:21** (2000) 4794–4797.
3. M. Biskup, C. Borgs, J.T. Chayes, and R. Kotecký, *Phase diagrams of Potts models in external fields: I. Real fields*, in preparation.
4. M. Biskup, C. Borgs, J.T. Chayes, and R. Kotecký, *Phase diagrams of Potts models in external fields: II. One complex field*, in preparation.
5. M. Biskup, C. Borgs, J.T. Chayes, and R. Kotecký, *Partition function zeros at first-order phase transitions: Pirogov-Sinai theory*, J. Statist. Phys. (to appear).
6. C. Borgs and J.Z. Imbrie, *A unified approach to phase diagrams in field theory and statistical mechanics*, Commun. Math. Phys. **123** (1989) 305–328.
7. C. Borgs and R. Kotecký, *A rigorous theory of finite-size scaling at first-order phase transitions*, J. Statist. Phys. **61** (1990) 79–119.
8. S.-C. Chang and R. Shrock, *Ground state entropy of the Potts antiferromagnet on strips of the square lattice*, Physica A **290** (2001) 402–430.
9. S.-C. Chang and R. Shrock, $T = 0$ *partition functions for Potts antiferromagnets on lattice strips with fully periodic boundary conditions*, Physica A **292** (2001) 307–345.
10. C.-N. Chen, C.-K. Hu, and F.Y. Wu, *Partition function zeros of the square lattice Potts model*, Phys. Rev. Lett. (1996) 169–172.
11. R.L. Dobrushin, *Estimates of semiinvariants for the Ising model at low temperatures*, In: R.L. Dobrushin et al. (eds), Topics in statistical and theoretical physics. F. A. Berezin memorial volume, Transl. Ser. 2, vol. 177(32), Amer. Math. Soc. (1996) pp. 59–81.
12. B.P. Doland and D.A. Johnston, *One dimensional Potts model, Lee-Yang edges, and chaos*, Phys. Rev. E **65** (2002) 057103.
13. H. Federer, *Geometric Measure Theory*, Springer-Verlag, Berlin, 1996.
14. M.E. Fisher, *The nature of critical points*, In: W.E. Brittin (ed), Lectures in Theoretical Physics, vol 7c (Statistical physics, weak interactions, field theory), University of Colorado Press, Boulder, 1965, pp. 1-159.
15. S. Friedli and C.-E. Pfister, *On the singularity of the free energy at first order phase transitions*, Commun. Math. Phys. **245** (2004) 69–103.
16. T.W. Gamelin, *Complex Analysis*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2001.
17. J.W. Gibbs, *Elementary Principles of Statistical Mechanics*, The Collected Works of J.W. Gibbs, vol. II, Yale University Press, New Haven, CT, 1948.
18. Z. Glumac and K. Uzelac, *The partition function zeros in the one-dimensional* q*-state Potts model*, J. Phys. A: Math. Gen. **27** (1994) 7709–7717.

19. S.N. Isakov, *Nonanalytic features of the first-order phase transition in the Ising model*, Commun. Math. Phys. **95** (1984), 427–443.
20. W. Janke and R. Kenna, *Phase transition strengths from the density of partition function zeroes*, Nucl. Phys. Proc. Suppl. **106** (2002) 905–907.
21. R. Kenna and C.B. Lang, *Scaling and density of Lee-Yang zeros in the four-dimensional Ising model*, Phys. Rev. E **49** (1994) 5012–5017.
22. S.-Y. Kim and R.J. Creswick, *Yang-Lee zeros of the q-state Potts model in the complex magnetic field plane*, Phys. Rev. Lett. **81** (1998) 2000–2003.
23. R. Kotecký and D. Preiss, *Cluster expansion for abstract polymer models*, Commun. Math. Phys. **103** (1986) 491–498.
24. K.-C. Lee, *Generalized circle theorem on zeros of partition functions at asymmetric first-order transitions*, Phys. Rev. Lett. **73** (1994) 2801–2804.
25. T.D. Lee and C.N. Yang, *Statistical theory of equations of state and phase transitions: II. Lattice gas and Ising model*, Phys. Rev. **87** (1952) 410–419.
26. E.H. Lieb and A.D. Sokal, *A general Lee-Yang theorem for one-component and multi-component ferromagnets*, Commun. Math. Phys. **80** (1981) 153–179.
27. W.T. Lu and F.Y. Wu, *Partition function zeroes of a self-dual Ising model*, Physica A **258** (1998) 157–170.
28. V. Matveev and R. Shrock, *Complex-temperature properties of the Ising model on 2D heteropolygonal lattices*, J. Phys. A: Math. Gen. **28** (1995) 5235–5256.
29. V. Matveev and R. Shrock, *Some new results on Yang-Lee zeros of the Ising model partition function*, Phys. Lett. **A 215** (1996) 271–279.
30. J.W. Milnor, *Topology from the Differentiable Viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997.
31. H. Nashimori and R.B. Griffiths, *Structure and motion of the Lee-Yang zeros*, J. Math. Phys. **24** (1983) 2637–2647.
32. C.M. Newman, *Zeros of the partition function for generalized Ising systems*, Commun. Pure Appl. Math. **27** (1974) 143–159.
33. S.A. Pirogov and Ya.G. Sinaĭ, *Phase diagrams of classical lattice systems* (Russian), Theor. Math. Phys. **25** (1975) no. 3, 358–369.
34. S.A. Pirogov and Ya.G. Sinaĭ, *Phase diagrams of classical lattice systems. Continuation* (Russian), Theor. Math. Phys. **26** (1976), no. 1, 61–76.
35. D. Ruelle, *Extension of the Lee-Yang circle theorem*, Phys. Rev. Lett. **26** (1971) 303–304.
36. W. van Saarloos and D.A. Kurtze, *Location of zeros in the complex temperature plane: Absence of Lee-Yang theorem*, J. Phys. A: Math. Gen. **18** (1984) 1301–1311.
37. J. Salas and A.D. Sokal, *Transfer matrices and partition-function zeros for antiferromagnetic Potts models. I. General theory and square-lattice chromatic polynomial*, J. Statist. Phys. **104** (2001), no. 3-4, 609–699.
38. R. Shrock, *Exact Potts model partition functions on ladder graphs*, Physica A **283** (2000) 388–446.
39. R. Shrock and S.-H. Tsai, *Exact partition functions for Potts antiferromagnets on cyclic lattice strips*, Physica A **275** (2000) 429–449.
40. A.D. Sokal, *Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions*, Combin. Probab. Comput. **10** (2001), no. 1, 41–77.
41. A.D. Sokal, *Chromatic roots are dense in the whole complex plane*, Combin. Probab. Comput. (to appear).
42. C.N. Yang and T.D. Lee, *Statistical theory of equations of state and phase transitions: I. Theory of condensation*, Phys. Rev. **87** (1952) 404-409.
43. M. Zahradník, *An alternate version of Pirogov-Sinai theory*, Commun. Math. Phys. **93** (1984) 559–581.
44. M. Zahradník, *Analyticity of low-temperature phase diagrams of lattice spin models*, J. Statist. Phys. **47** (1987) 725–455.

Communicated by M. Aizenman