# SHARP ASYMPTOTIC FOR THE CHEMICAL DISTANCE IN LONG-RANGE PERCOLATION

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**Abstract:** We consider instances of long-range percolation on $\mathbb{Z}^d$ and $\mathbb{R}^d$, where points at distance $r$ get connected by an edge with probability proportional to $r^{-s}$, for $s \in (d, 2d)$, and study the asymptotic of the graph-theoretical (a.k.a. chemical) distance $D(x,y)$ between $x$ and $y$ in the limit as $|x-y| \to \infty$. For the model on $\mathbb{R}^d$ we show that, in probability as $r \to \infty$ for any nonzero $x \in \mathbb{R}^d$, the distance $D(0,xr)$ is asymptotic to $\phi(r)(\log r)^\Delta$, where $\phi$ is a positive, continuous function obeying $\phi(r^\gamma) = \phi(r)$, with $\gamma := s/(2d)$, for all $r > 1$, and $\Delta := 1/\log_2(1/\gamma)$. For the model on $\mathbb{Z}^d$ we show that $D(0,xr)$ is with probability tending to one squeezed between two positive multiples of $(\log r)^\Delta$. The proof of the asymptotic scaling is based on a subadditive argument along a continuum of doubly-exponential sequences of scales. The results strengthen considerably the conclusions obtained earlier by the first author. Still, significant open questions remain.

## 1. Introduction

### 1.1 The model and main results.

Long-range percolation is a tool to expand connectivity of a given graph by adding, at random, edges between far-away vertices. Although arising from questions in mathematical physics (Dyson [15], Fröhlich and Spencer [16]), the problem was recognized quickly to pose interesting challenges for probability (Schulman [21], Newman and Schulman [20], Aizenman and Newman [1], Aizenman, Kesten and Newman [2]). More recently, instances of long-range percolation have been used as an ambient medium for other stochastic processes (e.g., Berger [6], Benjamini, Berger and Yadin [4], Crawford and Sly [11, 12], Misumi [19], Kumagai and Misumi [17]). The overarching theme here is the geometry of random networks.

In this paper we consider two models of long-range percolation on $\mathbb{R}^d$. One of these is set on the hypercubic lattice $\mathbb{Z}^d$ (endowed, *a priori*, with its nearest-neighbor edge structure) augmented by adding an edge between any non-neighboring vertices $x$ and $y$ with probability

$$p_{x,y} := 1 - \exp\{-\beta|x-y|^{-s}\} \tag{1.1}$$

independently of all other edges. Here $\beta > 0$ and $s > 0$ are parameters while $|\cdot|$ is any norm of choice. Our main point of interest is the behavior of the graph-theoretical distance $D^{\text{dis}}(x,y)$, defined as the minimal number of edges used in any path in that connects $x$ to $y$, in the limit as the separation between $x$ and $y$ tends to infinity.

The question of distance scaling in long-range percolation has been studied quite intensely in the past and this has revealed five regimes of typical behavior, $s < d$, $s = d$, $d < s < 2d$, $s = 2d$ and $s > 2d$, with rather different kinds of asymptotic behavior. Deferring the discussion of the specifics and requisite references until the end of this section, let us focus attention directly on the regime $d < s < 2d$. Here the first author [8,9] showed

$$D^{\text{dis}}(0,x) = (\log|x|)^{\Delta+o(1)}, \qquad |x| \to \infty, \tag{1.2}$$

where

$$\Delta := \frac{1}{\log_2(1/\gamma)} \quad \text{for} \quad \gamma := \frac{s}{2d}, \tag{1.3}$$

and where $o(1) \to 0$ in probability. The proof worked for more general connection probabilities than (1.1); in fact, it was enough that $p_{xy} = |x-y|^{-s+o(1)}$.

The question we wish to resolve here is whether assuming the "perfect" scaling (1.1) yields a sharper version of the asymptotic (1.2). Our first result in this regard is the subject of:

**Theorem 1.1** *Consider the long-range percolation on $\mathbb{Z}^d$ with connection probabilities* (1.1) *for $\beta > 0$ and $s \in (d,2d)$ and let $D^{\text{dis}}(x,y)$ denote the chemical distance between $x$ and $y$. There are $c, C \in (0,\infty)$ depending only on $\beta$, $s$ and the underlying norm $|\cdot|$ such that*

$$\lim_{|x|\to\infty} P\bigl(c(\log|x|)^\Delta \le D^{\text{dis}}(0,x) \le C(\log|x|)^\Delta\bigr) = 1, \tag{1.4}$$

*where $\Delta$ is as in* (1.3).

As soon as we accept the bound (1.4), a natural next step is the consideration of possible distributional limits of $D^{\text{dis}}(0,x)/(\log|x|)^\Delta$ as $|x| \to \infty$. We are able to argue that if a distributional limit exists along a particular lattice direction, then it has to be non-random. Unfortunately, the proof of existence of the limit remains elusive, despite multiple attempts. Ultimately, we were led to the consideration of a model on $\mathbb{R}^d$ where progress could eventually be made.

To define long-range percolation over $\mathbb{R}^d$, fix $\beta > 0$ and consider a sample $\mathscr{I}_\beta$ from the Poisson process on $\mathbb{R}^d \times \mathbb{R}^d$ with ($\sigma$-finite) intensity measure

$$\mu_{s,\beta}(\mathrm{d}x\,\mathrm{d}y) := 1_{\{|x|_2<|y|_2\}} \frac{\beta}{|x-y|^s} \mathrm{d}x\,\mathrm{d}y, \tag{1.5}$$

where $|\cdot|$ is the norm from (1.1) while $|\cdot|_2$ is, here and henceforth, the Euclidean norm on $\mathbb{R}^d$. Let us write $\text{Sym}(\mathscr{I}_\beta) := \mathscr{I}_\beta \cup \{(y,x) \colon (x,y) \in \mathscr{I}_\beta\}$ for the symmetrized version of $\mathscr{I}_\beta$. We regard a "point" $(x,y) \in \text{Sym}(\mathscr{I}_\beta)$ as an *undirected edge* connecting the two points. Given $x, y \in \mathbb{R}^d$, we then proclaim

$$D(x,y) := \inf\Bigl\{n + \sum_{i=0}^n |x_{i+1} - y_i| \colon n \ge 0,\, \{(x_i,y_i) \colon i = 1,\dots,n\} \subset \text{Sym}(\mathscr{I}_\beta)\Bigr\} \tag{1.6}$$

with the convention $y_0 := x$ and $x_{n+1} := y$, to be the *chemical distance* between points $x, y \in \mathbb{R}^d$ in the graph with edges $\mathscr{I}_\beta$.

We will at times refer to the sequence $\{(x_i,y_i) \colon i = 1,\dots,n\}$ as a *path* and call $x_{i+1} - y_i$ the $i$-th *linear segment*. Note that the infimum is over a non-empty set as the empty path, i.e., the one with $n = 0$, is always included. Note also that edges $(x_i,y_i)$ with $|x_i - y_i| < 1$ need not be considered as their removal decreases (thanks to the triangle inequality for the norm $|\cdot|$) the expression in the infimum. The main result of the present paper is then:

**Theorem 1.2** *For each $s \in (d, 2d)$, each $\beta > 0$ and each choice of the norm $|\cdot|$ there is a positive and continuous function $\phi\colon (1,\infty) \to (0,\infty)$ satisfying*

$$\phi(r^\gamma) = \phi(r), \quad r \geq 1, \tag{1.7}$$

*with $\gamma$ as in* (1.3)*, such that for each $x \in \mathbb{R}^d \smallsetminus \{0\}$,*

$$\frac{D(0,rx)}{\phi(r)(\log r)^\Delta} \underset{r\to\infty}{\longrightarrow} 1, \quad \textit{in probability}. \tag{1.8}$$

*Moreover, $t \mapsto \phi(\mathrm{e}^t)t^\Delta$ is convex throughout* $[0,\infty)$.

Note that we claim existence and continuity of $\phi$ on $(1,\infty)$ only. Actually, $\phi$ has a continuous extension to $r = 1$ if and only if it is constant.

### 1.2 Remarks and open questions.

We continue with some remarks and open questions. First we note that the mode of convergence in (1.8) cannot generally be improved to almost sure. This is best seen in $d = 1$ by the following argument: A ball of radius $r^\gamma$ centered at the origin will meet an edge of length of order $r$ (in fact, even up to lengths $r^{1/(2\gamma-1)}$) with a uniformly positive probability. If we parametrize the nearer endpoint of this edge as $r^\gamma x$ and write $ry$ for the farther endpoint of this edge, then the assumption of a.s. convergence in Theorem 1.2 would tell us

$$\begin{aligned} D(0,ry) &\leq 1 + D(0, r^\gamma x) \\ &= 1 + \phi(r^\gamma)(\log r^\gamma)^\Delta\bigl(1+o(1)\bigr) \\ &= \frac{1}{2}\phi(r)(\log r)^\Delta\bigl(1+o(1)\bigr), \end{aligned} \tag{1.9}$$

a contradiction with our very assumption. (We used (1.7) and the fact that $\gamma^\Delta = \frac{1}{2}$.)

Next, although this may not be quite apparent at first sight, the distance on $\mathbb{R}^d$ is actually quite closely related to the chemical distance on $\mathbb{Z}^d$. Indeed, replacing the Lebesgue measure on the right of (1.5) by the counting measure on $\mathbb{Z}^d$, the case when $|\cdot|$ is the $\ell^1$-norm on $\mathbb{R}^d$ reduces *exactly* to distance $D^{\,\mathrm{dis}}(x,y)$ with $x, y \in \mathbb{Z}^d$ connected with probability $p_{x,y}$ as in (1.1). However, this does not seem to help in extending the sharp asymptotic (1.8) to the model on $\mathbb{Z}^d$.

Another remark concerns the function $\phi$ which encodes the dependence of the limit on $\beta$ and the underlying norm $|\cdot|$. We in fact believe:

**Conjecture 1.3** *The function $\phi$ above is constant for each $\beta > 0$.*

This is because $\phi$ seems to appear largely as an artifact of our method which uses subadditivity arguments to relate the chemical distances at scales of the form $\{r^{\gamma^{-n}}\colon n \geq 1\}$ for a fixed choice of $r > 1$. The growth rates of this sequence for two distinct $r, r' \in [\mathrm{e}^\gamma, \mathrm{e})$ are so incommensurate that the same proof would apply even if the intensity measure (1.5) were modulated depending on which of the two sequences $|x-y|$ is closer to. In that situation, we would actually *not* expect the corresponding $\phi$ to take the same value at $r$ and $r'$.

The dependence of $D(0,rx)$ on $x$ is another interesting problem. As shown in Theorem 1.2, there is no such dependence in the leading order. Still, regardless on how the above conjecture gets resolved, formal expansions suggest:

**Conjecture 1.4** *For any $x \neq 0$ we have*

$$D(0,rx) = \phi(r)(\log r)^{\Delta} + \bigl(1+o(1)\bigr)\psi(r)(\log|x|)(\log r)^{\Delta-1} \tag{1.10}$$

*where $o(1) \to 0$ in probability as $r \to \infty$ and where $\psi$ is again a positive and continuous function satisfying the kind of "periodicity" requirement* (1.7).

This would in particular imply that balls in the chemical distance are close to those in the norm $|\cdot|$. However, at this point we lack good ideas how to tackle this question rigorously.

### 1.3 Earlier work and connections.

We will now give the promised connections to the existing literature on the scaling of the chemical distance in long-range percolation on $\mathbb{Z}^d$. In the regime $s < d$ the chemical distance approaches a deterministic finite number at large spatial scales; namely, $\lceil \frac{d}{s-d} \rceil$ (Benjamini, Kesten, Peres and Schramm [5]). When $s = d$, the chemical distance between points at Euclidean distance $N$ grows as $(\log N)/\log\log N$ (Coppersmith, Gamarnik and Sviridenko [10]) while, as already mentioned, for $d < s < 2d$ we get (1.2) (Biskup [8, 9]). For $s > 2d$, the chemical distance resumes linear scaling with the Euclidean distance (Berger [7]). These asymptotics extend even to some inhomogenous versions of long-range percolation (Deprez, Hazra and Wüttrich [13]).

The most interesting case is that of $s = 2d$, where the model is scale invariant. Some aspects of the $d = 1$ situation have been clarified already by Benjamini and Berger [3] but it was not until recently that Ding and Sly [14] established the existence of an exponent $\theta(\beta) \in (0,1)$ such that $D(0,N) \asymp N^{\theta(\beta)}$ in $d = 1$. Interestingly, also here subadditivity arguments play a prominent role. The existence of a sharp asymptotic in this case remains open.

### 1.4 Outline.

The rest of this note is organized as follows. In Section 2 we define the notion of a restricted distance $\widetilde{D}$ and show (in Proposition 2.7) that it obeys a stochastic subadditivity bound that will drive all subsequent derivations in this paper. This bound involves $\widetilde{D}$ at randomized locations and so its recursive use naturally leads, in Section 3, to the consideration of a random variable $W$ that is a fixed point for the randomized locations. This closes the recursion and permits extraction (in Proposition 3.7) of the limit asymptotic of $r \mapsto \widetilde{D}(0,rW)$. The key technical steps in this are finiteness (derived in Lemma 3.4) of the sum of conditional variances (given $W$) of $2^{-n}\widetilde{D}(0,r^{\gamma^{-n}}W)$ for any $r \geq 1$ and the use of Dini's theorem to obtain uniformity of these estimates in $r$. In Section 4 we then show that the same asymptotic applies to distance $D$ as well.

## 2. Restricted distance

We are now ready to commence the proofs. A majority of the work will be done directly for the model on $\mathbb{R}^d$ although we do use the model on $\mathbb{Z}^d$ in the proof of positivity of $\phi$. In this section we focus on an auxiliary quantity, called the restricted distance, that is better behaved under subadditivity arguments. We will return to the full distance in Section 4.

**2.1 Definition and comparisons.**

Let us write $B(x,r) := \{y \in \mathbb{R}^d\colon |x-y| < r\}$ for the open ball in the norm $|\cdot|$. Given $x,y \in \mathbb{R}^d$ we then define their *restricted distance* by constraining the infimum to paths that do not leave the ball $B(x,2|x-y|)$, i.e.,

$$\widetilde{D}(x,y) := \inf\left\{ n + \sum_{i=0}^{n} |x_{i+1} - y_i| \colon \begin{array}{l} n \ge 0,\, \{(x_i,y_i)\colon i = 1,\dots,n\} \subset \mathscr{I}_\beta \\ x_i, y_i \in B\bigl(x, 2|x-y|\bigr)\, \forall i = 1,\dots,n \end{array} \right\}, \tag{2.1}$$

where, as before, we set $y_0 := x$ and $x_{n+1} := y$. We caution the reader that $\widetilde{D}(\cdot,\cdot)$ is not a metric as it is neither symmetric nor obeying the triangle inequality. The following properties of the restricted distance will be important in the sequel:

**Lemma 2.1** *Let $D(x,y)$ be as in* (1.6) *and $\widetilde{D}(x,y)$ as in* (2.1). *Then*

(1) *for any $x,y \in \mathbb{R}^d$,*
$$D(x,y) \le \widetilde{D}(x,y) \le |x-y|, \tag{2.2}$$
(2) *the law of $\widetilde{D}$ is translation invariant,*
$$\bigl\{\widetilde{D}(x,y)\colon x,y \in \mathbb{R}^d\bigr\} \overset{\text{law}}{=} \bigl\{\widetilde{D}(x+z,y+z)\colon x,y \in \mathbb{R}^d\bigr\}, \qquad z \in \mathbb{R}^d, \tag{2.3}$$
(3) *$x \mapsto \widetilde{D}(0,x)$ is stochastically continuous in $x$ (i.e., the law of $\widetilde{D}(0,x)$ is continuous in $x$ in the topology of weak convergence of measures), and*

(4) *for any $x,y,\tilde{x},\tilde{y} \in \mathbb{R}^d$*
$$|x-\tilde{x}| > 2|x-y| + 2|\tilde{x}-\tilde{y}| \quad\Rightarrow\quad \widetilde{D}(x,y) \perp\!\!\!\perp \widetilde{D}(\tilde{x},\tilde{y}) \tag{2.4}$$

*Proof.* The inequalities (2.2) are checked by comparison of (1.6) with (2.1) and the fact that the path with no edges is included on the right of (2.1). The translation invariance in (2) is a consequence of the corresponding property of the intensity measure (1.5).

In order to prove (3), consider a path minimizing $\widetilde{D}(0,x)$. (Such a path exists as $B(0,2|x|)$ contains only a finite number of edges of length in excess of one, a.s.) The continuity of the law of the underlying point process and the fact that $B(0,2|x|)$ is open ensure that the minimizing path is a.s. unique and that the same sequence of edges are used by the minimizer of $\widetilde{D}(0,x+z)$ for all $|z|$ sufficiently small. It follows that, for every $y \in \mathbb{R}^d$, $x \mapsto \widetilde{D}(0,x)$ is continuous at $x := y$ a.s. This yields stochastic continuity via the Bounded Convergence Theorem.

The independence claimed in (2.4) follows from the independence of Poisson processes over disjoint sets. □

Let us write $D_\beta$ if need arises to mark explicitly the dependence of the law random variable $D$ on $\beta$. The following comparisons then hold:

**Lemma 2.2** *For all $\beta > 0$, all $a \ge 1$ and all $x \in \mathbb{R}^d$,*
$$D_\beta(0,ax) \overset{\text{law}}{\le} D_{a^{s-2d}\beta}(0,ax) \overset{\text{law}}{\le} aD_\beta(0,x). \tag{2.5}$$
*The same conclusions apply to the restricted distance $\widetilde{D}_\beta$ as well.*

*Proof.* Let $\mathscr{I}_\beta = \{(x_i,y_i)\colon i \in \mathbb{N}\}$ denote a sample from the point process on $\mathbb{R}^d \times \mathbb{R}^d$ with intensity measure (1.5). For any $a > 0$, the process $\mathscr{I}'_\beta := \{(ax_i,ay_i)\colon i \in \mathbb{N}\}$ is then equidistributed to $\mathscr{I}_{\beta(a)}$ where $\beta(a) := a^{s-2d}\beta$. Pick a path $\pi$ connecting 0 to $x$ using the edges in $\mathrm{Sym}(\mathscr{I}_\beta)$ and let $n(\pi)$ denote the number of edges and $\rho(\pi)$ the total length of the linear segments in $\pi$. Now consider the path $\pi'$ built using the corresponding edges in $\mathrm{Sym}(\mathscr{I}_\beta')$, and let $n(\pi')$ and $\rho(\pi')$ denote the corresponding quantities for $\pi'$. Then

$$n(\pi') = n(\pi) \quad \text{and} \quad \rho(\pi') = a\rho(\pi). \tag{2.6}$$

Assuming $a \geq 1$, it follows that

$$n(\pi') + \rho(\pi') \leq a\bigl[n(\pi) + \rho(\pi)\bigr]. \tag{2.7}$$

The left-hand side is bounded by $D(0,ax)$ from below; optimizing over $\pi$ then implies the inequality on the right of (2.5). The left inequality in (2.5) is a consequence of Poisson thinning: As $\beta(a) \leq \beta$ for $a \geq 1$, the process with parameter $\beta(a)$ can be realized as a pointwise subset of the process with parameter $\beta$. In this coupling, every path contributing to $\widetilde{D}_{\beta(a)}(0,ax)$ will contribute to $\widetilde{D}_\beta(0,ax)$ as well.

The inequalities extend to $\widetilde{D}$ as the additional restriction imposed on paths there scales proportionally to the distance between endpoints. □

For distance $D$ we can also get comparisons under rotations:

**Lemma 2.3** *For each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in \mathbb{R}^d \smallsetminus \{0\}$,*

$$|x|_2 = |y|_2 \quad \& \quad \frac{|x-y|_2}{|x|_2} < \delta \quad \Rightarrow \quad D_{(1+\varepsilon)\beta}(0,x) \overset{\text{law}}{\leq} (1+\varepsilon)D_\beta(0,y). \tag{2.8}$$

*Proof.* Thanks to all norms on $\mathbb{R}^d$ being continuous with respect to one another, for each $\varepsilon > 0$ there is $\delta > 0$ such that for any rotation $R \in \mathrm{SO}(d)$ which is close to the identity in the sense that $|Rx - x|_2 < \delta|x|_2$ for all non-zero $x \in \mathbb{R}^d$, we have

$$(1+\varepsilon)|x| \geq |Rx| \geq (1+\varepsilon)^{-1/s}|x|, \qquad x \in \mathbb{R}^d \smallsetminus \{0\}. \tag{2.9}$$

The inequality on the right shows that $\mu_{s,\beta(1+\varepsilon)} - \mu_{s,\beta} \circ R^{-1}$ is a positive measure. The additivity of Poisson processes implies that a sample $\mathscr{I}_{\beta(1+\varepsilon)}$ from the Poisson process with the intensity $\mu_{s,\beta(1+\varepsilon)}$ contains a sample $\mathscr{I}_\beta'$ from the process with intensity $\mu_{s,\beta}$ rotated by $R$. Pick a path $\pi'$ using the edges in $\mathrm{Sym}(\mathscr{I}_\beta')$ from 0 to $R^{-1}x$ and let $\pi$ be its rotation by $R$. Then, in the notation from the previous proof, $n(\pi) = n(\pi')$ while, by the left inequality in (2.9), $\rho(\pi) \leq (1+\varepsilon)\rho(\pi')$. Optimizing over $\pi'$ we get

$$D_{\beta(1+\varepsilon)}(0,x) \overset{\text{law}}{\leq} (1+\varepsilon)D_\beta(0,R^{-1}x) \tag{2.10}$$

for every $x \in \mathbb{R}^d$. Realizing $y$ as $R^{-1}x$, this yields (2.8). □

We can even get comparisons with the distance on $\mathbb{Z}^d$, writing again $D_\beta^{\text{dis}}$ to denote the distance on $\mathbb{Z}^d$ with connection probabilities (1.1) for parameter $\beta$:

**Lemma 2.4** *For each $\beta > 0$ there is $c = c(\beta) \in (0,1]$ such that for all $x, y \in \mathbb{Z}^d$,*

$$cD^{\text{dis}}_{c^{-1}\beta}(x,y) \overset{\text{law}}{\leq} D_\beta(x,y). \tag{2.11}$$

*Proof.* Denote $B := [-1/2, 1/2)^d$ and define the coupling of the process on $\mathbb{Z}^d$ and the process on $\mathbb{R}^d$ as follows. Given a sample $\mathscr{I}_\beta$ from the Poisson process with intensity $\mu_{s,\beta}$, place an edge between distinct non-nearest neighbors $x \in \mathbb{Z}^d$ and $y \in \mathbb{Z}^d$ whenever there is an edge $(x', y') \in \mathrm{Sym}(\mathscr{I}_\beta)$ with $x' - x, y' - y \in B$. Distinct vertices $x, y \in \mathbb{Z}^d$ are then connected by an edge with probability

$$1 - \exp\Big\{-\beta \int_{B\times B} \frac{\mathrm{d}z\,\mathrm{d}z'}{|x-y+z-z'|^s}\Big\} \tag{2.12}$$

independent of all other edges. Note that (since $s > d$) the integral diverges for any two $x, y \in \mathbb{Z}^d$ within $\ell^\infty$-distance one which ensures these points are connected almost surely. As is readily checked, the resulting process on $\mathbb{Z}^d$ stochastically dominates the process defined in (1.1) with $\beta$ multiplied by a sufficiently large constant.

Now consider a path $\pi$ contributing to $D_\beta(x,y)$ and use the above coupling to project it to a path $\pi'$ on $\mathbb{Z}^d$ while replacing each linear segments of $\pi$ by a shortest nearest-neighbor path on $\mathbb{Z}^d$ between the corresponding vertices on $\mathbb{Z}^d$. An edge in $\pi$ then gives rise to an edge in $\pi'$ or no edge at all. A linear segment in $\pi$ of length $L$ corresponds to a "segment" on $\mathbb{Z}^d$ of $\ell^1$-distance $L'$ between the endpoints or no segment at all. The fact that the $\ell^1$-distance is comparable with the norm $|\cdot|$ ensures that $L \ge cL'$ for some $c > 0$ small enough. The claim then follows. □

We will find the lower bound by distance on $\mathbb{Z}^d$ particularly useful in light of the following result by the first author that itself draws on earlier work by Trapman [22]:

**Theorem 2.5** *For each $\beta > 0$ there are $c_1, c_2 \in (0,\infty)$ such that for all $n \ge 1$ and all $x \in \mathbb{Z}^d$,*

$$P\big(D^{\,\mathrm{dis}}(0,x) \le n\big) \le c_1 \frac{\mathrm{e}^{c_2 n^{1/\Delta}}}{|x|^s}\,. \tag{2.13}$$

*Proof.* This is proved by following, nearly verbatim, the proof of [9, Theorem 3.1] while setting $s' := s$ and $\Delta' := \Delta$. (Note that $s'$ is introduced in [9] in order to reduce the asymptotic form $|x-y|^{-s+o(1)}$ assumed there for $p_{xy}$ to the sharp asymptotic (1.1) with $s'$ instead of $s$. The rest of the proof then uses the sharp asymptotic form only.) □

From here we get one half of Theorem 1.1 of the present paper:

**Corollary 2.6** *For each $\beta > 0$ there is $c = c(\beta) > 0$ such that*

$$\lim_{|x|\to\infty} P\big(D^{\,\mathrm{dis}}(0,x) \le c(\log|x|)^\Delta\big) = 0. \tag{2.14}$$

*Proof.* Substitute $n := c(\log|x|)^\Delta$ into (2.13) and observe that, thanks to $s > d$, the resulting probability is summable on $x \in \mathbb{Z}^d$ once $c$ is sufficiently small. This implies the claim. □

### 2.2 Subadditivity bound.

Our next task is to derive a subadditivity relation for the restricted distance. This relation will play a fundamental role in all derivations to come. We remark that $D$, being a metric, satisfies the "ordinary" subadditivity estimate

$$D\big(0,(n+m)x\big) \le D(0,nx) + D\big(nx,(n+m)x\big). \tag{2.15}$$

However, this estimate is not useful for our purposes because $n \mapsto D(0,nx)$ turns out to be sublinear a.s. Our subadditivity bound will thus have to be tailored to the polylogarithmic growth of $x \mapsto D(0,x)$. It will also be derived only for the restricted distance because that, unlike $D$, obeys the independence statement in Lemma 2.1(4).

**Proposition 2.7** (Subadditivity for restricted distance) *Fix $\eta \in (0,1)$ and let $Z,Z'$ be i.i.d. $\mathbb{R}^d$-valued random variables with common law*

$$P(Z \in B) = \sqrt{\eta\beta}\int_B \mathrm{e}^{-\eta\beta c_0|z|^{2d}}\mathrm{d}z, \tag{2.16}$$

*where*

$$c_0 := \int_{|z|^{2d}+|\tilde{z}|^{2d}\le 1}\mathrm{d}z\,\mathrm{d}\tilde{z}. \tag{2.17}$$

*Let $\widetilde{D}'$ be an independent copy of $\widetilde{D}$ with both quantities independent of $Z$ and $Z'$. For each $\gamma_1,\gamma_2 \in (0,\frac{1}{2}(1+\gamma))$ with $\gamma_1+\gamma_2 = 2\gamma$, there are $c_1,c_2 \in (0,\infty)$ and, for each $x \in \mathbb{R}^d$, there is an event $A(x) \in \sigma(Z,Z')$ with*

$$P\big(A(x)\big) \le c_1\mathrm{e}^{-c_2|x|^\theta} \tag{2.18}$$

*for $\theta := 2d[\frac{1+\gamma}{2} - \gamma_1 \vee \gamma_2]$ such that*

$$\widetilde{D}(0,x) \overset{\text{law}}{\le} \widetilde{D}\big(0,|x|^{\gamma_1}Z\big) + \widetilde{D}'\big(0,|x|^{\gamma_2}Z'\big) + 1 + |x|1_{A(x)} \tag{2.19}$$

*holds true for every $x \in \mathbb{R}^d$.*

**Remark 2.8** It may not appear at all obvious that the above choice of $c_0$ makes (2.16) a probability; this will be seen from formula (2.25) in the proof below. We will use this proposition mostly in the case when $\gamma_1 = \gamma_2 = \gamma$. The main reason for our consideration of the more general setting is the proof of continuity of the limit in Theorem 1.2 which requires (small but non-trivial) perturbations about the symmetric case as well. The choice of $\eta$ will be immaterial in what follows. We will therefore suppress $\eta$ from the notation wherever possible.

*Proof of Proposition 2.7.* The main idea of the proof is simple: We first pick an edge $(X,Y)$, with $X$ closest to 0 and $Y$ closest to $x$ according to criteria to be specified later. Then we pick a shortest path from 0 to $X$ and a shortest path from $x$ to $Y$, demanding in addition that the first path stay in $2|X|^{\gamma_1}$-neighborhood of 0 and the second in $2|x-Y|^{\gamma_2}$-neighborhood of $x$. Assuming $|x| \gg 1$, concatenating the two paths with $(X,Y)$ yields a path from 0 to $x$ not leaving $2|x|$-neighborhood of 0. Writing $Z$ for $|x|^{-\gamma_1}X$ and $Z'$ for $|x|^{-\gamma_2}(Y-x)$, a pointwise version of (2.19) follows. A key technical point is to choose the selection criteria for $(X,Y)$ to ensure independence of $Z$ and $Z'$ and (conditionally on $X$ and $Y$) the distances $\widetilde{D}(0,X)$ and $\widetilde{D}(x,Y)$.

Fix $\eta \in (0,1)$. There is nothing to prove when $x = 0$ so let us also assume that $x \in \mathbb{R}^d \smallsetminus \{0\}$. Recall that $a \vee b$ denotes $\max\{a,b\}$. The proof comes in three steps.

*STEP 1: Construction of $(X,Y)$*: We begin by constructing the aforementioned edge. Note that, for any $\tilde{x},\tilde{y} \in \mathbb{R}^d$ with $|\tilde{x}| \vee |\tilde{y}-x| \le |x|^{\frac{1}{2}(1+\gamma)}$ we have

$$|\tilde{x}-\tilde{y}| \le |x+\tilde{x}+\tilde{y}-x| \le |x| + 2|x|^{\frac{1}{2}(1+\gamma)} = |x|\big(1+2|x|^{-\frac{1}{2}(1-\gamma)}\big). \tag{2.20}$$

Recalling the intensity measure $\mu_{s,\beta}$ from (1.5), define

$$\mu'_{s,\beta}(\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}) := \eta\beta 1_{\{|\tilde{x}|_2<|\tilde{y}|_2\}} 1_{\left\{|\tilde{x}|\vee|\tilde{y}-x|\le|x|^{\frac{1}{2}(1+\gamma)}\right\}} \frac{\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}}{|x|^s}. \tag{2.21}$$

Then, as soon as $x$ is so large that $1+2|x|^{-\frac{1}{2}(1-\gamma)} \le \eta^{-1/s}$ (recall that $\gamma\in(0,1)$ and $\eta<1$), the inequality (2.20) ensures that $\mu''_{s,\beta} := \mu_{s,\beta}-\mu'_{s,\beta}$ is a positive measure. This permits us to represent $\mathscr{I}_\beta$ as the sum of two independent Poisson processes $\mathscr{I}'_\beta$ and $\mathscr{I}''_\beta$ with intensities $\mu'_{s,\beta}$ and $\mu''_{s,\beta}$, respectively. Considering also the measure

$$\mu'''_{s,\beta}(\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}) := \eta\beta\left(1-1_{\{|\tilde{x}|_2<|\tilde{y}|_2\}} 1_{\left\{|\tilde{x}|\vee|\tilde{y}-x|\le|x|^{\frac{1}{2}(1+\gamma)}\right\}}\right) \frac{\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}}{|x|^s}, \tag{2.22}$$

let $\mathscr{I}'''_\beta$ denote a sample of the Poisson process with intensity $\mu'''_{s,\beta}$. We regard $\mathscr{I}'_\beta$, $\mathscr{I}''_\beta$ and $\mathscr{I}'''_\beta$ as independent of one another.

As is directly checked from (2.21–2.22), $\mathscr{I}'_\beta\cup\mathscr{I}'''_\beta$ is a homogeneous Poisson process on $\mathbb{R}^d\times\mathbb{R}^d$ with density $\eta\beta|x|^{-s}$ and so, in particular, $\mathscr{I}'_\beta\cup\mathscr{I}'''_\beta\neq\emptyset$ a.s. The process is also locally finite and so there is (a.s.) a unique pair $(X,Y)\in\mathscr{I}'_\beta\cup\mathscr{I}'''_\beta$ minimizing the function

$$f_x(\tilde{x},\tilde{y}) := \bigl(|x|^{-\gamma_1}|\tilde{x}|\bigr)^{2d} + \bigl(|x|^{-\gamma_2}|\tilde{y}-x|\bigr)^{2d}. \tag{2.23}$$

Setting

$$Z_x := |x|^{-\gamma_1}X \quad\text{and}\quad Z'_x := |x|^{-\gamma_2}(Y-x), \tag{2.24}$$

and noting that $d\gamma_1+d\gamma_2=s$, the law of $(Z_x,Z'_x)$ is given by

$$P\bigl(Z_x\in\mathrm{d}z,\,Z'_x\in\mathrm{d}z'\bigr) = \eta\beta\exp\Bigl\{-\eta\beta\int\mathrm{d}\tilde{z}\,\mathrm{d}\tilde{z}'\,1_{\{|\tilde{z}|^{2d}+|\tilde{z}'|^{2d}\le|z|^{2d}+|z'|^{2d}\}}\Bigr\}\mathrm{d}z\,\mathrm{d}z'. \tag{2.25}$$

Scaling the variables in the inner integral by $(|z|^{2d}+|z'|^{2d})^{\frac{1}{2d}}$ and invoking (2.17) shows that $(Z_x,Z'_x)$ are i.i.d. with law as in (2.16).

*STEP 2: Definition of $A(x)$ and pointwise inequality*: We will now define $A(x)$ and prove a pointwise version of the inequality (2.19). For $x$ so large that $1+2|x|^{-\frac{1}{2}(1-\gamma)}\le\eta^{-1/s}$ and $4|x|^{\frac{1}{2}(1+\gamma)}<|x|$ hold true, we set

$$A(x) := \bigl\{|Z_x|>|x|^{\frac{1}{2}(1+\gamma)-\gamma_1}\bigr\}\cup\bigl\{|Z'_x|>|x|^{\frac{1}{2}(1+\gamma)-\gamma_2}\bigr\} \tag{2.26}$$

and otherwise set $A(x)$ to be the sample space carrying the three Poisson processes above. We now claim the pointwise inequality

$$\widetilde{D}(0,x)\le\widetilde{D}\bigl(0,|x|^{\gamma_1}Z_x\bigr)+\widetilde{D}\bigl(x,x+|x|^{\gamma_2}Z'_x\bigr)+1+|x|1_{A(x)}, \tag{2.27}$$

where all instances of $\widetilde{D}$ are defined using the symmetrized version of $\mathscr{I}_\beta=\mathscr{I}'_\beta\cup\mathscr{I}''_\beta$. Since $\widetilde{D}(0,x)\le|x|$, (2.27) holds whenever $A(x)$ occurs and so we just need to verify (2.27) under the conditions

$$1+2|x|^{-\frac{1}{2}(1-\gamma)}\le\eta^{-1/s},\quad 4|x|^{\frac{1}{2}(1+\gamma)}<|x|\quad\text{and}\quad |X|,|x-Y|\le|x|^{\frac{1}{2}(1+\gamma)}. \tag{2.28}$$

Noting that $\mu'_{s,\beta}$ and $\mu'''_{s,\beta}$ have disjoint supports, the last two conditions ensure $(X,Y)\in\mathscr{I}'_\beta\subset\mathscr{I}_\beta$ a.s. and so $(X,Y)$ is allowed to enter a path contributing to the distance on the left of (2.27). Fix any $\varepsilon > 0$ and consider a path in $B(0,2|X|)$ from 0 to $X$ of length at most $\widetilde{D}(0,X)+\varepsilon$ and then a path in $B(x,2|x-Y|)$ from $x$ to $Y$ of length at most $\widetilde{D}(x,Y)+\varepsilon$. Since (2.28) ensures

$$B(0,2|X|) \subseteq B(0,2|x|^{\frac{1}{2}(1+\gamma)}) \subseteq B(0,2|x|) \tag{2.29}$$

and

$$B(x,2|x-Y|) \subseteq B(x,2|x|^{\frac{1}{2}(1+\gamma)}) \subseteq x+B(0,|x|) \subseteq B(0,2|x|), \tag{2.30}$$

concatenating the former path with edge $(X,Y)$ and then adjoining the latter path after $Y$, we get a path contributing potentially to the infimum defining $\widetilde{D}(0,x)$ and having length at most $\widetilde{D}(0,X)+\widetilde{D}(x,Y)+2\varepsilon+1$. As $\varepsilon$ was arbitrary, (2.27) follows via (2.24).

*STEP 3: Reduction to independent variables*: Let us now see how (2.27) reduces to (2.19). Enlarge the probability space so that it holds two independent copies $\widetilde{D}'$ of $\widetilde{D}''$ of random variable $\widetilde{D}$, that are independent of the processes $\mathscr{I}'_\beta, \mathscr{I}''_\beta, \mathscr{I}'''_\beta$ and thus of the random objects $\widetilde{D}$, $Z_x$ and $Z'_x$. Under the restrictions on $x$ from (2.28) we have

$$B\big(0,2|x|^{\frac{1}{2}(1+\gamma)}\big) \cap B\big(x,2|x|^{\frac{1}{2}(1+\gamma)}\big) = \emptyset. \tag{2.31}$$

It follows that, conditional on $A(x)^{\mathrm{c}}$, the triplet of families of random variables

$$\big\{\widetilde{D}(0,z)\colon |z| < 2|x|^{\frac{1}{2}(1+\gamma)}\big\}, \quad \big\{\widetilde{D}(x,x+z)\colon |z| < 2|x|^{\frac{1}{2}(1+\gamma)}\big\} \quad \text{and} \quad \{X,Y\} \tag{2.32}$$

are independent. Moreover, $\widetilde{D}(x,x+z) \overset{\text{law}}{=} \widetilde{D}''(0,z)$ by translation symmetry of the underlying process. Since, as before, (2.27) holds trivially when $A(x)$ occurs, it suffices to check (2.19) conditionally on $A(x)^{\mathrm{c}}$. In that case the independence of the objects in (2.32) permits us to swap $\widetilde{D}(0,|x|^{\gamma_1}Z_x)$ for $\widetilde{D}'(0,|x|^{\gamma_1}Z_x)$ and $\widetilde{D}(x,x+|x|^{\gamma_2}Z'_x)$ for $\widetilde{D}''(0,|x|^{\gamma_2}Z'_x)$ without affecting the (conditional) law of the right-hand side of (2.27). Then (2.19) follows from (2.27).

In order to complete the proof, it remains to verify the bound (2.18). Assuming the first two conditions in (2.28) hold, $A(x)$ will occur only if one of $Z_x$ or $Z'_x$ exceeds the stated bounds. The formula (2.25) then readily shows (2.18) in this case. We then adjust the constant $c_1$ so that (2.18) holds even when the first two conditions in (2.28) fail. □

## 3. Limit considerations

The main goal of this section is to establish the limit claim from Theorem 1.1 for the restricted distance. Due to our lack of a suitable substitute for the Subadditive Ergodic Theorem, we will extract the result by controlling the expectation and the variance of the restricted distance. Throughout this section we fix $\beta > 0$ and $\eta \in (0,\infty)$ and suppress them from all formal statements.

### 3.1 Convergence along doubly-exponential sequences.

We begin by noting that iterations of (2.19) naturally lead to the consideration of *randomized* locations to which the restricted distance is to be computed. In order to get a closed-form expression, a natural idea is to work with a fixed point of the randomization. This leads to:

**Lemma 3.1** *Let $Z_0, Z_1, \dots$ be i.i.d. copies of the random variable from* (2.16)*. Then the infinite product in*

$$W := Z_0 \prod_{k=1}^{\infty} |Z_k|^{\gamma^k}, \tag{3.1}$$

*converges in $(0,\infty)$ a.s. Moreover, $W$ has continuous, non-vanishing probability density and has all moments. Furthermore, if $Z$ has the law as in* (2.16)*, then*

$$Z \perp\!\!\!\perp W \quad \Rightarrow \quad |W|^{\gamma} Z \overset{\text{law}}{=} W \tag{3.2}$$

*Proof.* The random variable $|\log Z|$ has exponential tails and so $k \mapsto |\log Z_k|$ grows at most polylogarithmically fast a.s. Since $k \mapsto \gamma^k$ decays exponentially, the infinite product converges to a number in $(0,\infty)$ a.s. This, along with the fact that $Z$ has continuous and positive density, implies that $W$ has continuous and positive density as well.

To control the upper tail of $W$, observe that $\sum_{k\ge 0}(k+1)\gamma^k = (1-\gamma)^{-2}$. Hence, if $|W| > t^{(1-\gamma)^{-2}}$, then we must have $|Z_k| > t^{k+1}$ for at least one $k \ge 0$. Hereby we get

$$P\bigl(|W| > t\bigr) \le \sum_{k\ge 0} P\bigl(|Z_k| > t^{(k+1)(1-\gamma)^2}\bigr), \qquad t > 0. \tag{3.3}$$

As the tails of $Z$ are no heavier than Gaussian in all $d \ge 1$, the claim follows. The distributional identity $|W|^{\gamma} Z \overset{\text{law}}{=} W$ for $Z \perp\!\!\!\perp W$ is checked directly from the definition of $W$. □

The identity (3.2) shows that $W$ is indeed a fixed point for the *random* arguments of the restricted distance under iterations of (2.19). This enables us to extract our first limit claim:

**Lemma 3.2** *Assume $W$ from* (3.1) *is independent of $\widetilde{D}$. Then for each $r \ge 1$, the limit*

$$L(r) := \lim_{n\to\infty} \frac{E\,\widetilde{D}(0, r^{\gamma^{-n}} W)}{2^n}, \tag{3.4}$$

*exists. Moreover, $r \mapsto L(r)$ is upper-semicontinuous on $[1,\infty)$ and positive on $(1,\infty)$.*

*Proof.* We will apply Proposition 2.7 for the choices $\gamma_1 = \gamma_2 = \gamma$. Let $W$ be the random variable independent of $\widetilde{D}$, $\widetilde{D}'$, $Z$ and $Z'$ in (2.19). Plugging $r^{\gamma^{-n}} W$ for $x$ in (2.19) and invoking $|W|^{\gamma} Z \overset{\text{law}}{=} W$ along with the bound (2.18) yields

$$E\,\widetilde{D}(0, r^{\gamma^{-n}} W) \le 2E\,\widetilde{D}(0, r^{\gamma^{-n+1}} W) + c, \tag{3.5}$$

where $c := 1 + c_1 \sup_{x\in\mathbb{R}^d} |x| \mathrm{e}^{-c_2|x|^{\theta}}$, for $c_1$, $c_2$ and $\theta$ as in Proposition 2.7. This shows that

$$a_n(r) := 2^{-n}\bigl[E\,\widetilde{D}(0, r^{\gamma^{-n}} W) + c\bigr] \tag{3.6}$$

is non-increasing and, being non-negative, $\lim_{n\to\infty} a_n(r)$ exists. The limit in (3.4) then exists as well and takes the same value. By Lemma 2.1(3), $r \mapsto a_n(r)$ is continuous and so $r \mapsto L(r)$ is is upper semicontinuous, being a decreasing limit of continuous functions. The positivity of $L(r)$ for $r > 1$ follows from Lemma 2.4, Theorem 2.5 and the fact that $[\log(r^{\gamma^{-n}})]^{\Delta} = 2^n (\log r)^{\Delta}$. □

We now augment the convergence of expectations to:

**Proposition 3.3** *For any $r \ge 1$ and Lebesgue a.e. $x \in \mathbb{R}^d$,*

$$\frac{\widetilde{D}(0, r^{\gamma^{-n}} x)}{2^n} \underset{n\to\infty}{\longrightarrow} L(r) \quad P\text{-a.s.} \tag{3.7}$$

*In particular, $L(r)$ defined in* (3.4) *does not depend on the choice of $\eta$.*

The main ingredient of the proof is:

**Lemma 3.4** *Suppose $\widetilde{D}$, $Z$ and $W$ are independent with distribution as above. Let $\sigma(W)$ denote the sigma algebra generated by $W$. Then for any $r \geq 1$,*

$$\sum_{n=1}^{\infty} E\biggl(\mathrm{Var}\Bigl(2^{-n}\widetilde{D}\bigl(0, r^{\gamma^{-n}}Z|W|^\gamma\bigr)\,\Big|\,\sigma(W)\Bigr)\biggr) < \infty. \tag{3.8}$$

*Proof.* Fix $r \geq 1$. Plugging $x := rW$ in (2.19), squaring both sides and taking expectations we get

$$E\bigl(\widetilde{D}(0,rW)^2\bigr) \leq 2E\bigl(\widetilde{D}(0,r^\gamma W)^2\bigr) + 2E\Bigl(\bigl[E\bigl(\widetilde{D}(0,r^\gamma Z|W|^\gamma)\,|\,\sigma(W)\bigr)\bigr]^2\Bigr) + F_0\,, \tag{3.9}$$

where, using $\widetilde{D}(0,x) \leq |x|$ and $Z|W|^\gamma \overset{\text{law}}{=} W$, the error term is given by

$$F_0 = F_0(r) := E\bigl([1 + r|W|1_{A(rW)}]^2\bigr) + 4E\widetilde{D}(0, r^\gamma W) + 4E\bigl[(r|W|)^{1+\gamma}1_{A(rW)}\bigr]\,. \tag{3.10}$$

Next we rewrite the second term on the right of (3.9) using conditional variance, and then subtract suitable terms on both sides to get

$$\begin{aligned}\mathrm{Var}(\widetilde{D}(0,rW)) \leq 2\mathrm{Var}(\widetilde{D}(0,r^\gamma W)) + 2\mathrm{Var}\Bigl(\bigl[E(\widetilde{D}(0,r^\gamma Z|W|^\gamma)|\sigma(W))\bigr]^2\Bigr)&\\ + 4E\bigl(\widetilde{D}(0,r^\gamma W)\bigr)^2 - E\bigl(\widetilde{D}(0,rW)\bigr)^2 + F_0.&\end{aligned} \tag{3.11}$$

Replacing $W$ by $Z|W|^\gamma$ in the first two variances above and using the standard identity

$$\mathrm{Var}(X) = E\bigl(\mathrm{Var}(X|Y)\bigr) + \mathrm{Var}\bigl(E(X|Y)\bigr) \tag{3.12}$$

yields

$$\begin{aligned}\mathrm{Var}\bigl(E(\widetilde{D}(0,rZ|W|^\gamma)\,|\,\sigma(W)\bigr) + E\bigl(\mathrm{Var}(\widetilde{D}(0,rZ|W|^\gamma)\,|\,\sigma(W))\bigr)&\\ \leq 4\mathrm{Var}\bigl(E(\widetilde{D}(0,r^\gamma Z|W|^\gamma)\,|\,\sigma(W))\bigr) + 2E\bigl(\mathrm{Var}(\widetilde{D}(0,r^\gamma Z|W|^\gamma)\,|\,\sigma(W))\bigr)&\\ + 4E\bigl(\widetilde{D}(0,r^\gamma W)\bigr)^2 - E\bigl(\widetilde{D}(0,rW)\bigr)^2 + F_0\,.&\end{aligned} \tag{3.13}$$

Abbreviating

$$\begin{aligned}A_n &:= \frac{1}{4^n}\mathrm{Var}\Bigl(E\bigl(\widetilde{D}(0,r^{\gamma^{-n}}Z|W|^\gamma)\,|\,\sigma(W)\bigr)\Bigr)\\ B_n &:= \frac{1}{4^n}E\Bigl(\mathrm{Var}\bigl(\widetilde{D}(0,r^{\gamma^{-n}}Z|W|^\gamma)\,|\,\sigma(W)\bigr)\Bigr)\\ C_n &:= \frac{1}{4^n}E\bigl(\widetilde{D}(0,r^{\gamma^{-n}}W)\bigr)^2\end{aligned} \tag{3.14}$$

the inequality (3.13) gives

$$A_n + B_n + C_n \leq A_{n-1} + \frac{1}{2}B_{n-1} + C_{n-1} + \frac{F_n}{4^n}\,, \tag{3.15}$$

where $F_n(r) := F_0(r^{\gamma^{-n}})$. Iterating shows

$$A_n + \frac{1}{2}B_n + C_n \leq A_0 + \frac{1}{2}B_0 + C_0 - \frac{1}{2}\sum_{k=1}^{n} B_k + \sum_{k=1}^{n}\frac{F_k}{4^k}. \tag{3.16}$$

Thanks to (2.18) and (3.5) we have $\sup_{n\geq 1} F_n/2^n < \infty$. Since $A_n, B_n, C_n \geq 0$ and $F_n/4^n$ is summable on $n \geq 0$, the sum of $B_k$ must remain bounded uniformly in $n$. □

As a direct consequence we get:

**Corollary 3.5** *Assume $\widetilde{D}$ and $W$ are independent with distributions as above. Then*

$$\sup_{r\in[e^\gamma,e)}\sup_{n\geq 1} E\biggl(\Bigl(\frac{\widetilde{D}(0,r^{\gamma^{-n}}W)}{2^n}\Bigr)^2\biggr) < \infty. \tag{3.17}$$

*Proof.* In the notation of the previous proof, the expectation equals $A_n+B_n+C_n$ which is bounded uniformly in $n$ thanks to (3.16). As $\sup_{n\geq 1}(F_n(r)/2^n)$ is bounded uniformly on compact intervals of $r$, the expectation is bounded also uniformly in $r$ on the stated interval. □

We are now ready to give:

*Proof of Proposition 3.3.* Consider again the independent copies $\widetilde{D}'$ and $Z'$ of the quantities $\widetilde{D}$ and $Z$, respectively. Formula (3.8) then reads

$$\sum_{n=1}^\infty E\biggl[\Bigl(\frac{\widetilde{D}'(0,r^{\gamma^{-n}}Z'|W|^\gamma)}{2^n}-\frac{\widetilde{D}(0,r^{\gamma^{-n}}Z|W|^\gamma)}{2^n}\Bigr)^2\biggr] < \infty. \tag{3.18}$$

Pick a compact set $U\subset\mathbb{R}^d\setminus\{0\}$ with non-empty interior, denote its Lebesgue measure by $|U|$ and let $\varepsilon\in(0,1)$. From the fact that $Z$ has a continuous nonvanishing density $f$, there is a constant $c=c(U,\varepsilon)>0$ such that

$$z|w|^\gamma\in U \quad \& \quad |w|<1/\varepsilon \quad\Rightarrow\quad f(z)|w|^{-d\gamma}\geq c\frac{1}{|U|}. \tag{3.19}$$

Restricting the expectation to the event $\{Z|W|^\gamma\in U\}\cup\{|W|<1/\varepsilon\}$, this bound permits us to change variables from $z$ to $x:=z|w|^\gamma$ and conclude that for $X$ uniform on $U$, and independent of all other random objects, we have

$$\sum_{n=1}^\infty E\biggl[\Bigl(\frac{\widetilde{D}'(0,r^{\gamma^{-n}}Z'|W|^\gamma)}{2^n}-\frac{\widetilde{D}(0,r^{\gamma^{-n}}X)}{2^n}\Bigr)^2\,\bigg|\,|W|<1/\varepsilon\biggr] < \infty \tag{3.20}$$

where we also used that $P(|W|<1/\varepsilon)>0$ for $\varepsilon\in(0,1)$. Using Jensen's inequality, we can now pass the expectation over $\widetilde{D}$, $Z'$ and $W$ inside the square to get

$$\sum_{n=1}^\infty E\biggl[\biggl(E\Bigl[\frac{\widetilde{D}(0,r^{\gamma^{-n}}Z|W|^\gamma)}{2^n}\,\Big|\,|W|<1/\varepsilon\Bigr]-\frac{\widetilde{D}(0,r^{\gamma^{-n}}X)}{2^n}\biggr)^2\biggr] < \infty \tag{3.21}$$

By the Monotone Convergence Theorem, this implies

$$\frac{\widetilde{D}(0,r^{\gamma^{-n}}X)}{2^n}-E\Bigl[\frac{\widetilde{D}(0,r^{\gamma^{-n}}Z|W|^\gamma)}{2^n}\,\Big|\,|W|<1/\varepsilon\Bigr]\underset{n\to\infty}{\longrightarrow}0,\text{a.s.} \tag{3.22}$$

with the exceptional set not depending on $\varepsilon$.

Let $\tilde{c}$ denote the quantity in Corollary 3.5. Denoting $q_\varepsilon := P(|W|\geq 1/\varepsilon)$, from Cauchy-Schwarz we have

$$\biggl|(1-q_\varepsilon)E\Bigl[\frac{\widetilde{D}(0,r^{\gamma^{-n}}Z|W|^\gamma)}{2^n}\,\Big|\,|W|<1/\varepsilon\Bigr]-E\Bigl[\frac{\widetilde{D}(0,r^{\gamma^{-n}}W)}{2^n}\Bigr]\biggr| \leq \sqrt{\tilde{c}q_\varepsilon} \tag{3.23}$$

As $q_\varepsilon \to 0$ when $\varepsilon \downarrow 0$, we thus get (3.7) for Lebesgue a.e. $x \in U$. Since $U$ was arbitrary (compact), the same applies to a.e. $x \in \mathbb{R}^d$. $\square$

**Remark 3.6** The reader may wonder why the passage through an a.s. limit for Lebesgue a.e. $x$ has been used instead of trying to prove the a.s. convergence of $X_n := 2^{-n}\widetilde{D}(0, r^{\gamma^{-n}}W)$ directly. (The convergence $X_n \to L(r)$ a.s. does hold by (3.7) and the fact that $W$ has a density w.r.t. the Lebesgue measure.) This is because Lemma 3.4 only controls the conditional variances of $X_n$ given $W$, and not the full variances $\mathrm{Var}(X_n)$. We will in fact show $\mathrm{Var}(X_n) \to 0$ in the proof of Proposition 3.7, but that only with the help of (3.7).

### 3.2 Full limit for the restricted distance.

We now proceed to extend the limit from multiples of the argument by terms from $\{r^{\gamma^{-n}} \colon n \ge 0\}$ to multiples ranging continuously through positive reals. However, for reasons described after Theorem 1.2, such a limit can generally be claimed only in probability. It will also suffice to show this for $x$ replaced by random variable $W$. This is the content of:

**Proposition 3.7** *Suppose $\widetilde{D}$ and $W$ are independent with distributions as above. Then*

$$\frac{\widetilde{D}(0, rW)}{L(r)} \underset{r\to\infty}{\longrightarrow} 1 \quad \textit{in probability}. \tag{3.24}$$

As we will see, a key point in proving Proposition 3.7 is:

**Lemma 3.8** *The identity $L(r) = 2L(r^\gamma)$ holds for all $r \ge 1$ and $t \mapsto L(\mathrm{e}^t)$ is convex on $[0,\infty)$. In particular, $r \mapsto L(r)$ is continuous, strictly increasing on $[1,\infty)$. The function*

$$\phi(r) := L(r)(\log r)^{-\Delta}, \qquad r > 1, \tag{3.25}$$

*obeys the conditions stated in Theorem 1.2.*

*Proof.* First, $L(r) = 2L(r^\gamma)$ is a consequence of the limit definition of $L$ in Lemma 3.4. Let $\gamma_1, \gamma_2$ be such that $0 < \gamma_1, \gamma_2 < \frac{1+\gamma}{2}$ and $d\gamma_1 + d\gamma_2 = s$. Plugging $r^{\gamma^{-n}}x$ for $x$ in (2.19) yields

$$\widetilde{D}(0, r^{\gamma^{-n}}x) \overset{\text{law}}{\le} \widetilde{D}\big(0, r^{\gamma_1\gamma^{-n}}|x|^\gamma Z\big) + \widetilde{D}'\big(0, r^{\gamma_2\gamma^{-n}}|x|^{\gamma_2} Z'\big) + 1 + r^{\gamma^{-n}}|x| 1_{A(r^{\gamma^{-n}}x)}. \tag{3.26}$$

Dividing the expression by $2^n$, applying Proposition 3.3 and noting that, by (2.18), the last two terms tend to zero in probability as $n \to \infty$ gives

$$L(r) \le L(r^{\gamma_1}) + L(r^{\gamma_2}). \tag{3.27}$$

Now set $\mathrm{e}^{t_1} := r^{\gamma_1/\gamma}$ and $\mathrm{e}^{t_2} := r^{\gamma_2/\gamma}$ and observe that then $r = \mathrm{e}^{\frac12(t_1+t_2)}$. The identity $2L(r^\gamma) = L(r)$ and the fact that the constraints on $\gamma_1, \gamma_2$ will be satisfied if $|\gamma_1 - \gamma_2| < 1 - \gamma$ then imply

$$\forall t_1, t_2 \ge 0\colon \quad 0 < \frac{|t_1 - t_2|}{t_1 + t_2} < \frac{1-\gamma}{2\gamma} \quad \Rightarrow \quad L\big(\mathrm{e}^{\frac12(t_1+t_2)}\big) \le \frac{L(\mathrm{e}^{t_1}) + L(\mathrm{e}^{t_2})}{2}, \tag{3.28}$$

i.e., a local mid-point convexity of $t \mapsto L(\mathrm{e}^t)$. The upper semicontinuity of $L$ from Lemma 3.4 then implies continuity of $r \mapsto L(r)$ on $[1,\infty)$, and subsequently also the convexity of $t \mapsto L(\mathrm{e}^t)$ on $[0,\infty)$. The strict monotonicity arises from convexity and the fact that $L(1) = 0$ while $L(r) > 0$ for $r > 1$, by Lemma 3.2. The conditions for $\phi$ in Theorem 1.2 are checked directly. $\square$

We will also need a uniform bound on third moments of $2^{-n}\widetilde{D}(0,r^{\gamma^{-n}}W)$:

**Lemma 3.9** *Assume $\widetilde{D}$ and $W$ are independent with distributions as above. Then*

$$\sup_{r\in[\mathrm{e}^\gamma,\mathrm{e})}\sup_{n\ge1} E\biggl(\Bigl(\frac{\widetilde{D}(0,r^{\gamma^{-n}}W)}{2^n}\Bigr)^3\biggr)<\infty. \tag{3.29}$$

*Proof.* Consider the setup of Proposition 2.7 with $\gamma_1=\gamma_2:=\gamma$. Taking the third power of both sides of (2.19) and setting with $x:=rW$ yields

$$E\bigl(\widetilde{D}(0,rW)^3\bigr)\le E\Bigl(\bigl[\widetilde{D}(0,r^\gamma|W|^\gamma Z)+\widetilde{D}'(0,r^\gamma|W|^\gamma Z)\bigr]^3\Bigr)+G_0(r), \tag{3.30}$$

where

$$\begin{aligned}G_0(r):=3E\Bigl(\bigl[\widetilde{D}(0,r^\gamma|W|^\gamma Z)+\widetilde{D}'(0,r^\gamma|W|^\gamma Z)\bigr]^2\Bigr)\\+3E\Bigl(\widetilde{D}(0,r^\gamma|W|^\gamma Z)+\widetilde{D}'(0,r^\gamma|W|^\gamma Z)\Bigr)+1+3E\Bigl(\bigl[1+2r^\gamma|W|^\gamma\bigr]^2r|W|1_{A(rW)}\Bigr)\\+3E\Bigl(\bigl[1+2r^\gamma|W|^\gamma\bigr]r^2|W|^21_{A(rW)}\Bigr)+E\Bigl(r^3|W|^31_{A(rW)}\Bigr)\end{aligned} \tag{3.31}$$

With the help of Hölder's inequality and the fact that $Z|W|^\gamma\overset{\text{law}}{=}W$, we then get

$$E\bigl(\widetilde{D}(0,r^{\gamma^{-n}}W)^3\bigr)\le 8E\bigl(\widetilde{D}(0,r^{\gamma^{-n+1}}W)^3\bigr)+G_n(r), \tag{3.32}$$

where, as before, $G_n(r):=G_0(r^{\gamma^{-n}})$. Corollary 2.6 and (2.18) ensure that $G_n(r)/4^n$ is bounded uniformly in $n\ge1$ and $r\in[\mathrm{e}^\gamma,\mathrm{e})$. The claim follows. □

We are now ready to give:

*Proof of Proposition 3.7.* Abbreviate $X_n:=2^{-n}\widetilde{D}(0,r^{\gamma^{-n}}W)$. By Lemma 3.2, $E(X_n)\to L(r)$. Lemma 3.9 and the almost sure convergence in Proposition 3.3 in turn show $E(X_n^2)\to L(r)^2$. Using the quantities from (3.14), we can alternatively write $E(X_n^2)=A_n+B_n+C_n$. Since $C_n=[EX_n]^2$, the above shows $C_n\to L(r)^2$ and so $A_n+B_n\to0$ for each $r\ge1$.

We claim that the convergence of the moments is uniform in $r$ on compact subsets of $[1,\infty)$. Starting with the former, observe that (3.15) in fact shows that

$$n\mapsto A_n+B_n+C_n-\sum_{k=n+1}^\infty\frac{F_k}{4^k} \tag{3.33}$$

is non-increasing, with the sum absolutely convergent. The functions $A_n,B_n,C_n,F_n$ are continuous and so is the limit $L(r)^2$, by Lemma 3.8. Dini's Theorem then ensures that the convergence $A_n+B_n\to0$ is indeed uniform on compact sets of $r$.

The argument for the first moments is similar; the proof of Lemma 3.2 shows that, for some constant $c>0$, the sequence $E(X_n)+c2^{-n}$ decreases to $L(r)$ pointwise. Since both the sequence and the limit are continuous, Dini's Theorem again implies local uniformity.

As $A_n+B_n=\mathrm{Var}(X_n)$, we have $\mathrm{Var}(X_n)\to0$ locally uniformly in $r$. In light of the similar uniformity of $E(X_n)\to L(r)$, for each $\varepsilon>0$ there is $n_0\ge1$ such that

$$\sup_{r\in[\mathrm{e}^\gamma,\mathrm{e})}\sup_{n\ge n_0} E\biggl(\Bigl|\frac{\widetilde{D}(0,r^{\gamma^n}W)}{2^n}-L(r)\Bigr|^2\biggr)<\varepsilon. \tag{3.34}$$

Since $L(r^{\gamma^{-n}}) = 2^n L(r)$, dividing the expression by $L(r)$ we get

$$\sup_{r\ge \mathrm{e}^{\gamma^{1-n_0}}} E\left(\left|\frac{\widetilde{D}(0,rW)}{L(r)}-1\right|^2\right) < \frac{\varepsilon}{L(\mathrm{e}^\gamma)}, \tag{3.35}$$

where we also used that $L(r) \ge L(\mathrm{e}^\gamma)$ for all $r \in [\mathrm{e}^\gamma, \mathrm{e})$. This implies $\widetilde{D}(0,rW)/L(r) \to 1$ in $L^2$ and thus in probability. □

**Remark 3.10** With some extra work, we could show that the limit in (3.24) also exists in probability conditional on $W$. As $W$ is continuously distributed with support $\mathbb{R}^d$, we could then replace $W$ by Lebesgue a.e. $x \in \mathbb{R}^d$ and, finally, use monotonicity arguments to extend to conclusion to all non-zero $x \in \mathbb{R}^d$. However, the same arguments will (have to) be applied to the full distance treated in the next section and, since the full distance is what we are interested in, we refrain from making them here.

## 4. Full distance scaling

We are now ready to return to the full distance $D(x,y)$ associated with long-range percolation on $\mathbb{R}^d$ and prove its asymptotic stated in Theorem 1.2. We begin by extending the conclusions of Proposition 3.7 to the full distance:

**Proposition 4.1** *Suppose $D$ and $W$ are independent with distributions as above. Then*

$$\frac{D(0,rW)}{L(r)} \underset{r\to\infty}{\longrightarrow} 1 \quad \textit{in probability}. \tag{4.1}$$

For this, we will need to expand the notion of the restricted distance to a whole family of "distances" $D_k$ indexed by $k \in \{0,1,\dots\}$ as follows. Abbreviating $\tilde{\gamma} := \frac{1}{2}(1+\gamma)$, we set

$$\widetilde{D}_k(x,y) := \inf\left\{ n + \sum_{i=0}^n |x_{i+1}-y_i| : \begin{array}{l} n \ge 0,\ \{(x_i,y_i)\colon i=1,\dots,n\} \subset \mathscr{I}_\beta \\ x_i,y_i \in B\bigl(x, 2|x-y|^{\tilde{\gamma}^{-k}}\bigr)\ \forall i=1,\dots,n \end{array}\right\}, \tag{4.2}$$

where, as before, we set $y_0 := x$ and $x_{n+1} := y$. We have

$$D(x,y) \le \dots \le \widetilde{D}_{k+1}(x,y) \le \widetilde{D}_k(x,y) \le \dots \le \widetilde{D}_1(x,y) \le \widetilde{D}_0(x,y) = \widetilde{D}(x,y)\,. \tag{4.3}$$

Our first observation is:

**Lemma 4.2** *Let $W$ be independent of the distances $\widetilde{D}_k$ and $D$. There is $k \in \mathbb{N}$ such that*

$$\lim_{r\to\infty} P\bigl(\widetilde{D}_k(0,rW) \ne D(0,rW)\bigr) = 0. \tag{4.4}$$

*Proof.* Pick $x \in \mathbb{R}^d$ and let $c$ denote the diameter of $[0,1)^d$ in $|\cdot|$-norm. Note that, as soon as $r$ is sufficiently large, on $\{\widetilde{D}_k(0,rx) \ne D(0,rx)\}$ there must be a point

$$y \in \mathbb{Z}^d \smallsetminus \Bigl[B\bigl(0,(r|x|)^{\tilde{\gamma}^{-k}}\bigr) \cup B\bigl(x,(r|x|)^{\tilde{\gamma}^{-k}}\bigr)\Bigr] \tag{4.5}$$

for which $D(0,y) \le \widetilde{D}(0,rx)+c$ and $D(rx,y) \le \widetilde{D}(0,rx)+c$ occur using disjoint collections of edges in the underlying sample of the Poisson process. Given any $C > 0$ and assuming that $r$ is so large that $C(\log r)^\Delta > 2c$, the van den Berg-Kesten inequality then shows

$$P\Big(\widetilde{D}_k(0,rx)\neq D(0,rx),\, \widetilde{D}(0,rx)\le C(\log r)^\Delta\Big) \le \sum_{\substack{y\in\mathbb{Z}^d\\ |y|\wedge|\lfloor rx\rfloor-y|\ge(r|x|)^{\tilde\gamma^{-k}}}} P\big(D(0,y)\le 2C(\log r)^\Delta\big)P\big(D(\lfloor rx\rfloor,y)\le 2C(\log r)^\Delta\big)\,. \tag{4.6}$$

where $a\wedge b:=\min\{a,b\}$ and where $\lfloor rx\rfloor$ is the closest point on $\mathbb{Z}^d$ to $rx$. Our aim is to show that the sum vanishes as $r\to\infty$ once $k$ is large enough.

In light of the domination bound in Lemma 2.4, we can replace the continuum distance $D$ by the discrete distance $D^{\,\mathrm{dis}}$ at the cost of changing $C$ and $\beta$ by multiplicative constants. We may thus estimate the above sum for the model on $\mathbb{Z}^d$ instead, writing temporarily just $x$ for $\lfloor rx\rfloor$ and $n$ for $2C(\log r)^\Delta$. The bound in Theorem 2.5 then shows

$$\sum_{\substack{y\in\mathbb{Z}^d\\ |y|\wedge|y-x|\ge|x|^{\tilde\gamma^{-k}}}} P\big(D^{\,\mathrm{dis}}(0,y)\le n\big)P\big(D^{\,\mathrm{dis}}(y,x)\le n\big) \le c_1^2\mathrm{e}^{2c_2n^{1/\Delta}}\sum_{\substack{y\in\mathbb{Z}^d\\ |y|\wedge|y-x|\ge|x|^{\tilde\gamma^{-k}}}}\frac{1}{|y|^s|x-y|^s}\le\tilde c_1\frac{\mathrm{e}^{2c_2'n^{1/\Delta}}}{|x|^{s\tilde\gamma^{-k}}} \tag{4.7}$$

for some $\tilde c_1\in(0,\infty)$ independent of $x$. Returning to the continuum problem with $n:=C(\log r)^\Delta$, there is thus a constant $\tilde c\in(0,\infty)$ such that, for some $c_2'$ proportional to $c_2$,

$$P\Big(\widetilde{D}_k(0,rx)\neq D(0,rx),\, \widetilde{D}(0,rx)\le C(\log r)^\Delta\Big)\le\tilde c\,\frac{|x|^{-s\tilde\gamma^{-k}}}{r^{s\tilde\gamma^{-k}-2c_2'C^{1/\Delta}}}\,. \tag{4.8}$$

The exponent of $r$ in the denominator is positive once $k$ is taken sufficiently large (depending only on $C$). Plugging in $x:=W$, choosing $C>\max\phi$ for $\phi$ as in Lemma 3.8, adjusting $k$ accordingly and invoking Proposition 3.7, we get (4.4) on the event $\{|W|>\varepsilon\}$, for any $\varepsilon>0$. But $W$ is continuously distributed and so the claim follows by noting that $P(|W|\le\varepsilon)\to0$ as $\varepsilon\downarrow0$. □

Next we observe:

**Lemma 4.3** *Let $k\in\mathbb{N}$ and suppose $W$ and $\widetilde{D}_k$ are independent with distributions as above. Then for every $k\ge0$,*

$$\liminf_{r\to\infty}\frac{E\widetilde{D}_k(0,rW)}{L(r)}\ge1. \tag{4.9}$$

The proof will be based on perturbations of the underlying model in $\beta$. For this reason, let $L_\beta(r)$ henceforth mark the explicit dependence of the limit in Lemma 3.2 on $\beta$ and define, as before, $\phi_\beta(r):=L_\beta(r)/(\log r)^\Delta$. We note one useful fact:

**Lemma 4.4** *The function $(\beta,r)\mapsto\phi_\beta(r)$ is jointly continuous on $(0,\infty)\times(1,\infty)$.*

*Proof.* From (2.5) and the existence of the limit in Lemma 3.2 we get

$$\forall a\ge1:\quad L_\beta(r)\le L_{a^{s-2d}\beta}(r)\le aL_\beta(r). \tag{4.10}$$

The continuity of $\beta \mapsto L_\beta(r)$ for each $r \ge 1$ is then readily inferred. The continuity and monotonicity of $r \mapsto L_\beta(r)$ then yields the joint continuity of $(\beta, r) \mapsto L_\beta(r)$ on $(0,\infty)\times[1,\infty)$. The claim follows by applying the definition of $\phi_\beta$. $\square$

*Proof of Lemma 4.3.* The argument hinges on a subadditivity bound of the kind derived in Proposition 2.7 which links expectations of $\widetilde{D}_k$ and $\widetilde{D}_{k+1}$ albeit at slightly different values of $\beta$. The proof of this bound follows closely that of the above proposition, although it is simpler as here we can efficiently use additivity of Poisson processes.

Fix $\beta > 0$ and let $W$ be the random variable associated with parameters $\beta$ and $\eta := 1$ as defined in Lemma 3.1. Let $\varepsilon \in (0,1/2)$. Writing $E_\beta$ for the expectation with respect to the point process $\mathscr{I}_\beta$ with intensity measure $\mu_{s,\beta}$, we will show later that, for some $c = c(\beta,\varepsilon)\in(0,\infty)$,

$$E_\beta\big[D_k(0, r\varepsilon^{-\frac{1}{2d}}W)\big] \le 2E_{\beta(1-2\varepsilon)}\big[D_{k+1}(0, r^\gamma\varepsilon^{-\frac{1}{2d}}W)\big] + c \tag{4.11}$$

holds for all $r \ge 1$. This is sufficient to prove the claim by induction. Indeed, the factors $\varepsilon^{-\frac{1}{2d}}$ can seamlessly be absorbed into $r$ by noting that $L_\beta(ar)/L(r) \to 1$ as $r\to\infty$ for any $a>0$ thanks to the continuity of $r \mapsto \phi_\beta(r)$. Assuming (4.9) for some $k\in\mathbb{N}$, then dividing (4.11) by $L_\beta(r) = 2L_\beta(r^\gamma)$ and relabeling $\beta(1-2\varepsilon)$ for $\beta$ yields

$$\liminf_{r\to\infty}\frac{E_\beta\big[D_{k+1}(0,rW)\big]}{L_\beta(r)} \ge \inf_{r\in[\mathrm{e}^\gamma,\mathrm{e})}\frac{\phi_{\beta/(1-2\varepsilon)}(r)}{\phi_\beta(r)} \tag{4.12}$$

Taking $\varepsilon \downarrow 0$ and invoking the continuity from Lemma 4.4, we then get (4.9) for $k+1$ as well. Since Lemma 3.2 ensures (4.9) for $k := 0$, we get it for all $k \ge 0$.

It remains to prove (4.11). Abbreviate

$$\beta' := 2\varepsilon\beta \quad\text{and}\quad \beta'' := (1-2\varepsilon)\beta. \tag{4.13}$$

A sample $\mathscr{I}_\beta$ of the Poisson process with intensity measure $\mu_{s,\beta}$ can then be written as the union $\mathscr{I}_{\beta'}\cup\mathscr{I}_{\beta''}$ of two independent processes with intensities $\mu_{s,\beta'}$ and $\mu_{s,\beta''}$, respectively. Fix $x\in\mathbb{R}^d$. Following the proof of Proposition 2.7 (with $\eta$ there set to $1/2$), under the condition

$$1 + 2|x|^{\tilde\gamma-1} \le 2^{1/s} \tag{4.14}$$

we can further decompose $\mathscr{I}_{\beta'}$ into the union of independent processes $\mathscr{I}'_{\beta'}$ and $\mathscr{I}''_{\beta'}$, with their respective intensity measures given by

$$\mu'_{s,\beta'}(\mathrm{d}\tilde x\,\mathrm{d}\tilde y) := \varepsilon\beta 1_{\{|\tilde x|_2<|\tilde y|_2\}}1_{\{|\tilde x|\vee|x-\tilde y|<|x|^{\tilde\gamma}\}}\frac{\mathrm{d}\tilde x\,\mathrm{d}\tilde y}{|x|^s} \tag{4.15}$$

and $\mu''_{s,\beta'} := \mu_{s,\beta'} - \mu'_{s,\beta'}$. (The condition (4.14) ensures that $\mu''_{s,\beta'}$ is a positive measure.) We also introduce an auxiliary independent process $\mathscr{I}'''_{\beta'}$ with intensity measure

$$\mu'''_{s,\beta'}(\mathrm{d}\tilde x\,\mathrm{d}\tilde y) := \varepsilon\beta\Big(1 - 1_{\{|\tilde x|_2<|\tilde y|_2\}}1_{\{|\tilde x|\vee|x-\tilde y|<|x|^{\tilde\gamma}\}}\Big)\frac{\mathrm{d}\tilde x\,\mathrm{d}\tilde y}{|x|^s}. \tag{4.16}$$

As is directly checked, $\mathscr{I}'_{\beta'}\cup\mathscr{I}'''_{\beta'}$ is a homogenous Poisson process with intensity $\varepsilon\beta|x|^{-s}$.

Now define a pair of random variables $(X,Y)$ as the minimizer of $f_x(\tilde x,\tilde y) := |\tilde x|^{2d} + |x-\tilde y|^{2d}$ among all points of $\mathscr{I}'_{\beta'}\cup\mathscr{I}'''_{\beta'}$. Set

$$Z := \varepsilon^{\frac{1}{2d}}|x|^{-\gamma}X \quad\text{and}\quad Z' := \varepsilon^{\frac{1}{2d}}|x|^{-\gamma}(x-Y) \tag{4.17}$$

and note that, by the calculation in (2.25) and a simple scaling arguement, $Z,Z'$ are i.i.d. with common law (2.16) for $\eta := 1$. Given $k\in\mathbb{N}$, let $D_k(0,x)$ be defined using the full process $\mathscr{I}_\beta$ and let $D''_{k+1}(\cdot,\cdot)$ be defined using the process $\mathscr{I}_{\beta''}$. We now claim

$$D_k(0,x)\le D''_{k+1}\bigl(0,\varepsilon^{-\frac{1}{2d}}|x|^\gamma Z\bigr)+D''_{k+1}\bigl(x,x+\varepsilon^{-\frac{1}{2d}}|x|^\gamma Z'\bigr)+1+|x|1_{A'(x)} \tag{4.18}$$

where we set

$$A'(x):=\bigl\{|Z|\vee|Z'|\ge\tfrac12\varepsilon^{\frac{1}{2d}}|x|^{\gamma-\tilde\gamma}\bigr\}. \tag{4.19}$$

whenever $|x|$ is so large that (4.14) and

$$|x|^{1-1/\tilde\gamma^k}+2^{1-1/\tilde\gamma^{k+1}}\le 2,\qquad k\ge 0, \tag{4.20}$$

hold, and put $A'(x)$ to the whole sample space otherwise. To see this we note that, on $A'(x)$ the inequality follows from $D_k(0,x)\le|x|$ and so we just need to prove this on $A'(x)^{\mathrm{c}}$. Here we observe that $|X|\vee|x-Y|\le\frac12|x|^{\tilde\gamma}$ and so $(X,Y)\in\mathscr{I}_{\beta'}$. A path minimizing $D''_{k+1}(0,X)$ will then lie in $B(0,2|X|^{\tilde\gamma^{-k-1}})\subseteq B(0,2|x|^{\tilde\gamma^{-k}})$ while the path minimizing $D''_{k+1}(x,Y)$ will lie in

$$B\bigl(x,2|x-Y|^{\tilde\gamma^{-k-1}}\bigr)\subseteq B\bigl(x,2^{1-\tilde\gamma^{-k-1}}|x|^{\tilde\gamma^{-k}}\bigr)\subseteq B\bigl(0,2|x|^{\tilde\gamma^{-k}}\bigr), \tag{4.21}$$

where the last inclusion is inferred from (4.20). The concatenation of these paths with edge $(X,Y)$ then produces a path entering the infimum defining $D_k(0,x)$. Hence (4.18) follows.

Noting that the probability of $A'(x)$ decays stretched-exponentially with $|x|$, plugging $W$ on both sides of (4.18), taking expectation and using that $|W|^\gamma Z\overset{\text{law}}{=}W$ then yields (4.11). □

Armed with the above lemmas, we can now give:

*Proof of Proposition 4.1.* Abbreviate $X(r):=\widetilde{D}_k(0,rW)/L(r)$. Corollary 3.5 and $D(0,rW)\le\widetilde{D}(0,rW)$ show $\sup_{r\ge\mathrm{e}^\gamma}E[X(r)^2]<\infty$ and so $\{X(r)\colon r\ge\mathrm{e}^\gamma\}$ is uniformly integrable. Proposition 3.7 in turn implies $P(X(r)>1+\varepsilon)\to0$ as $r\to\infty$ for every $\varepsilon>0$ and so we have $E[X(r)1_{\{X(r)>1+\varepsilon\}}]\to0$ as well. Lemma 4.3 then gives $E[X(r)]\to1$. Since $X(r)\ge0$, it follows that the mass of $X(r)$ must asymptotically concentrate at 1. This proves the claim for $\widetilde{D}_k(0,rW)$; Lemma 4.2 then extends it to $D(0,rW)$. □

This makes us finally ready to complete the proof of our main results:

*Proof of Theorem 1.2.* The definition and properties of function $\phi$ have already been established, so we just have to prove the limit claim (1.8). This will be derived from Proposition 4.1 and some perturbation arguments. Write $P_\beta$ for the law of the Poisson process with intensity $\mu_{s,\beta}$. Fix $x\in\mathbb{R}^d\smallsetminus\{0\}$ and note that, by the stochastic domination bounds in Lemmas 2.2–2.3, for each $\varepsilon$ there is $\delta$ such that for all $y\in\mathbb{R}^d$ and all $t>0$,

$$|y-x|<\delta|x|\quad\Rightarrow\quad P_\beta\bigl(D(0,x)\le t\bigr)\le P_{\beta(1+\varepsilon)}\bigl(D(0,y)\le(1+\varepsilon)t\bigr). \tag{4.22}$$

Let $W$ be independent of $D$ with the distribution as above and pick any $\zeta\in(0,1)$. Noting that $P(|W-x|<\delta|x|)>0$, for the above $\varepsilon$ and $\delta$ we then get

$$\begin{aligned} &P_\beta\bigl(D(0,rx)\le(1-\zeta)L_\beta(r)\bigr)\\ &\qquad=P_\beta\Bigl(D(0,rx)\le(1-\zeta)L_\beta(r)\,\Big|\,|W-x|<\delta|x|\Bigr)\\ &\qquad\le P_{\beta(1+\varepsilon)}\Bigl(D(0,rW)\le(1-\zeta)(1+\varepsilon)L_\beta(r)\,\Big|\,|W-x|<\delta|x|\Bigr)\end{aligned} \tag{4.23}$$

Lemma 4.4 permits us to pick $\varepsilon$ so small that

$$(1-\zeta)(1+\varepsilon)\inf_{r\in[\mathrm{e}^\gamma,\mathrm{e})}\frac{\phi_\beta(r)}{\phi_{\beta(1+\varepsilon)}(r)}<1-\varepsilon. \tag{4.24}$$

The right-hand side of (4.23) then tends to zero by Proposition 4.1. The argument for the other bound is completely analogous and so we omit it. □

*Proof of Theorem 1.1.* The lower bound was already shown in Corollary 2.6. For the the upper bound we first use Theorem 1.2 and the comparisons in Lemma 2.4 to prove the claim for $x$ of the form $x:=r\mathrm{e}_i$, where $\mathrm{e}_i$ is one of the coordinate vectors. Then we use the triangle inequality for $D^{\,\mathrm{dis}}$ to get the full limit as $|x|\to\infty$. □

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