PHASE TRANSITION AND CRITICAL BEHAVIOR IN HIERARCHICAL INTEGER-VALUED GAUSSIAN AND COULOMB GAS MODELS

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ABSTRACT. Given a square box $\Lambda_n \subseteq \mathbb{Z}^2$ of side-length L^n with $L, n > 1$, we study hierarchical random fields $\{\phi_x \colon x \in \Lambda_n\}$ with law proportional to $\mathrm{e}^{\frac{1}{2}\beta(\varphi,\Delta_n\phi)} \prod_{x\in\Lambda_n} \nu(\mathrm{d}\phi_x)$, where $\beta > 0$ is the inverse temperature, Δ_n is a hierarchical Laplacian on Λ_n, and ν is a non-degenerate 1-periodic measure on $\mathbb{R}$. Our setting includes the integer-valued Gaussian field (a.k.a. DG-model or Villain Coulomb gas) and the sine-Gordon model. Relying on renormalization group analysis we derive sharp asymptotic formulas, in the limit as $n \to \infty$, for the covariance $\langle \phi_x \phi_y \rangle$ and the fractional charge $\langle \mathrm{e}^{2\pi\mathrm{i}\alpha(\phi_x - \phi_y)} \rangle$ in the subcritical $\beta < \beta_\mathrm{c} := \pi^2/\log L$, critical $\beta = \beta_\mathrm{c}$ and slightly supercritical $\beta > \beta_\mathrm{c}$ regimes. The field exhibits logarithmic correlations throughout albeit with a distinct β-dependence of the variance and fractional-charge exponents in the sub/supercritical regimes. Explicit logarithmic corrections appear at the critical point.

1. INTRODUCTION AND RESULTS

1.1 The model and assumptions.

The aim of this paper is to study a class of random fields on $\mathbb{Z}^2$ with periodically modulated values. The general setting of these models is as follows: Fix an integer $L \geqslant 2$ and, for each integer $n \geqslant 1$, let $\Lambda_n := \{0, \dots, L^n - 1\}^2$ be a box of side-length L^n in $\mathbb{Z}^2$. Then consider a family $\{\phi_x \colon x \in \Lambda_n\}$ of real-valued random variables with joint law

$$P_{n,\beta}(\mathrm{d}\phi) := \frac{1}{Z_n(\beta)} \mathrm{e}^{\frac{1}{2}\beta(\phi,\Delta_n\phi)} \prod_{x\in\Lambda_n} \nu(\mathrm{d}\phi_x), \tag{1.1}$$

where $\beta > 0$ is the inverse temperature, $Z_n(\beta)$ is a normalization constant, $(\cdot,\cdot)$ denotes the canonical inner product in $\ell^2(\Lambda_n)$ and Δ_n is a Laplacian or, in probabilistic terms, the generator of a Markov chain on Λ_n. The modulation comes via ν which is assumed to be a 1-periodic locally-finite positive Borel measure on $\mathbb{R}$.

Throughout we focus on hierarchical models, for which the Markov chain defined by Δ_n jumps from x to y at a rate that depends only on the coefficients in base-L expansion of the coordinates of x and y. To state this precisely, set $b := L^2$ and identify Λ_n with the set of sequences $(x_1, \dots, x_n) \in \{0, \dots, b-1\}^n$. For $x = (x_1, \dots, x_n) \in \Lambda_n$, let $\mathcal{B}_k(x)$ denote the set of $y = (y_1, \dots, y_n)$ such that $y_i = x_i$ for $i = 1, \dots, n-k$. Then take Δ_n to be a hierarchical Laplacian on Λ_n that acts on $f \colon \Lambda_n \to \mathbb{R}$ as

$$\Delta_n f(x) := -\mathfrak{c}_{n+1} f(x) + \sum_{k=1}^{n} \sum_{y\in\mathcal{B}_k(x)} \mathfrak{c}_k \big[f(y) - f(x) \big], \tag{1.2}$$

where $\{\mathfrak{c}_k\}_{k=1}^{n+1}$ is a sequence of positive numbers subject to specific decay conditions; see Assumption 1.1 below. The positivity ensures that Δ_n is strictly negative definite on $\ell^2(\Lambda_n)$ which is needed for $Z_n(\beta)$ to be finite.

A well studied example of above field is the hierarchical *Gaussian Free Field* (GFF) for which ν is the Lebesgue measure on $\mathbb{R}$. An example of our prime interest in this work is the hierarchical *integer-valued Gaussian* model, a.k.a. *DG-model*, for which ν is the counting measure on $\mathbb{Z}$. The two models are interpolated by a continuous family of *sine-Gordon models* defined by

$$\nu(\mathrm{d}\phi) := \mathrm{e}^{-\kappa[1-\cos(2\pi\phi)]}\mathrm{d}\phi, \tag{1.3}$$

where $\kappa > 0$ is a parameter. Indeed, the GFF corresponds to $\kappa = 0$ while the DG-model arises via the weak limit as $\kappa \to \infty$ under the scaling of ν by $(2\pi\kappa)^{1/2}$.

The models (1.1) turn out to be dual to Coulomb gas systems whenever the Fourier coefficients of ν are non-negative. A remarkable fact is that two-dimensional Coulomb gas models, and thus also our fields, undergo a *BKT phase transition* at some β_{c} (named after Berezinskii [13], Kosterlitz and Thouless [44]) as soon as ν is distinct from the Lebesgue measure; see Section 2.1 for more discussion. Various aspects of this transition have previously been addressed in hierarchical models (e.g., by Benfatto, Gallavotti and Nicolò [11], Marchetti and Perez [49], Benfatto and Renn [12], Guidi and Marchetti [39]) albeit subject to limitations that generally exclude the DG-model. Our aim here is to provide a robust treatment of the transition and establish heretofore uncontrolled aspects of the critical and near-critical behavior.

Similarly to references [11,12,39,49], our analysis relies on the renormalization-group technique whose implementation requires some regularity of the coefficients $\{\mathfrak{c}_k\}_{k=1}^{n+1}$. We collect these requirements in:

Assumption 1.1 *There exists a positive sequence $\{\mathfrak{d}_k\}_{k\geqslant 0}$ satisfying $\sum_{k\geqslant 0}\mathfrak{d}_k < \infty$ such that, for each $n \geqslant 1$, the sequence $\{\mathfrak{c}_k\}_{k=1}^{n+1}$ takes the form*

$$\mathfrak{c}_{n+1} = \left(\sum_{j=0}^{n} b^j\sigma_j^2\right)^{-1} \tag{1.4}$$

and

$$\mathfrak{c}_k = \frac{1}{b^k}\left[\left(\sum_{j=0}^{k-1} b^j\sigma_j^2\right)^{-1} - \left(\sum_{j=0}^{k} b^j\sigma_j^2\right)^{-1}\right], \quad k = 1,\dots,n, \tag{1.5}$$

for a strictly positive sequence $\{\sigma_k^2\}_{k=0}^n$ satisfying

$$\left|\sigma_k^2 - 1\right| \leqslant \mathfrak{d}_{\min\{k,n-k\}}, \quad k = 0,\dots,n. \tag{1.6}$$

Moreover, we have $\inf_{n\geqslant k\geqslant 0}\sigma_k^2 > 0$.

In addition, we also need a bit of regularity of the measure ν:

Assumption 1.2 *ν is a 1-periodic Borel measure on $\mathbb{R}$ whose Fourier coefficients defined by $a(q) := \int_{[0,1)} \mathrm{e}^{-2\pi\mathrm{i}qz}\nu(\mathrm{d}z)$ are real-valued, strictly positive and satisfy*

$$a(-q) = a(q), \quad q \in \mathbb{Z}, \tag{1.7}$$

along with

$$\sup_{q\geqslant 0}\frac{a(q+1)}{a(q)}<\infty. \tag{1.8}$$

In particular, ν is reflection symmetric and 1-periodic but not $1/p$-periodic for any $p\geqslant 2$.

Note that, while the analysis by way of the renormalization technique is easiest when $\{\sigma_k^2\}_{k=0}^n$ are all equal to a positive constant (which we take to be 1), permitting more general coefficients allows us to remain flexible in what specific operator we take for a hierarchical Laplacian; see Remark 3.2. The formulation using $\{\mathfrak{d}_k\}_{k\geqslant 0}$ is done to ensure uniformity. Observe also that (1.6) along with $\mathfrak{d}_k\to 0$ imply that $\mathfrak{c}_k$, for both k and $n-k$ large, decays proportionally to $k\mapsto b^{-2k}$. The term $\mathfrak{c}_{n+1}$ scales only as b^{-n} due to its role of a "mass"; see again Remark 3.2. (The same asymptotic arises if we think of $\mathfrak{c}_{n+1}$ as an aggregate killing rate $\sum_{k>n}\mathfrak{c}_k b^k$ for a Markov chain on $\mathbb{Z}^2$ with conductances $\{\mathfrak{c}_k\}_{k>n}$ given as in (1.5).)

As to the conditions on measure ν, here the DG model corresponds to $a(q)=1$ for each $q\in\mathbb{Z}$ while for the sine-Gordon model (1.3) we get

$$a(q)=\sum_{\ell=0}^{\infty}\frac{(\kappa/2)^{2\ell+|q|}}{(\ell+|q|)!\ell!},\quad q\in\mathbb{Z}. \tag{1.9}$$

In particular, these models satisfy the conditions (1.7–1.8). The GFF is excluded but so is unfortunately the *hard-core Coulomb gas* that corresponds to

$$\nu(\mathrm{d}\phi):=[1+2\kappa\cos(2\pi\phi)]\mathrm{d}\phi, \tag{1.10}$$

where $\kappa\in[0,1/2]$. While our conclusions (to be stated next) definitely fail for the GFF, we still expect them to apply to the model (1.10).

1.2 Covariance structure.

Our first result concerns the asymptotic covariance structure of the field. Recall that b denotes the "branching number" of the hierarchical model which in the description based on a box in $\mathbb{Z}^2$ relates to the base scale L of Λ_n as $b=L^2$. The representation of elements of Λ_n as sequences leads to a hierarchical metric on Λ_n defined for any two distinct vertices $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ by

$$d(x,y):=b^{\frac{1}{2}(n-\min\{j=0,\dots,n:\,x_j\neq y_j\})} \tag{1.11}$$

with the convention $x_0=y_0:=0$, and by $d(x,y):=0$ when $x=y$. Under the natural embedding of Λ_n into $\mathbb{Z}^2$ we have $d(x,y)\geqslant\|x-y\|_\infty$ for all $x,y\in\Lambda_n$ with both sides comparable for generic x and y.

As a consequence of the hierarchical structure of Δ_n, all of our models undergo a phase transition at the same value of the inverse temperature; namely, at

$$\beta_{\mathrm{c}}:=\frac{2\pi^2}{\log b}. \tag{1.12}$$

We will henceforth write $\langle - \rangle_{n,\beta}$ to denote expectation with respect to $P_{n,\beta}$. Our result on the covariance structure of $P_{n,\beta}$ is then as follows:

Theorem 1.3 (Covariance structure) *There exists a map $\sigma^2\colon \mathbb{R}_+ \to \mathbb{R}_+$ with*

$$\sigma^2(\beta) \begin{cases} = 1/\beta, & \text{if } \beta \leqslant \beta_{\rm c}, \\ < 1/\beta, & \text{if } \beta > \beta_{\rm c}, \end{cases} \tag{1.13}$$

such that the following holds for all models satisfying Assumptions 1.1–1.2 with $\{\mathfrak{d}_k\}_{k\geqslant 0}$ decaying exponentially when $\beta > \beta_{\rm c}$: There exists $\epsilon > 0$ and, for all $\beta > 0$ with $1/\beta > 1/\beta_{\rm c} - \epsilon$, all $n \geqslant 1$ and all $x, y \in \Lambda_n$,

$$\langle \phi_x \phi_y \rangle_{n,\beta} = \begin{cases} \sigma^2(\beta) \log_{b^{1/2}}\Bigl(\dfrac{\operatorname{diam}(\Lambda_n)}{1+d(x,y)}\Bigr) + O(1), & \text{if } \beta \neq \beta_{\rm c}, \\ \dfrac{1}{\beta_{\rm c}} \log_{b^{1/2}}\Bigl(\dfrac{\operatorname{diam}(\Lambda_n)}{1+d(x,y)}\Bigr) - \bar{c}\log\Bigl(\dfrac{\log\operatorname{diam}(\Lambda_n)}{\log[2+d(x,y)]}\Bigr) + O(1), & \text{if } \beta = \beta_{\rm c}, \end{cases} \tag{1.14}$$

where

$$\bar{c} := \frac{8\pi^2}{\beta_{\rm c}^2} \frac{b(b^3-1)}{(b-1)^3(b+1)^2} \tag{1.15}$$

and $O(1)$ are quantities bounded uniformly in $n \geqslant 1$ and $x, y \in \Lambda_n$.

The above shows that models (1.1) subject to Assumptions 1.1–1.2 exhibit logarithmic decay of correlations at all $\beta > 0$ (with $1/\beta > 1/\beta_{\rm c} - \epsilon$). This makes them qualitatively similar to the GFF, for which the covariances behave exactly as in the $\beta < \beta_{\rm c}$ regime above. The connection to GFF at $\beta < \beta_{\rm c}$ is very tight; indeed, in our earlier work [16] we showed that one can couple $P_{n,\beta}$ to the law of GFF so closely that the two fields are within order-unity of each other at most (and at typical) points.

For $\beta > \beta_{\rm c}$, (1.13) shows that the overall scale of the fluctuations is strictly smaller than what GFF would give and, indeed, $P_{n,\beta}$ is far from the law of GFF both in terms of global scaling properties as well as other correlations (see Theorem 1.4 below). A reader looking for exponential decay for $\beta > \beta_{\rm c}$ should note that the Laplacian (1.2) is long range with the matrix coefficient for the pair x and y decaying proportionally to $d(x,y)^{-4}$ (see Section 2.3) so exponential decay is not to be expected.

The difference in the overall variance scale arises from the fact that, at $\beta > \beta_{\rm c}$, the field "feels" the 1-periodicity of ν at all spatial scales. Technically, this is seen in renormalization group iterations that draw the system towards a "non-trivial" fixed point — meaning one that does not correspond to GFF — rather than the "trivial" one as happens for $\beta \leqslant \beta_{\rm c}$. The quantity $\sigma^2(\beta)$ admits a formula, see (4.47), that makes the inequality in (1.13) quite apparent. We even get the asymptotic expansion

$$\sigma^2(\beta) = \frac{1}{\beta} - \frac{32\pi^4}{\beta_{\rm c}^4} \frac{b(b^3-1)}{(b-1)^3(b+1)^2}(\beta - \beta_{\rm c}) + O\bigl((\beta-\beta_{\rm c})^{3/2}\bigr), \quad \beta \downarrow \beta_{\rm c}, \tag{1.16}$$

see Remark 4.8. In particular, $\beta \mapsto \sigma^2(\beta)$ is not differentiable at $\beta_{\rm c}$. The apparent numerical closeness of (1.15) to the coefficient of $\beta - \beta_{\rm c}$ in (1.16) is not a coincidence; see Remark 4.9 for an explanation.

The behavior at $\beta_{\rm c}$ is yet different as an iterated-log correction arises in the covariance structure. This can be attributed to the fact that, while the renormalization group iterations still draw the model towards the "trivial" fixed point, the convergence is polynomially slow and a residue of 1-periodicity of ν survives to the macroscopic scale. We expect that the iterated-log correction is reflected in the extremal behavior of the field. For instance, the maximum $\max_{x\in\Lambda_n}\phi_x$, which for $\beta<\beta_{\rm c}$ scales exactly as that of the GFF (see [16, Corollary 2.2]), should have a different second-order (i.e., order $\log n$) term at $\beta=\beta_{\rm c}$. Controlling the maximum at (and beyond) the critical β seems to be an interesting open problem; see Remark 4.10 for further discussion.

As far as we know, the asymptotic covariance structure of the models (1.1) has not been studied previously. We suspect that this is because the standard approach based on incorporating observables into the renormalization-group flow does not fare too well for extensive quantities; i.e., those that scale with the system size.

1.3 Fractional charge asymptotic.

The connection of our model with Coulomb gas naturally leads us to the so-called fractional charge correlation $\langle \mathrm{e}^{2\pi\mathrm{i}\alpha(\phi_x-\phi_y)}\rangle_{n,\beta}$, where α is a parameter that, due to the underlying 1-periodicity and also interpretation as an electric charge, is taken generally real-valued; see Section 2.4. Here we get:

Theorem 1.4 (Fractional charge) *There exists a map* $\kappa\colon(0,1/2)\times\mathbb{R}_+\to\mathbb{R}_+$ *with*

$$\kappa(\alpha,\beta)\begin{cases}=\frac{4\pi^2}{\beta}\alpha^2, & \text{if } \beta\leqslant\beta_{\rm c},\\ <\frac{4\pi^2}{\beta}\alpha^2, & \text{if } \beta>\beta_{\rm c},\end{cases}\tag{1.17}$$

such that the following is true for all models satisfying Assumptions 1.1–1.2 with $\{\mathfrak{d}_k\}_{k\geqslant0}$ *decaying exponentially when* $\beta>\beta_{\rm c}$ *and obeying* $\sum_{j\geqslant1}\mathfrak{d}_j\log(j)<\infty$ *when* $\beta=\beta_{\rm c}$*: For all* $\alpha_0\in(0,1/2)$ *there exists* $\epsilon>0$ *and, for all* $\alpha\in(0,\alpha_0]$*, all* $\beta>0$ *with* $1/\beta>1/\beta_{\rm c}-\epsilon$ *and all* $n\geqslant1$*, there exists* $C_n\in(0,\infty)$ *satisfying* $0<\inf_{n\geqslant1}C_n\leqslant\sup_{n\geqslant1}C_n<\infty$ *such that*

$$\big\langle \mathrm{e}^{2\pi\mathrm{i}\alpha(\phi_x-\phi_y)}\big\rangle_{n,\beta}=\big[C_n+o(1)\big]\begin{cases}d(x,y)^{-\kappa(\alpha,\beta)}, & \text{if } \beta\neq\beta_{\rm c},\\ d(x,y)^{-\kappa(\alpha,\beta)}[\log d(x,y)]^{\tau(\alpha)}, & \text{if } \beta=\beta_{\rm c},\end{cases}\tag{1.18}$$

holds for all $x,y\in\Lambda_n$ *with* $x\neq y$*, where*

$$\tau(\alpha):=2\frac{b^3-1}{(b-1)(b+1)^3}\left[\frac{b-1}{b^{1+2\alpha}-1}+\frac{b-1}{b^{1-2\alpha}-1}-2\right]\tag{1.19}$$

and where $o(1)\to0$ *in the limit as* $\min\{d(x,y),\mathrm{diam}(\Lambda_n)/d(x,y)\}\to\infty$.

The n-dependence of C_n stems from potential variability in n of the sequence $\{\sigma_k^2\}_{k=0}^n$. The quantity $\kappa(\alpha,\beta)$ is determined, albeit somewhat implicitly, by (5.113). As discussed in Remark 5.13, we have the asymptotic form

$$\kappa(\alpha,\beta)=\frac{4\pi^2}{\beta}\alpha^2-\frac{4\pi^2}{\beta_{\rm c}^2}\tau(\alpha)(\beta-\beta_{\rm c})+O\big((\beta-\beta_{\rm c})^2\big),\quad \beta\downarrow\beta_{\rm c},\tag{1.20}$$

where $\tau(\alpha)$ is as in (1.19). It is easy to check that $\tau(\alpha)>0$ once $\alpha\neq0$, which is how we prove the inequality in (1.17). (As $\kappa(\alpha,\beta)\geqslant0$, the fact that $\tau(\alpha)$ diverges as $|\alpha|$ increases to $1/2$ only attests the lack of uniformity.) Again, the apparent numerical closeness of the critical and near-critical asymptotic is not a coincidence; see Remark 5.13.

In the language of Coulomb gas models, $\log\langle \mathrm{e}^{2\pi\mathrm{i}\alpha(\phi_x-\phi_y)}\rangle_{n,\beta}$ represents the energetic cost of inserting a charge α at x and a charge $-\alpha$ at y into a system of integer-valued charges kept at thermal equilibrium. Inserting just a single charge α at x has energetic cost $\log\langle \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_x}\rangle_{n,\beta}$ for which our proof similarly shows

$$\bigl\langle \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_x}\bigr\rangle_{n,\beta} = \bigl[C_n' + o(1)\bigr]\begin{cases} N^{-\kappa(\alpha,\beta)/2}, & \text{if } \beta\neq\beta_{\mathrm{c}},\\ N^{-\kappa(\alpha,\beta)/2}[\log N]^{\tau(\alpha)/2}, & \text{if } \beta=\beta_{\mathrm{c}}, \end{cases} \tag{1.21}$$

where we abbreviated $N := \mathrm{diam}(\Lambda_n)$ and where $o(1)\to 0$ as $N\to\infty$; see Remark 5.14. The drop in the value of $\kappa(\alpha,\beta)$ marked by the inequality in (1.17) is indicative of a *charge screening* taking place above β_{c} which (unlike for the lattice model) is only partial due to the long-range structure of the hierarchical Laplacian. See again Section 2.4.

We note that some aspects of the above result are already known. For instance, the subcritical regime $\beta<\beta_{\mathrm{c}}$ appears as an upper bound in Marchetti and Perez [49, Theorem 4.3], albeit assuming that ν is suitably close to the Lebesgue measure when β is close to β_{c}. For $\beta\gtrsim\beta_{\mathrm{c}}$, [49, Theorem 5.1] shows existence and stability of a non-trivial renormalization-group fixed point and, for the model with ν corresponding to the fixed point, compute the leading order expansion of the fractional charge exponent as $\beta\downarrow\beta_{\mathrm{c}}$, albeit somewhat less explicitly than (1.20). (The paper [49] works in the language of Coulomb gasses, so translations described in Section 2.4 are needed to identify their result with ours.)

Another relevant paper is that by Benfatto and Renn [12] who (while working in our framework) established existence of a non-trivial renormalization fixed point for $\beta\gtrsim\beta_{\mathrm{c}}$ and studied the *integer-charge correlations*; namely, truncated correlations of 1-periodic functions f of the field for the model with ν corresponding to the renormalization fixed point. In this case they proved that (for such generic f)

$$\bigl\langle f(\phi_x)f(\phi_y)\bigr\rangle_{n,\beta} - \bigl\langle f(\phi_x)\bigr\rangle_{n,\beta}\bigl\langle f(\phi_y)\bigr\rangle_{n,\beta} \asymp d(x,y)^{-2} \tag{1.22}$$

as $d(x,y)\to\infty$ regardless of $\beta\gtrsim\beta_{\mathrm{c}}$. This coincides with the behavior of the massive hierarchical GFF. It will be of interest to find an argument that proves the same for more general initial ν.

1.4 Summary and main ideas.

Theorems 1.3 and 1.4 capture the character of the phase transition in $\mathbb{Z}$-modulated hierarchical fields by way of asymptotic form of two important correlation functions. The main novelty is uniformity in the underlying model, and thus *universality*, which we achieve (in Theorems 3.4–3.6) by relying on Fourier representation of the exponential of the renormalized potentials, rather than the potentials themselves. This avoids arguments based on linearization, whose accuracy deteriorates close to the critical point, and/or significant restrictions on the model taken in earlier work. Our control thus extends all the way to and even slightly beyond the critical point revealing heretofore unattended aspects of the critical behavior.

Our conclusions for the subcritical and critical regimes apply solely under Assumptions 1.1 and 1.2. In the supercritical regime we restrict to $\beta-\beta_{\mathrm{c}}$ small, but we think of this as a mere technicality whose purpose is to keep (already very long) proof of Theorem 3.6 to a manageable length. The restriction to exponentially decaying $\{\mathfrak{d}_k\}_{k\geqslant 0}$ and the minor restriction in the critical case in Theorem 1.4 are imposed to allow for a comfortable control of the error terms. Another technical restriction (for all β) comes in the assumption that the Fourier coefficients of ν obey Assumption 1.2. This is natural for the connection with Coulomb gas but not necessarily so for the field itself. We take this as a price to pay for the precision of our conclusions.

As was just noted, our proofs hinge on tracking the "flow" of the Fourier coefficients of effective potentials under renormalization-group iterations. A key observation, stated in Lemma 3.7 which itself draws on [16, Lemma 4.2], is that the iterations preserve the structure in Assumption 1.2 and, in fact, improve the estimate on the ratios in (1.8). For $\beta \leqslant \beta_{\mathrm{c}}$ this leads to a full asymptotic analysis while, for $\beta > \beta_{\mathrm{c}}$, we at least eventually dominate the ratios by a quantity of order $\sqrt{\beta - \beta_{\mathrm{c}}}$. Assuming that to be small, a suitable fixed-point argument then extracts the desired limit behavior.

The proofs of Theorems 1.3 and 1.4 rely on the observation that the Gibbs measure (1.1) can be viewed as the law of a tree-indexed Markov chain after n steps. The transition probabilities of this (time-inhomogeneous) chain are simple functions of the effective potentials, see (3.15), and so one can extract a good amount of information about the chain just from the asymptotic behavior of the effective potentials. The details unfortunately still require some lengthy calculations.

1.5 Outline.

The remainder of this paper is organized as follows. First, in Section 2, we discuss the broader context of the above models while providing additional (or missing) details for various remarks made in the text above. In Section 3 we then introduce the renormalization-group approach and state the corresponding convergence theorems; see Theorems 3.4–3.6 in Section 3.2. Sections 4 and 5 are devoted to the proofs of our main results (namely, Theorems 1.3 and 1.4) based on these convergence theorems. The final section (Section 6) supplies the proof of Theorem 3.6 on supercritical renormalization-group flow which, unlike Theorems 3.4–3.5, could not be efficiently reduced to estimates proved in our previous work [16].

2. Connections and references

We proceed to discuss the broader context of our work; specifically, connections to lattice interface models, the BKT transition, Coulomb gas systems and hierarchical models. This will also give us the opportunity to cite additional relevant literature.

2.1 Lattice interface models.

The Gibbsian distributions of the kind (1.1) arise as models of fluctuating interfaces in statistical mechanics, albeit with the "harmonic" energy term $\frac{1}{2}(\phi, -\Delta_n\phi)$ often generalized to the "anharmonic" expression of the form

$$\frac{1}{2}\sum_{x,y\in\Lambda_n} V_{x,y}(\phi_x - \phi_y) \tag{2.1}$$

for some collection of potentials $\{V_{xy}\colon x,y\in\Lambda_n\}$ — typically, convex, translation invariant and decaying sufficiently fast with $|x-y|$; see e.g., Velenik [53], Funaki [36] or Sheffield [51]. In this language our setting corresponds to

$$V_{x,y}(\eta) := \mathfrak{c}(x,y)\eta^2 \tag{2.2}$$

for a collection $\{\mathfrak{c}(x,y) = \mathfrak{c}(y,x)\colon x,y \in \Lambda_n\}$ of non-negative quantities called conductances, due to a natural connection of this problem to resistor-network theory (see, e.g., Biskup [14] for a review).

There are two canonical choices for the "single-spin" measure ν: the Lebesgue measure on $\mathbb{R}$ and the counting measure on $\mathbb{Z}$. In the former case, the field corresponding to (2.2) is the *Gaussian Free Field* (GFF) associated with the generator

$$\mathcal{L}f(x) := \sum_{y\in\Lambda_n} \mathfrak{c}(x,y)[f(y) - f(x)] \tag{2.3}$$

of a Markov chain defined by the conductances $\{\mathfrak{c}(x,y)\colon x,y \in \Lambda_n\}$. Here, often but not always, $\mathfrak{c}(x,y) = 1$ when x and y neighbors and zero otherwise.

The GFF is special among above models for the fact that many relevant quantities are explicitly computable. A continuum version of the GFF also arises as the limit process at large spatial scales for many of the above models. This was first shown for models with uniformly strictly convex potentials by Naddaf and Spencer [50] and Giacomin, Olla and Spohn [38] and later extended to various cases beyond; e.g., Biskup and Spohn [20], Brydges and Spencer [26], Cotar, Deuschel and Müller [27], Ye [54], Adams, Buchholtz, Kotecký and Müller [1,2], Dario [28,29] and Armstrong and Wu [5].

The integer-valued models (i.e., for ν being the counting measure on $\mathbb{Z}$) exhibit richer behavior and are thus less well understood. One clear distinction is that any perturbation of a ground state costs a uniformly positive amount of energy. A Peierls-type argument then shows that, for β very large, a sample from the corresponding Gibbs measure deviates from a ground state configuration only by localized perturbations whose density decreases exponentially with their size. In particular, two-point correlations decay exponentially and we have

$$\sup_{n\geqslant 1}\ \sup_{x,y\in\Lambda_n} \langle(\phi_x - \phi_y)^2\rangle_{n,\beta} < \infty, \tag{2.4}$$

for all β large.

As it turns out, for $\mathbb{Z}$-valued fields over $\mathbb{Z}^d$ with $d \geqslant 3$, the salient part of the previous conclusion is not limited to large β. Indeed, the interface is expected to be *localized* in the sense (2.4) for all $\beta > 0$; see, e.g., Bricmont, Fontaine and Lebowitz [10] for a proof for the SOS model (where $V_{x,y}(\eta) := |\eta|$ for nearest neighbors and zero otherwise). On the other hand, in spatial dimension $d = 1$ the interface is always *delocalized* in the sense that the $\lim_{n\to\infty}\langle(\phi_x - \phi_y)^2\rangle_{n,\beta}$ grows linearly with $|x - y|$.

2.2 Roughening transition for 2D interfaces.

The behavior of integer-valued models and even just $\mathbb{Z}$*-modulated* ones, for which ν is a 1-periodic measure, in spatial dimension $d = 2$ is special and has been the source of much interest and effort of mathematical physicists and probabilists alike. Indeed, here one expects both types of behavior to arise depending on the value of β. Specifically, localization in the sense (2.4) should occur for $\beta > \beta_{\mathrm{c}}$ and delocalization for $\beta < \beta_{\mathrm{c}}$, where β_{c} is a positive and finite critical value. The phase transition at β_{c} is referred to as *roughening*; see e.g. [10] for a discussion of this phenomenon.

The roughening transition bears a close connection to another remarkable transition in two-dimensional models; namely, the *Berezinskii-Kosterlitz-Thouless* (BKT) phase transition predicted independently by Berezinskii [13] and Kosterlitz and Thouless [44] for, e.g., the XY-model on $\mathbb{Z}^2$. A common point is a power-law decay of correlations on one side of β_c in contrast to exponential decay on the other side. In the XY-model the power-law decay occurs at β large as a "residue" of long-range order which does occur in these models in $d \geqslant 3$ but is impossible in $d = 2$ due to the Mermin-Wagner phenomenon. In $\mathbb{Z}$-modulated interface models a power-law decay takes place at β small where it reflects on the discrete nature of the fields being washed out at large spatial scales.

The first mathematical treatment of a BKT phase transition was achieved by Fröhlich and Spencer [35] who proved that, in the DG-model as well as sine-Gordon and other models of this type, the fractional charge correlations,

$$x, y \mapsto \langle \mathrm{e}^{2\pi \mathrm{i}\alpha(\phi_x - \phi_y)} \rangle_{n,\beta} \tag{2.5}$$

with α small, exhibit power-law decay in $|x-y|$ when $n \gg |x-y| \gg 1$ at high temperatures; i.e., for β small. (The decay is exponential when β is large.) The argument of [35] was later extended throughout the "asymptotic subcritical regime" by Marchetti and Klein [48] although this is not the same as controlling the model up to the conjectural critical value β_c. Alternative presentations appeared in PhD thesis of Braga [21] and a recent paper by Kharash and Peled [43].

Fröhlich and Spencer's result (see [43, Theorem 1.1]) implies that ϕ is logarithmically-correlated at small β while it exhibits exponential decay of correlations at large β. A different point of view has been pursued by Lammers [45] and Aizenman, Harel, Peled and Shapiro [4] who focus on the asymptotic properties of the variance function

$$x \mapsto \langle \phi_x^2 \rangle_{n,\beta} \tag{2.6}$$

in the limit as $n \to \infty$. By way of monotonicity arguments they established existence of a threshold $\tilde{\beta}_c \in (0,\infty)$ such that, for x deep inside Λ_n,

$$\lim_{n\to\infty} \langle \phi_x^2 \rangle_{n,\beta} \begin{cases} < \infty, & \text{if } \beta > \tilde{\beta}_c, \\ = \infty, & \text{if } \beta < \tilde{\beta}_c. \end{cases} \tag{2.7}$$

Still, the transition at $\tilde{\beta}_c$ has yet to be linked to the (conjectural) threshold β_c for polynomial decay-rate of the fractional charge.

The extreme ends of the two phases have in the meantime been studied by perturbative methods. As mentioned earlier, the very low-temperature regime ($\beta \gg 1$) can be analyzed by contour expansions. (Notably, an interesting remnant of the GFF-connection persists in the behavior of the maximum; see Lubetzky, Martinelli and Sly [46].) Important inroads have also been made into the high-temperature regime ($\beta \ll 1$) using the renormalization group method, where the field is expected to scale to a continuum Gaussian Free Field, albeit at some effective inverse temperature. For the sine-Gordon model (1.3) with small κ this was shown by Dimock and Hurd [31] and for the DG-model by Bauerschmidt, Park and Rodriguez [7,8].

The behavior at β_c is yet different. Indeed, the convergence to continuum Gaussian Free Field is expected to persist but only with additional logarithmic corrections popping up in correlation functions. The only context in which this seems to have been controlled mathematically is the remarkable work of Falco [33, 34] who determined the asymptotic form of the fractional charge for the lattice sine-Gordon model (1.3) at $\beta = \beta_c(\kappa)$ for $\kappa > 0$ small.

2.3 Hierarchical models.

The present work focuses on $\mathbb{Z}$-modulated interface models with interactions having a hierarchical structure. Hierarchical models were originally introduced by Dyson [32] as systems that are friendly to coarse-graining arguments. They soon became a testing ground for the study of critical behavior (e.g., Bleher and Sinai [23, 24]). For similar reasons, they also served well in the analysis of interacting fields using the real-space renormalization group method; see e.g., Brydges [25].

Mathematicians often resort to hierarchical models when the actual model of interest is just too hard but one still wishes to make serious predictions about its behavior. This was the case in, e.g., the classical studies of "triviality" of the four-dimensional φ^4 and Ising models (Gawędzki and Kupiainen [37], Hara, Hattori and Watanabe [40]) whose lattice counterparts have now been established as well, albeit along rather different lines. The trend to test a hierarchical setting first continues; see e.g., Hutchcroft's recent work [41, 42] on hierarchical critical percolation. The present paper is a similar attempt for two-dimensional $\mathbb{Z}$-modulated interface models.

Our hierarchical models fall under the umbrella of GFF-like interface systems discussed after (2.1) but with the conductances of the associated Markovian generator (2.3) taking constant values on annuli $\mathcal{B}_k(x) \smallsetminus \mathcal{B}_{k-1}(x)$; namely,

$$\mathfrak{c}(x,y) := \mathfrak{c}_k \quad \text{for} \quad k := \log_{b^{1/2}} d(x,y) \tag{2.8}$$

whenever $x \neq y$. For $\{\mathfrak{c}_k\}_{k\geqslant 1}$ as in (1.5) of Assumption 1.1, calculations show $\mathfrak{c}_k \asymp b^{-2k}$ and so we have

$$\mathfrak{c}(x,y) \asymp d(x,y)^{-4} \tag{2.9}$$

at large separations of x and y. (Recall that $d(x,y)$ is comparable with $\|x-y\|_\infty$ at typical vertices of Λ_n.) It is worth noting that long-range conductance/percolation models over $\mathbb{Z}^2$ with this kind of decay are known to exhibit interesting scaling phenomena; e.g., in the scaling of the graph-theoretical distance (Bäumler [9]) and, conjecturally, in superdiffusive behavior of random walks on such percolation graphs. The polynomial decay built into the interaction naturally amplifies the critical properties of two-dimensional hierarchical interface models.

2.4 Duality with Coulomb gas.

As noted earlier, the $\mathbb{Z}$-modulated interface models are dual to Coulomb gas models, which describe systems of charged particles interacting via Coulomb forces. A configuration of such a system is an assignment $\{q_x \colon x \in \Lambda_n\}$ of $\mathbb{Z}$-valued electrostatic charges to vertices of Λ_n. The Coulomb electrostatic energy is then given by $\frac{1}{2}(q,(-\Delta_n)^{-1}q)$ and the equilibrium distribution of the charge configuration at inverse temperature β is thus given by the Gibbs law

$$\widetilde{P}_{n,\beta}(\mathrm{d}q) := \frac{1}{\widetilde{Z}_n(\beta)}\,\mathrm{e}^{\frac{\beta}{2}(q,\Delta_n^{-1}q)} \prod_{x\in\Lambda_n} \omega(\mathrm{d}q_x), \tag{2.10}$$

where ω is an *a priori* measure on charge configurations at each vertex. Physical reasons dictate that ω is concentrated on $\mathbb{Z}$ and obeys $\omega(-\mathrm{d}q) = \omega(\mathrm{d}q)$.

The link between (1.1) and (2.10) is facilitated by the so-called *sine-Gordon transformation* (also known as Siegert transformation after [52]) which amounts to the following: For any $f\colon \Lambda_n \to \mathbb{R}$ a calculation shows

$$\left\langle \mathrm{e}^{2\pi \mathrm{i}(\phi,f)}\right\rangle_{n,\beta} = \int_{\mathbb{Z}^{\Lambda_n}} \mathrm{e}^{\frac{\beta'}{2}(q+f,\,\Delta_n^{-1}(q+f))} \prod_{x\in\Lambda_n} \omega(\mathrm{d}q_x), \tag{2.11}$$

where

$$\beta' := 4\pi^2/\beta \tag{2.12}$$

while

$$\omega(\mathrm{d}q) := a(q)\#(\mathrm{d}q) \tag{2.13}$$

for $\{a(q)\}_{q\in\mathbb{Z}}$ the Fourier coefficients of ν and # the counting measure on $\mathbb{Z}$. Writing $\langle - \rangle^{\sim}_{n,\beta'}$ for expectation with respect to $\widetilde{P}_{n,\beta'}$, this becomes

$$\left\langle \mathrm{e}^{2\pi \mathrm{i}(\phi,f)}\right\rangle_{n,\beta} = \mathrm{e}^{\frac{\beta'}{2}(f,\,\Delta_n^{-1}f)} \left\langle \mathrm{e}^{\beta'(q,\Delta_n^{-1}f)}\right\rangle^{\sim}_{n,\beta'}, \tag{2.14}$$

where we noted that taking $f := 0$ in (2.11) gives $\widetilde{Z}_n(\beta') = 1$. In particular, the measures $P_{n,\beta}$ and $\widetilde{P}_{n,\beta'}$ determine each other.

Through the above connection, the DG-model is dual to the so-called *Villain gas*, which corresponds to $a(q) = 1$ for all $q \in \mathbb{Z}$ and both ν and ω being the counting measure on $\mathbb{Z}$. For the sine-Gordon models (1.3) we get (1.9) while for the hard-core Coulomb gas (1.10) we get $a(0) := 1$, $a(\pm 1) := \kappa \in [0, 1/2]$ and $a(q) = 0$ for $q \neq -1, 0, +1$. (This is interpreted as a rule that at most one particle can appear at each vertex, giving the model its name.) Note that, by (2.12), the high-temperature regime of the fields corresponds to the low-temperature regime of the Coulomb gas, and *vice versa*.

The connection of our models to the Coulomb gas is a central motivation for the consideration (and reason for the name) of the fractional charge correlation (2.5). Indeed, setting $f := \alpha\delta_x - \alpha\delta_y$ in (2.11) gives

$$\left\langle \mathrm{e}^{2\pi \mathrm{i}\alpha(\phi_x-\phi_y)}\right\rangle_{n,\beta} = \int_{\mathbb{Z}^{\Lambda_n}} \mathrm{e}^{\frac{\beta'}{2}(q+\alpha\delta_x-\alpha\delta_y,\,\Delta_n^{-1}(q+\alpha\delta_x-\alpha\delta_y))} \prod_{x\in\Lambda_n} \omega(\mathrm{d}q_x). \tag{2.15}$$

The negative of the quantity in the exponent,

$$\frac{1}{2}\big(q + \alpha\delta_x - \alpha\delta_y,\, (-\Delta_n)^{-1}(q + \alpha\delta_x - \alpha\delta_y)\big), \tag{2.16}$$

has the interpretation of the Coulomb energy of the charge configuration $q + \alpha\delta_x - \alpha\delta_y$; namely, the fluctuating "background" distribution q with a "static" charge α inserted at x and a "static" charge $-\alpha$ inserted at y.

A power-law decay of the fractional charge correlation is indicative of a logarithimic growth of this energy as the separation of x and y increases to infinity, while an exponential decay to a non-zero constant (which is what is expected in lattice models) makes the energy gain bounded. The change in the behavior for β' small is explained by the so-called *Debye screening* which is a mechanism through which the ambient charges shield the monopole at x from the monopole at y to make their existence at large separation less costly than if these monopoles were placed in a vacuum.

As is well known (see Brydges [25, Section 3.1]), the Debye screening is far less pronounced in the hierarchical models than what is expected in lattice models. Indeed, as shown in Theorem 1.4, for $\beta > \beta_c$ the energy still increases logarithmically but now with a smaller overall scale than for the $\beta < \beta_c$, where it behaves as in a vacuum. An additional iterated-log correction to the energy appears at $\beta = \beta_c$ similarly as shown in the lattice sine-Gordon model with small κ by Falco [33,34].

3. Renormalization group flow

We are now ready to commence the proofs of Theorems 1.3–1.4. As noted earlier, we rely on the renormalization-group method that works particularly well in the hierarchical setting. Here we review the steps that turn the model (1.1) to the form amenable to analysis by this method and state the relevant conclusions. The $\beta \leqslant \beta_c$-part of these can largely be drawn from our earlier work [16] so we give the needed proofs here. The proofs for $\beta > \beta_c$ are deferred to Section 6.

3.1 Representation as a tree-indexed Markov chain.

The (x-space) renormalization-group analysis of a Gibbs measure of the form (1.1) typically consists of repeated applications of two steps: a coarse-graining step and a renormalization step. In the coarse-graining step we partition the system into disjoint blocks and integrate the configuration on each block conditional on a suitable "representative" value. The renormalization step then casts the integrated Gibbs weight (which is a function of the "representative" values) as the Gibbs weight for a new energy function with suitably adjusted, or "renormalized," potentials or coefficients. The hope is that the resulting "flow" of the energy functions captures the large-scale correlations of the original Gibbs measure.

For the coarse-graining step in the hierarchical models (1.1) we use blocks that are just balls $\mathcal{B}_k(x)$ in the ultrametric distance (1.11). Note that two such balls are either equal or disjoint and so Λ_n partitions into b^{n-k} of such disjoint balls which we will refer to as k-blocks. The choice of the "representative value" relies on a "finite-range" decomposition of the inverse Laplacian Δ_n^{-1} stated in:

Lemma 3.1 *Given $n \geqslant 1$, suppose that $\{\mathfrak{c}_k\}_{k=1}^{n+1}$ is related to a positive sequence $\{\sigma_k^2\}_{k=0}^n$ as in (1.4–1.5). Writing $Q_k f(x) := b^{-k}\sum_{y\in\mathcal{B}_k(x)} f(y)$ for the orthogonal projection of $f\colon \Lambda_n \to \mathbb{R}$ on its averages over k-blocks, we then have*

$$(-\Delta_n)^{-1} = \sigma_0^2 Q_0 + \sum_{k=1}^n \sigma_k^2\, b^k Q_k. \tag{3.1}$$

Proof. We start by recalling facts from the proof of [16, Lemma 3.1]: The family of operators $\{Q_k - Q_{k+1}\colon k = 0,\dots,n\}$, subject to the convention $Q_{n+1} := 0$, are orthogonal projections on orthogonal subspaces of $\ell^2(\Lambda_n)$ such that $Q_0 = \sum_{k=0}^n (Q_k - Q_{k+1})$ is the identity. As a consequence, any operator $\mathcal{L}_n$ on $\ell^2(\Lambda_n)$ of the form

$$\mathcal{L}_n := -u_n^{-1} Q_0 + \sum_{k=1}^n (u_{k-1}^{-1} - u_k^{-1})(Q_k - Q_0) \tag{3.2}$$

for some constants $\{u_k\}_{k=0}^n$ inverts to

$$\mathcal{L}_n^{-1} := -u_0 Q_0 - \sum_{k=1}^{n} (u_k - u_{k-1}) Q_k, \tag{3.3}$$

see [16, Eqs. (3.11–3.13)], provided that $u_0 > 0$ and $u_k > u_{k-1}$ for $k = 1, \dots, n$.

Now observe that the operator Δ_n from (1.2) takes the form (3.2) if

$$u_n^{-1} = \mathfrak{c}_{n+1} \tag{3.4}$$

and

$$u_{k-1}^{-1} - u_k^{-1} = b^k \mathfrak{c}_k, \quad k = 1, \dots, n, \tag{3.5}$$

while (3.3) matches (3.1) if

$$u_k = \sum_{j=0}^{k} b^j \sigma_j^2, \quad k = 0, \dots, n. \tag{3.6}$$

As a calculation shows, under (1.4–1.5) we get (3.4–3.6) as desired. □

Remark 3.2 In the literature (see, e.g., [6, Section 1.3]) the massive hierarchical Laplacian is sometimes presented in the form

$$-m^2 Q_0 - \sum_{k=0}^{n-1} L^{-2k} (Q_k - Q_{k+1}), \tag{3.7}$$

where m^2 is the "mass-squared" and where we write the coefficient using the scale L. Noting that the right-hand side of (3.2) rewrites as $-\sum_{k=0}^{n} u_k^{-1}(Q_k - Q_{k+1})$, the form (3.7) agrees with (3.2) provided we set $u_k^{-1} := m^2 + L^{-2k}$ for $k = 0, \dots, n-1$ and $u_n^{-1} := m^2$. (Observe that (3.4) then gives $\mathfrak{c}_{n+1} = m^2$.) This in turn matches (3.6) with $b := L^2$ provided that

$$\sigma_k^2 = \frac{L^{-4k}(L^2 - 1)}{(m^2 + L^{-2k})(m^2 + L^{2-2k})}, \quad k = 1, \dots, n-1, \tag{3.8}$$

with the "boundary" cases given as

$$\sigma_0^2 = \frac{1}{m^2 + 1} \quad \text{and} \quad \sigma_n^2 = \frac{L^{2-4n}}{m^2(m^2 + L^{2-2n})}. \tag{3.9}$$

Assuming that m^2/L^{-2n} is bounded between two positive constants uniformly in $n \geqslant 1$, a calculation shows that, for $k = 1, \dots, n-1$,

$$\sigma_k^2 - (1 - L^{-2}) = O(L^{-2(n-k)}) \tag{3.10}$$

while $\sigma_0^2 = 1 + O(L^{-2n})$ and $\sigma_n^2 = O(1)$ and so, modulo scaling by $1 - L^{-2}$, the operator (3.7) thus conforms to Assumption 1.1 with $\{\mathfrak{d}_k\}_{k \geqslant 0}$ decaying exponentially. This example is actually the prime motivation for the setting in Assumption 1.1.

The representation (3.1) allows us to view $\mathrm{e}^{\frac{1}{2}\beta(\varphi, \Delta_n \varphi)}$ in (1.1) as a convolution of $n+1$ probability densities of Gaussian fields on Λ_n with covariances $\sigma_0^2 Q_0, \sigma_1^2 b Q_1, \dots, \sigma_n^2 b^n Q_n$, respectively. (A caveat is that these densities are singular because the field with covariance Q_k is constant on each k-block.) Adding the integral over ϕ with respect to the product measure $\prod_{x \in \Lambda_n} \nu(\mathrm{d}\phi_x)$, we then perform one integral after another, starting from that for ϕ itself and proceeding to the field with covariance $\sigma_0^2 Q_0$, then to the field with covariance $\sigma_1^2 bQ_1$, etc. The "representative" value of a block is, at each step, the sum of the Gaussian fields yet to be integrated. (This field is constant on each k-block so we just take the value at any point in the k-block.)

As a consequence (see [16, Section 3.3]), after k integrals have been performed and the result has been expressed as a field on Λ_{n-k}, the resulting "renormalized" Gibbs measure admits a density with respect to the law of the Gaussian field $\varphi^{(k)}$ on Λ_{n-k} with covariance $\sigma_k^2 Q_0 + \sigma_{k+1}^2 bQ_1 + \cdots + \sigma_n^2 b^{n-k} Q_{n-k}$ that is proportional to

$$\exp\Big\{-\sum_{x\in\Lambda_{n-k}} bv_{k-1}(\varphi_x^{(k)})\Big\}. \tag{3.11}$$

Here $\{v_k\}_{k=0}^n$ is a sequence of potentials defined, for $k=0,\dots,n-1$, recursively by

$$\mathrm{e}^{-v_{k+1}(z)} := \int \mathrm{e}^{-bv_k(z+\zeta)}\mu_{\sigma_{k+1}^2/\beta}(\mathrm{d}\zeta) \tag{3.12}$$

where μ_{σ^2} denotes the law of $\mathcal{N}(0,\sigma^2)$, with the "initial value" set as

$$\mathrm{e}^{-v_0(z)} := \int \mathrm{e}^{-\frac12\beta\sigma_0^{-2}(z-\zeta)^2}\nu(\mathrm{d}\zeta). \tag{3.13}$$

The 1-periodicity of ν implies that all v_k's are 1-periodic. The renormalization group flow is thus encoded by the sequence $\{v_k\}_{k=0}^n$ of functions of one variable. (This sequence depends on n but we suppress that from the notation.)

The primary output of the above procedure is a representation of the normalization constant of the Gibbs measure (1.1) as

$$Z_n(\beta) = \frac{\beta^{\frac{|\Lambda_n|}{2}}\sqrt{\det(-\Delta_n)}}{\big(\frac{2\pi\sigma_0^2}{\beta}\big)^{\frac{|\Lambda_n|}{2}}}\,\mathrm{e}^{-v_n(0)}, \tag{3.14}$$

where the numerator, resp., the denominator in the prefactor are the quantities that normalize $\phi\mapsto \mathrm{e}^{\frac{\beta}{2}(\phi,\Delta_n\phi)}$, resp., $\phi\mapsto \mathrm{e}^{-\frac{\beta}{2}\sigma_0^{-2}\sum_{x\in\Lambda_n}\phi_x^2}$ into probability densities. (Note that the ν-dependence is now hidden inside v_n.) In order to control expectations of relevant local observables, standard treatments of rigorous renormalization group proceed by incorporating the observable into suitably modified potentials whose "flow" then needs to be controlled alongside $\{v_k\}_{k=0}^n$.

In our earlier work [16] we instead took a different approach that is based on representing the full Gibbs measure (1.1) via a *tree-indexed Markov chain*. Consider a b-ary rooted tree $\mathbb{T}_n$ of depth n with the root denoted by ϱ and note that, keeping the same root, $\mathbb{T}_n$ naturally embeds $\mathbb{T}_k$ for each $k=0,\dots,n$. Let $m\colon \mathbb{T}_n\smallsetminus\{\varrho\}\to\mathbb{T}_n$ be the map assigning to x the nearest vertex on the unique path from x to the root. For $k=0,\dots,n$, define the probability kernels

$$\mathfrak{p}_k(\mathrm{d}\varphi|\varphi') := \begin{cases} \mathrm{e}^{v_k(\varphi')-bv_{k-1}(\varphi)}\,\mu_{\sigma_k^2/\beta}(-\varphi'+\mathrm{d}\varphi), & \text{if } k\geqslant 1,\\ \mathrm{e}^{v_0(\varphi')-\frac{\beta}{2}\sigma_0^{-2}(\varphi-\varphi')^2}\,\nu(\mathrm{d}\varphi), & \text{if } k=0.\end{cases} \tag{3.15}$$

Now generate a family of random variables

$$\{\varphi_x\colon x\in\mathbb{T}_n\} \tag{3.16}$$

as follows: Sample φ_ϱ from $\mathfrak{p}_n(\cdot|\,0\,)$. Then, for each $k = 1,\dots,n$, assuming that the values of φ on $\mathbb{T}_{k-1}$ have already been sampled, draw φ_x for each $x \in \mathbb{T}_k \smallsetminus \mathbb{T}_{k-1}$ from $\mathfrak{p}_{n-k}(\cdot|\varphi_{m(x)})$, independently for different x. We then have:

Lemma 3.3 *Under the canonical identification of Λ_n with the leaves of $\mathbb{T}_n$, the restriction of the family* (3.16) *to Λ_n is distributed according to $P_{n,\beta}$ from* (1.1).

Proof. This is a restatement of [16, Lemma 3.2] modulo the fact that there the proof was performed only for $\{\sigma_k^2\}_{k=0}^n$ equal to one. We leave the modifications to the reader. □

Note that the definition implies that the values of (3.16) along any path from the root to a leaf-vertex is an ordinary (time-inhomogeneous) Markov chain with transition probabilities (3.15). This will be very useful in our later calculations.

3.2 Results for renormalization group iterations.

Our ability to control the above tree-indexed Markov chain depends very strongly on our ability to control the differences $v_k(\varphi') - bv_{k-1}(\varphi)$ for large values of k (and n). In high-temperature approaches to this problem (see, e.g., Bauerschmidt and Bodineau [6]) this is done by linearization of (3.12). However, linearization becomes inefficient if we want to work uniformly up to β_c, or even beyond, so in [16] we instead followed the flow of the Fourier coefficients of e^{-v_k}, defined for $k = 0,\dots,n$ by

$$a_k(q) := \int_0^1 \mathrm{e}^{-v_k(z)-2\pi\mathrm{i}qz}\,\mathrm{d}z. \tag{3.17}$$

As shown in [16, Lemma-4.1], in light of (3.12) these coefficients iterate as

$$a_{k+1}(q) := \sum_{\substack{\ell_1,\dots,\ell_b\in\mathbb{Z}\\ \ell_1+\dots+\ell_b=q}} \left[\prod_{i=1}^b a_k(\ell_i)\right]\theta_{k+1}^{q^2}, \quad q\in\mathbb{Z}, \tag{3.18}$$

where

$$\theta_k := \mathrm{e}^{-\frac{2\pi^2}{\beta}\sigma_k^2} \tag{3.19}$$

and where the "initial" value is set as

$$a_0(q) := \sqrt{\frac{2\pi\sigma_0^2}{\beta}}\,a(q)\,\theta_0^{q^2} \quad \text{for} \quad a(q) := \int_{[0,1)} \mathrm{e}^{-2\pi\mathrm{i}qz}\nu(\mathrm{d}z). \tag{3.20}$$

As also noted in [16, Lemma-4.1] (whose proof only needs that $0 \leqslant \theta_{k+1} < 1$), the conditions in Assumption 1.2 ensure that $a_k(q) > 0$ for all $k \geqslant 0$ and $q \in \mathbb{Z}$ and that $\{a_k(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ for all $k = 0,\dots,n$. (As for the v_k's, the a_k's also depend on n but we do not mark that explicitly in the notation.)

We will now state our results concerning the iterations (3.12) and (3.18). Our first theorem concerns the subcritical β:

Theorem 3.4 (Subcritical flow) *Suppose that Assumptions 1.1–1.2 hold. For each $b \geqslant 2$ and each $\beta > 0$ with $\beta < \beta_c$ there exist $\eta > 0$ and $C > 0$ such that for all $n \geqslant k \geqslant 0$,*

$$0 < \frac{a_k(q)}{a_k(0)} \leqslant C\mathrm{e}^{-\eta(k|q|+q^2)}, \quad q\in\mathbb{Z}. \tag{3.21}$$

Moreover, the v_k's are C^∞-functions with

$$\sup_{z,z'\in\mathbb{R}}\bigl|v_{k+1}(z)-bv_k(z')\bigr|\leqslant C\mathrm{e}^{-\eta k} \tag{3.22}$$

for all $n>k\geqslant 0$, and

$$\sup_{z\in\mathbb{R}}\max\bigl\{|v_k'(z)|,|v_k''(z)|\bigr\}\leqslant C\mathrm{e}^{-\eta k} \tag{3.23}$$

for all $n\geqslant k\geqslant 0$.

The next result provides a similar statement at β_c, where control of the iterations is more subtle than in the subcritical cases.

Theorem 3.5 (Critical flow) *Suppose that Assumptions 1.1–1.2 hold and assume $\beta=\beta_\mathrm{c}$. For each $b\geqslant 2$ there exist $\eta>0$ and $C>0$ such that for all $n\geqslant k\geqslant 0$,*

$$0<\frac{a_k(q)}{a_k(0)}\leqslant\frac{C\mathrm{e}^{-\eta q^2}}{(1+\sqrt{k})^{|q|}},\quad q\in\mathbb{Z}. \tag{3.24}$$

Moreover, the v_k's are C^∞-functions with

$$\sup_{z,z'\in\mathbb{R}}\bigl|v_{k+1}(z)-bv_k(z')\bigr|\leqslant\frac{C}{\sqrt{1+k}} \tag{3.25}$$

for all $n>k\geqslant 0$, and

$$\sup_{z\in\mathbb{R}}\max\bigl\{|v_k'(z)|,|v_k''(z)|\bigr\}\leqslant\frac{C}{\sqrt{1+k}} \tag{3.26}$$

for all $n\geqslant k\geqslant 0$. Furthermore, we have

$$\frac{a_k(1)}{a_k(0)}=\frac{1}{\sqrt{k}}\biggl[\frac{b^3-1}{(b-1)^2(b+1)^3}\biggr]^{1/2}+O(k^{-1})+O\biggl(k^{-1/2}\sum_{j\geqslant\min\{\sqrt{k},n-k\}}\mathfrak{d}_j\biggr), \tag{3.27}$$

where $\{\mathfrak{d}_j\}_{j\geqslant 0}$ is the sequence from Assumption 1.1. Consequently,

$$v_k'(z)=\frac{4\pi}{\sqrt{k}}\biggl[\frac{b^3-1}{(b-1)^2(b+1)^3}\biggr]^{1/2}\sin(2\pi z)+\frac{o(1)}{\sqrt{k}} \tag{3.28}$$

where $o(1)\to 0$ as $\min\{k,n-k\}\to\infty$, uniformly in $z\in\mathbb{R}$.

The asymptotic (3.28) implies that $\{|v_k'|^2\}_{k=0}^n$ is not summable uniformly in n which, as we will see in Section 4, is the root cause of the doubly-logarithmic correction to the covariance structure at $\beta=\beta_\mathrm{c}$. The logarithmic correction to the fractional charge asymptotic at $\beta=\beta_\mathrm{c}$ can in turn be traced to (3.27).

Our final theorem in this section deals with supercritical β. The statement only applies to β slightly over β_c. Denote

$$\theta:=\mathrm{e}^{-\frac{2\pi^2}{\beta}} \tag{3.29}$$

and observe that $\beta>\beta_\mathrm{c}$ is equivalent to $b\theta>1$. We then have:

Theorem 3.6 (Supercritical flow) *Suppose Assumption 1.1 holds. For each $b \geqslant 2$ there exists $\epsilon > 0$ and, for all $\beta > 0$ with $1 < b\theta < 1 + \epsilon$, there exist $\eta > 0$, $C > 0$ and a sequence $\{\lambda_\star(q)\}_{q\in\mathbb{Z}}$ with $\lambda_\star(0) = 1$ and*

$$0 < \lambda_\star(q) = \lambda_\star(-q) \leqslant \bigl(2b^{1/2}\sqrt{b\theta - 1}\,\bigr)^{|q|}, \quad q \in \mathbb{Z}, \tag{3.30}$$

for which the following is true: For all initial ν subject to Assumption 1.2 there exists $k_0 \geqslant 0$ such that for all n and k with $\min\{k, n-k\} \geqslant k_0$ and all $q \in \mathbb{Z}$,

$$a_k(q) \leqslant a_k(0)\bigl(2b^{1/2}\sqrt{b\theta - 1}\,\bigr)^{|q|} \tag{3.31}$$

and

$$\left|\frac{a_k(q)}{a_k(0)} - \lambda_\star(q)\right| \leqslant C^{|q|}\left[\mathrm{e}^{-\eta k} + \sum_{j=0}^{k} \mathrm{e}^{-\eta(k-j)}\, \mathfrak{d}_{\min\{j,n-j\}}\right] \tag{3.32}$$

hold with $\{\mathfrak{d}_j\}_{j\geqslant 0}$ denoting the sequence from Assumption 1.1. Moreover, the v_k's are C^∞-functions with $\{v_k'\}_{k\geqslant 0}$ and $\{v_k''\}_{k\geqslant 0}$ uniformly bounded and, defining $v_\star\colon \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{e}^{-v_\star(z)} := \Biggl(\sum_{\substack{q_1,\dots,q_b\in\mathbb{Z}\\ q_1+\dots+q_b=0}} \prod_{i=1}^{b} \lambda_\star(q_i)\Biggr)^{-\frac{1}{b-1}} \sum_{q\in\mathbb{Z}} \lambda_\star(q)\mathrm{e}^{2\pi \mathrm{i} qz}, \tag{3.33}$$

we have $v_k(z) - bv_{k-1}(z') \to v_\star(z) - bv_\star(z')$ and $v_k'(z) \to v_\star'(z)$ as $\min\{k, n-k\} \to \infty$, uniformly on $z, z' \in \mathbb{R}$. (The existence of $v_\star'$ is ensured by (3.30).) In addition, assuming $\{\mathfrak{d}_k\}_{k\geqslant 0}$ decays exponentially fast there exist $\eta' > 0$ and $C' > 0$ such that

$$\sup_{z,z'\in\mathbb{R}} \Bigl|v_k(z) - bv_{k-1}(z') - \bigl[v_\star(z) - bv_\star(z')\bigr]\Bigr| \leqslant C'\mathrm{e}^{-\eta' \min\{k,n-k\}} \tag{3.34}$$

and

$$\sup_{z\in\mathbb{R}} \bigl|v_k'(z) - v_\star'(z)\bigr| \leqslant C'\mathrm{e}^{-\eta' \min\{k,n-k\}} \tag{3.35}$$

hold for all $n \geqslant k \geqslant 0$.

We emphasize that $\{\lambda_\star(q)\}_{q\in\mathbb{Z}}$ and thus also $v_\star$ do not depend on ν; indeed, they represent a "nontrivial" fixed point of the renormalization group flow. This means that $v_\star$ is a non-zero solution to

$$\mathrm{e}^{-v_\star(z)} = \int \mathrm{e}^{-bv_\star(z+\zeta)}\mu_{1/\beta}(\mathrm{d}\zeta), \tag{3.36}$$

where, we recall, $\mu_{1/\beta}$ is the law of $\mathcal{N}(0, 1/\beta)$. As our proofs show (see Theorem 6.1), such a fixed point is *unique* and attractive to all the 1-periodic measures whose Fourier coefficients are positive and obey (1.7–1.8). (These assumptions are crucial; indeed, under $b\theta^{p^2} \leqslant 1$, functions that are $1/p$-periodic are still attracted to the "trivial" fixed point.) We do not have explicit expressions for $v_\star$ or $\{\lambda_\star(q)\}_{q\in\mathbb{Z}}$. The best we can offer is a characterization of their $b \to \infty$ limit; see Remark 6.15.

3.3 Proof of Theorem 3.4.

The above convergence statements for the subcritical and critical regimes require only relatively minor adaptations of the results already proved in [16], and so we prove them right away. The main new obstacle is the fact that [16] assumed $\sigma_k^2 = 1$ for all $k \geqslant 0$ while for us the equality holds only asymptotically. We will suppose that Assumptions 1.1–1.2 hold throughout and stop referencing these in the statements of lemmas.

We will start with Theorem 3.4. First we recall an observation from [16] that drives many of the subsequent arguments:

Lemma 3.7 *For all $\beta > 0$, $n > k \geqslant 0$ and $q \geqslant 0$,*

$$\frac{a_{k+1}(q+1)}{a_{k+1}(q)} \leqslant b\,\theta_{k+1}{}^{(q+1)^2-q^2} \sup_{\ell\geqslant 0} \frac{a_k(\ell+1)}{a_k(\ell)}. \tag{3.37}$$

Proof. This is a restatement of [16, Lemma 4.2] with θ allowed to depend on k. □

As a consequence we obtain:

Lemma 3.8 *For all $\beta > 0$, $n \geqslant k \geqslant 0$ and $q \in \mathbb{Z}$,*

$$\frac{a_k(q)}{a_k(0)} \leqslant \theta_k^{q^2}\biggl(\hat{c}\prod_{j=0}^{k-1}(b\theta_j)\biggr)^{|q|}, \tag{3.38}$$

where $\hat{c} := \sup_{\ell\geqslant 0}\frac{a(\ell+1)}{a(\ell)}$ for $\ell \mapsto a(\ell)$ being the Fourier coefficients of ν.

Proof. Denote $c_k := \sup_{\ell\geqslant 0}\frac{a_k(\ell+1)}{a_k(\ell)}$. Then (3.37) along with $(q+1)^2 - q^2 \geqslant 1$ for $q \geqslant 0$ gives $c_{k+1} \leqslant (b\theta_{k+1})c_k$ with $c_0 \leqslant \hat{c}\theta_0$. Iterating, we get $c_k \leqslant b^{-1}\hat{c}\prod_{j=0}^{k}(b\theta_j)$ for all $n \geqslant k \geqslant 0$. Plugging this in (3.37) and iterating yields the claim. □

We are now ready to give:

Proof of Theorem 3.4. Suppose $0 < \beta < \beta_c$. We start with (3.21), which for σ_k^2 equal to one was shown already in [16, Lemma 4.3]. Assumption 1.1 shows $\theta_{\max} := \sup_{n\geqslant k\geqslant 0}\theta_k < 1$ and $\prod_{j=0}^{k-1} b\theta_j \leqslant \theta^{-\tilde{c}}(b\theta)^k$ where $\tilde{c} := 2\sum_{j\geqslant 0}\mathfrak{d}_j$. Using (3.38) we then get

$$\frac{a_k(q)}{a_k(0)} \leqslant \theta_{\max}^{q^2}\bigl[\hat{c}\theta^{-\tilde{c}}(b\theta)^k\bigr]^{|q|}. \tag{3.39}$$

Setting, with some waste for a later convenience, $\mathrm{e}^{-\eta} := \max\{\sqrt{b\theta}, \sqrt{\theta_{\max}}\}$, we get (3.38) with $C := \sup_{q\geqslant 0}\theta_{\max}^{q^2/2}[\hat{c}\theta^{-\tilde{c}}]^q$. The positivity follows from iterations of (3.18) and the assumption that the Fourier coefficients of ν are strictly positive.

Concerning (3.22), we observe that, by $\{a_k(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$,

$$\mathrm{e}^{-v_k(z)} = \sum_{q\in\mathbb{Z}} a_k(q)\mathrm{e}^{2\pi\mathrm{i}qz} \tag{3.40}$$

with the left-hand side continuous and, by (3.12), strictly positive for all $z \in \mathbb{R}$. Hence, the v_k's are also continuous. The bound (3.21) gives

$$\bigl|\mathrm{e}^{-v_k(z)} - a_k(0)\bigr| \leqslant 2a_k(0)\sum_{q\geqslant 1} C\mathrm{e}^{-\eta(kq+q^2)} \leqslant \frac{2C\mathrm{e}^{-k\eta}}{1-\mathrm{e}^{-k\eta}}a_k(0). \tag{3.41}$$

For k so large that $\max\{1, C\}\mathrm{e}^{-k\eta} \leqslant 1/8$ we get

$$\sup_{z\in\mathbb{R}}\bigl|\,a_k(0)^{-1}\mathrm{e}^{-v_k(z)} - 1\bigr| \leqslant 3C\mathrm{e}^{-\eta k} \leqslant \frac{1}{2}. \tag{3.42}$$

Once k is so large that the quantity in the large parentheses in (3.38) is less than $\mathrm{e}^{-\eta k}$, this now implies (3.22) along the same argument that proved [16, Eq. (3.40)]. The remaining k and n are handled directly by noting that, under Assumption 1.1, e^{-v_k} is bounded above and below by positive constants that depend only on k. Hence $|v_k|$ is bounded uniformly in n with $n \geqslant k$ and so (3.22) follows by relabeling C.

For the corresponding bound on the derivatives of v_k, first note that (3.21) permits us to differentiate the series in (3.40) term-by-term to get

$$v_k'(z) = -\mathrm{e}^{v_k(z)} \sum_{q\in\mathbb{Z}} (2\pi\mathrm{i}q) a_k(q) \mathrm{e}^{2\pi\mathrm{i}qz}. \tag{3.43}$$

Using (3.42) along with the uniform boundedness of $a_k(0)\mathrm{e}^{v_k}$ for each k we conclude that the v_k' are continuous and bounded on $\mathbb{R}$, uniformly in $n \geqslant k \geqslant 0$. It thus suffices to prove (3.23) for k sufficiently large. Here we invoke the bound (3.21) to get

$$\left|v_k'(z)\right| \leqslant a_k(0)\mathrm{e}^{v_k(z)} \sum_{q\geqslant 1} 4\pi q C \mathrm{e}^{-\eta k q} \leqslant a_k(0)\mathrm{e}^{v_k(z)} \frac{4\pi C\mathrm{e}^{-\eta k}}{(1-\mathrm{e}^{-\eta k})^2}. \tag{3.44}$$

Since $a_k(0)\mathrm{e}^{v_k(z)} \leqslant 2$ whenever (3.42) is in force, the right-hand side is at most $32\pi C\mathrm{e}^{-\eta k}$ as soon as $\mathrm{e}^{-\eta k} \leqslant 1/8$. This proves (3.23) for the first derivative. For the bound on the second derivative we differentiate (3.43) one more time and apply a similar reasoning, along with the bound on the first derivative. We leave the details to the reader. □

3.4 Bounds on Fourier coefficients.

For the critical case, we first need to establish estimates on the Fourier coefficients $a_k(q)$. We start with a bound that drove the analysis of the critical case in [16]:

Lemma 3.9 *For all $\beta > 0$ and $n > k \geqslant 0$,*

$$\frac{a_{k+1}(1)}{a_{k+1}(0)} \leqslant \theta_{k+1} \frac{\dfrac{a_k(1)}{a_k(0)}}{1 + \dbinom{b}{2}\left(\dfrac{a_k(1)}{a_k(0)}\right)^2} + (b-1)\theta_{k+1} \sup_{\ell\geqslant 0} \frac{a_k(\ell+1)}{a_k(\ell)}. \tag{3.45}$$

Proof. This is a restatement of [16, Lemma 4.5] with θ allowed to depend on k. □

Next we show that the supremum on the right of (3.45) exhibits polynomial decay:

Lemma 3.10 *Assume $\beta = \beta_c$. Then there exists a constant $C > 0$ such that for all $n \geqslant k \geqslant 0$,*

$$\sup_{\ell\geqslant 0} \frac{a_k(\ell+1)}{a_k(\ell)} \leqslant \frac{C}{\sqrt{1+k}}. \tag{3.46}$$

Proof. We will adapt the proofs of [16, Lemma 4.6] and [16, Theorem 3.5] to allow σ_k^2 depend on n and k. Abbreviate the supremum in (3.46) as c_k and let α_k be the unique number in $(0, 1)$ such that

$$\alpha_k = \frac{1}{1 + \binom{b}{2} c_k^2 \alpha_k^2}. \tag{3.47}$$

Next we will prove that $k \mapsto c_k$ is bounded. Indeed, at $\beta = \beta_c$ we have $\theta_k = b^{-\sigma_k^2}$ and so $b\theta_k = b^{1-\sigma_k^2}$. The argument from the proof of Lemma 3.8 along with the inequality $c_0 \leqslant \theta_0\hat c \leqslant b\theta_0\hat c$ implied by (3.20) then show

$$c_k \leqslant \hat c \exp\biggl\{(\log b)\sum_{j=0}^k |1-\sigma_j^2|\biggr\} \tag{3.48}$$

with the sum is bounded uniformly in $n \geqslant k \geqslant 0$ thanks to Assumption 1.1. This along with $\alpha_k \leqslant 1$ implies $\inf_{n\geqslant k\geqslant 0}\alpha_k \geqslant [1+\binom{b}{2}(\sup_{n\geqslant k\geqslant 0} c_k)^2]^{-1} > 0$.

We will now repeat the argument from the proof of [16, Lemma 4.6] to get an iterative bound on c_k. We start by the inequality

$$\frac{a_{k+1}(1)}{a_{k+1}(0)} \leqslant \frac{\theta_{k+1}\, c_k}{1+\binom{b}{2}{c_k}^2{\alpha_k}^2} + (b-1)\theta_{k+1}c_k \tag{3.49}$$

which, for $a_k(1)/a_k(0) > \alpha_k c_k$, is obtained by bounding the denominator in (3.45) from below by $1+\binom{b}{2}\alpha_k^2c_k^2$ and then applying $a_k(1)/a_k(0) \leqslant c_k$ in the numerator. For $a_k(1)/a_k(0) \leqslant \alpha_k c_k$, we instead drop the denominator in (3.45) altogether, invoke $a_k(1)/a_k(0) \leqslant \alpha_k c_k$ in the numerator and then observe that, by (3.47), right-hand side of (3.49) equals $(b-1+\alpha_k)\theta_{k+1}c_k$. With (3.49) in hand, observe that Lemma 3.7 also gives

$$\frac{a_{k+1}(q+1)}{a_{k+1}(q)} \leqslant b\,{\theta_{k+1}}^{(q+1)^2-q^2}c_k \tag{3.50}$$

with $(q+1)^2-q^2 \geqslant 3$ for $q \geqslant 1$. Noting again that the right-hand side of (3.49) can been written as $(b-1+\alpha_k)\theta_{k+1}c_k$, we get

$$c_{k+1} \leqslant \frac{\theta_{k+1}\, c_k}{1+\binom{b}{2}{c_k}^2{\alpha_k}^2} + (b-1)\theta_{k+1}c_k \tag{3.51}$$

as soon as $\min\{k, n-k\}$ is so large that $b{\theta_{k+1}}^3 \leqslant (b-1+\alpha_{k+1})\theta_k$.

To deal with k dependence of θ_k and α_k in (3.51), denote

$$\tilde c_k := c_k \exp\biggl\{\log(1/\theta)\sum_{j=0}^k(\sigma_j^2-1)\biggr\}, \tag{3.52}$$

and abbreviate

$$\bar\alpha := \binom{b}{2}\bigl(\inf_{n\geqslant k\geqslant 0}\alpha_k^2\bigr)\exp\biggl\{-4\log(1/\theta)\sum_{j\geqslant 0}\mathfrak{d}_j\biggr\}. \tag{3.53}$$

The bound (3.51) then gives

$$\tilde c_{k+1} \leqslant \frac{\theta\,\tilde c_k}{1+\bar\alpha{\tilde c_k}^2} + (b-1)\theta\tilde c_k \tag{3.54}$$

once k and $n-k$ exceed some $k_0 \geqslant 1$.

Now observe that setting $\theta := 1/b$ reduces (3.54) to the conclusion of [16, Lemma 4.6]. The proof of [16, Theorem 3.5] then applies, resulting in the bound $\tilde c_k \leqslant C'(1+\sqrt{k})^{-1/2}$. Noting that $c_k/\tilde c_k$ is bounded uniformly in $n \geqslant k \geqslant 0$ by Assumption 1.1 then gives (3.46) once $\min\{k, n-k\} \geqslant k_0$. Thanks to $\sup_{n\geqslant k\geqslant 0} c_k < \infty$, the extension to small k is achieved by choosing C sufficiently large. The extension to k close to n is in turn supplied by Lemma 3.7 along with Assumption 1.1. $\square$

The next lemma shows that the supremum in (3.46) is actually order $k^{-1/2}$ and, in fact, so is even the ratio $a_k(1)/a_k(0)$:

Lemma 3.11 *Assume $\beta = \beta_c$. There exists a constant $C' > 0$ such that for all $n \geqslant k \geqslant 0$,*

$$\frac{a_k(1)}{a_k(0)} \geqslant \frac{C'}{\sqrt{1+k}}. \tag{3.55}$$

Proof. Let c_k continue to denote the supremum in (3.46). We first prove suitable bounds on $a_{k+1}(1)$ and $a_{k+1}(0)$ using (3.18). Indeed, neglecting all but the terms with just one ℓ_i non-zero in (3.18) yields the lower bound

$$a_{k+1}(1) \geqslant b\theta_{k+1}a_k(1)a_k(0)^{b-1}. \tag{3.56}$$

For an upper bound on $a_{k+1}(0)$, we note that the sum in (3.18) contains a term with all zeros, then terms with exactly two indices equal to ± 1 (and all other equal to zero), and then the remaining terms in which either three indices are non-zero or two are non-zero but at least one is at least two in absolute value. Invoking the bound $a_k(\ell) \leqslant c_k^{|\ell|}a_k(0)$ while assuming, thanks to Lemma 3.10, that k is so large that $c_k \leqslant 1/2$, the contribution of the last two cases is estimated by

$$\big[b(b-1)+b(b-1)(b-2)\big]c_k^3\Bigg(1+2\sum_{m\geqslant 1}c_k^m\Bigg)^b a_k(0)^b. \tag{3.57}$$

Here the prefactors dominate the number of ways the above sets of indices can appear in the given ordering of the b-tuple $\ell_1,\dots,\ell_b$ in (3.18) and the term in the large parentheses dominates the sum over the remaining indices after the restriction on $\ell_1+\dots+\ell_b$ has been dropped. Hence we get

$$a_{k+1}(0) \leqslant a_k(0)^b + b(b-1)a_k(1)^2a_k(0)^{b-2} + \alpha' c_k^3 a_k(0)^b, \tag{3.58}$$

where $\alpha' := b(b-1)^2 3^b$. Abbreviating

$$\lambda_k := \frac{a_k(1)}{a_k(0)}, \tag{3.59}$$

from (3.56) and (3.58) we then get

$$\lambda_{k+1} \geqslant \frac{b\theta_{k+1}\lambda_k}{1+b(b-1)\lambda_k^2+\alpha' c_k^3} \geqslant \frac{b\theta_{k+1}}{1+\alpha' c_k^3}\,\frac{\lambda_k}{1+b(b-1)\lambda_k^2}. \tag{3.60}$$

At $\beta = \beta_c$ we have $b\theta_{k+1} = b^{1-\sigma_{k+1}^2}$. Denote

$$\tilde\lambda_k := \lambda_k \prod_{j=1}^k \frac{1+\alpha' c_{j-1}^3}{b\theta_j} = \lambda_k \prod_{j=1}^k \frac{1+\alpha' c_{j-1}^3}{b^{1-\sigma_j^2}}, \tag{3.61}$$

and observe that $\lambda_k/\tilde\lambda_k$ is bounded from above and below uniformly in $n \geqslant k \geqslant 1$ thanks to Assumption 1.1 and $\prod_{j\geqslant 1}\big(1+j^{-3/2}\big) < \infty$. Abbreviating $r := b(b-1)\sup_{n\geqslant k\geqslant 0}\lambda_k/\tilde\lambda_k$, the inequality (3.60) then gives

$$\tilde\lambda_{k+1}\geqslant\frac{\tilde\lambda_k}{1+r\tilde\lambda_k^2}. \tag{3.62}$$

This now readily yields $\tilde\lambda_{k+1}^{-2}\leqslant\tilde\lambda_k^{-2}+r'$ for $k\geqslant 1$ and $r':=2r+r^2\sup_{n\geqslant k\geqslant 1}\tilde\lambda_k^2<\infty$. Hence we get that $\tilde\lambda_k^{-2}/k$ is bounded uniformly in $n\geqslant k\geqslant k_0$, for some $k_0\geqslant 1$. This, along with $\lambda_k/\tilde\lambda_k$ being bounded from below gives (3.55) for k sufficiently large. The extension to $k<k_0$ is routine from the positivity of $a_k(1)/a_k(0)$ which by Assumption 1.1 and (3.60) holds uniformly in n satisfying $n\geqslant k$. □

3.5 Proof of Theorem 3.5.

Moving to the proof of Theorem 3.5, we now cast the iterations of $\{\lambda_k\}_{k\geqslant 0}$ from (3.59) in a form that is amenable to asymptotic analysis. This will require tracking also the second-order Fourier coefficients in the form

$$\gamma_k:=\frac{a_k(2)}{a_k(0)} \tag{3.63}$$

which, as suggested by (3.46) and (3.55), decays proportionally to λ_k^2. Here we get:

Lemma 3.12 *Assume $\beta=\beta_{\rm c}$. For each $n\geqslant 1$ there exist positive sequences $\{r_k\}_{k=0}^{n-1}$, $\{s_k\}_{k=0}^{n-1}$ and $\{t_k\}_{k=0}^{n-1}$ that are bounded uniformly in n such that*

$$\begin{aligned}\lambda_{k+1}&=b\theta_{k+1}\frac{\lambda_k+(b-1)\lambda_k\gamma_k+\frac12(b-1)(b-2)\lambda_k^3+r_k\lambda_k^4}{1+b(b-1)\lambda_k^2+s_k\lambda_k^3}\\ \gamma_{k+1}&=b^4\theta_{k+1}^4\frac{b^{-3}\gamma_k+\frac12b^{-3}(b-1)\lambda_k^2+t_k\lambda_k^3}{1+b(b-1)\lambda_k^2+s_k\lambda_k^3}\end{aligned} \tag{3.64}$$

hold true for all $n>k\geqslant 0$, where λ_k is as in (3.59).

Proof. Note that, by Lemmas 3.10–3.11, when $\beta=\beta_{\rm c}$ the quantity λ_k is proportional to c_k while $a_k(q)/a_k(0)$ is bounded by $c_k^{|q|}$. Using this we now write (3.58) as equality

$$\frac{a_{k+1}(0)}{a_k(0)^b}=1+b(b-1)\lambda_k^2+s_k\lambda_k^3, \tag{3.65}$$

where $\{s_k\}_{k=0}^{n-1}$ is a positive sequence that is bounded uniformly in $n\geqslant 1$. Similarly, since the only integer-valued b-tuples $(\ell_1,\dots,\ell_b)$ with $\ell_1+\dots+\ell_b=1$ and $|\ell_1|+\dots+|\ell_b|<4$ are permutations of $(1,0,\dots,0)$, $(-1,2,0,\dots,0)$ and $(1,1,-1,0,\dots,0)$, we get

$$\frac{a_{k+1}(1)}{a_k(0)^b}=\theta_{k+1}\Bigl(b\lambda_k+b(b-1)\lambda_k\gamma_k+\tfrac12b(b-1)(b-2)\lambda_k^3+br_k\lambda_k^4\Bigr) \tag{3.66}$$

for a positive sequence $\{r_k\}_{k=0}^{n-1}$ that is bounded uniformly in $n\geqslant 1$. Dividing (3.66) by (3.65) then yields the first equality in (3.64).

For the second equality in (3.64) we first observe that the only integer-valued b-tuples $(\ell_1,\dots,\ell_b)$ with $\ell_1+\dots+\ell_b=2$ and $|\ell_1|+\dots+|\ell_b|<4$ are permutations of $(2,0,\dots,0)$

and $(1,1,0,\dots,0)$. This implies

$$\frac{a_{k+1}(2)}{a_k(0)^b} = \theta_{k+1}^4\bigl(b\gamma_k + \tfrac{1}{2}b(b-1)\lambda_k^2 + b^4 t_k\lambda_k^4\bigr) \tag{3.67}$$

for a positive sequence $\{t_k\}_{k=0}^{n-1}$ that is bounded uniformly in $n \geqslant 1$. Dividing this by (3.65) then gives the desired claim. $\square$

We are now finally ready to give:

Proof of Theorem 3.5. Assume $\beta = \beta_c$. Let us start with the bounds (3.24–3.26). First, using (3.46) in (3.37) and iterating yields

$$\frac{a_k(q)}{a_k(0)} \leqslant \theta_k^{q^2}\Bigl(\frac{bC}{\sqrt{1+k}}\Bigr)^{|q|} \tag{3.68}$$

whenever $k \geqslant 1$. (For $k = 0$ we note that $a_0(q)/a_0(0) \leqslant \theta_0^{q^2}\hat{c}$, for $\hat{c}$ as in Lemma 3.8.) Since Assumption 1.1 implies that $\theta_k^{q^2/2}(bC)^{|q|}$ is bounded uniformly in $n \geqslant k \geqslant 0$ and $q \in \mathbb{Z}$, this is sufficient for (3.24). The bound (3.25) is then proved using the same argument as in [16, Theorem 3.4]. For the bound (3.26), we plug (3.68) in (3.43) using the same argument as for the subcritical case.

The main point of the proof is the asymptotic (3.27) and (3.28). For the latter we isolate the terms $|q| = 1$ from the rest of the sum in (3.43) to get

$$v_k'(z) = 4\pi \mathrm{e}^{v_k(z)}a_k(0)\lambda_k\sin(2\pi z) - \sum_{|q|\geqslant 2}\mathrm{e}^{v_k(z)}a_k(0)2\pi i q\,\mathrm{e}^{2\pi i q z}\frac{a_k(q)}{a_k(0)}. \tag{3.69}$$

Proceeding as in (3.42) using (3.24) instead of (3.21), we get uniform convergence of $a_k(0)^{-1}\mathrm{e}^{-v_k(z)}$ to 1 with decay rate $(1+k)^{-1/2}$, which leads to

$$\bigl|v_k'(z) - 4\pi\lambda_k\sin(2\pi z)\bigr| \leqslant \frac{C}{k} \tag{3.70}$$

for some constant $C \in (0,\infty)$ uniformly in $z \in \mathbb{R}$. To get (3.28) it thus suffices to prove the asymptotic (3.27).

We will prove (3.27) by iterating the top line in (3.64) but for that we first have to show that γ_k/λ_k^2 is, for k large, close to a constant. Here (3.64) expresses $\gamma_{k+1}/\lambda_{k+1}^2$ as

$$(b\theta_{k+1})^2\Bigl(b^{-3}\frac{\gamma_k}{\lambda_k^2} + \frac{1}{2}b^{-3}(b-1) + t_k\lambda_k\Bigr)\frac{1 + b(b-1)\lambda_k^2 + s_k\lambda_k^3}{\bigl(1 + (b-1)\gamma_k + \frac{1}{2}(b-1)(b-2)\lambda_k^2 + r_k\lambda_k^3\bigr)^2} \tag{3.71}$$

which in light of γ_k/λ_k^2 being bounded from above gives

$$\frac{\gamma_{k+1}}{\lambda_{k+1}^2} = \bigl((b\theta_{k+1})^2 + t_k'\lambda_k\bigr)\biggl(b^{-3}\frac{\gamma_k}{\lambda_k^2} + \frac{1}{2}b^{-3}(b-1)\biggr) \tag{3.72}$$

for a sequence $\{t_k'\}_{k=0}^{n-1}$ that is bounded uniformly in $n \geqslant 1$. Since the prefactor is close to 1 for $\min\{k, n-k\}$ large, set $\delta_k := (b\theta_{k+1})^2 + t_k'\lambda_k - 1$ and observe that then

$$\left|\frac{\gamma_{k+1}}{\lambda_{k+1}^2} - \frac{1}{2}\frac{b-1}{b^3-1}\right| \leqslant \frac{1+\delta_k}{b^3}\left|\frac{\gamma_k}{\lambda_k^2} - \frac{1}{2}\frac{b-1}{b^3-1}\right| + \delta_k. \tag{3.73}$$

Iteration shows

$$\left|\frac{\gamma_k}{\lambda_k^2}-\frac{1}{2}\frac{b-1}{b^3-1}\right|\leqslant\sum_{j=1}^k\delta_{k-j}\prod_{i=1}^{j-1}\frac{1+\delta_{k-i}}{b^3}+\left(\prod_{j=0}^k\frac{1+\delta_j}{b^3}\right)\left|\frac{\gamma_0}{\lambda_0^2}-\frac{1}{2}\frac{b-1}{b^3-1}\right|. \tag{3.74}$$

Using $\lambda_k=O(k^{-1/2})$ and $b\theta_k-1=O(|\sigma_k^2-1|)$ we get $\delta_k=O(k^{-1/2})+O(|\sigma_{k+1}{}^2-1|)$ and so $1+\delta_k\leqslant\exp\{ck^{-1/2}+c|\sigma_{k+1}^2-1|\}$ for some constant $c>0$. It follows that, for some constants $c',c''>0$,

$$\prod_{i=1}^{j-1}\frac{1+\delta_{k-i}}{b^3}\leqslant b^{-3(j-1)}\exp\left\{c\sum_{i=0}^{j-1}(k-i)^{-1/2}+c\sum_{i=0}^n|\sigma_i^2-1|\right\}\leqslant c'b^{-3(j-1)}\exp\{c''j/k^{1/2}\}, \tag{3.75}$$

where in the second inequality we invoked Assumption 1.1 for the second sum and noticed that the first sum is bounded by a constant times $j/k^{1/2}$. The product is thus checked to decay at least as $O(b^{-2j})$ which then allows us to simplify (3.74) as

$$\frac{\gamma_k}{\lambda_k^2}=\frac{1}{2}\frac{b-1}{b^3-1}+O(k^{-1/2})+O\left(\sum_{j=0}^k b^{-2j}|\sigma_{k-j}^2-1|\right). \tag{3.76}$$

Here the implicit constants are uniform in $n\geqslant k\geqslant 0$.

Moving to the proof of (3.27), we temporarily denote

$$\tilde\lambda_k:=\lambda_k\prod_{j=0}^k(b\theta_j)^{-1} \tag{3.77}$$

and observe that the first line in (3.64) can concisely be written as

$$\tilde\lambda_{k+1}=\frac{\tilde\lambda_k}{\sqrt{1+\rho_k\lambda_k^2}}, \tag{3.78}$$

where

$$\rho_k:=\frac{1}{\lambda_k^2}\left[\left(\frac{1+b(b-1)\lambda_k^2+s_k\lambda_k^3}{1+(b-1)\gamma_k+\frac{1}{2}(b-1)(b-2)\lambda_k^2+r_k\lambda_k^3}\right)^2-1\right]. \tag{3.79}$$

The reason for writing the iteration this way is because (3.78) can now be cast as

$$\frac{1}{\tilde\lambda_{k+1}^2}=\frac{1}{\tilde\lambda_k^2}+\rho_k\prod_{j=0}^k(b\theta_j)^2. \tag{3.80}$$

Iterating we then get

$$\frac{1}{\lambda_k^2}=\frac{1}{\lambda_0^2}\left(\prod_{j=1}^k(b\theta_j)^{-2}\right)+\sum_{\ell=0}^{k-1}\left(\prod_{j=\ell+1}^k(b\theta_j)^{-2}\right)\rho_\ell, \tag{3.81}$$

where we already returned to the original variables.

In order to extract the leading asymptotic of the sum in (3.81), we note that (3.76) along with $\lambda_k = O(k^{-1/2})$ show

$$\rho_k = \frac{(b-1)^2(b+1)^3}{b^3-1} + O(k^{-1/2}) + O\biggl(\sum_{j=0}^{k} b^{-2j}|\sigma_{k-j}^2 - 1|\biggr). \tag{3.82}$$

Denote by $\rho_\star$ the first quantity on the right. Plugging (3.82) in (3.81) while noting that, thanks to Assumption 1.1, $\prod_{j=\ell+1}^{k}(b\theta_j)^{-2}$ is bounded uniformly in $n \geqslant k > \ell \geqslant 0$ we get

$$\lambda_k^{-2} = O(1) + \sum_{\ell=0}^{k-1}\prod_{j=\ell+1}^{k}(b\theta_j)^{-2}\rho_\star + O(k^{1/2}) + O\biggl(\sum_{j=0}^{k-1}|\sigma_j^2 - 1|\biggr). \tag{3.83}$$

The last term is again $O(1)$ by Assumption 1.1 so we just need to control the middle term. Here we separate the terms with $\ell \leqslant \sqrt{k}$ at the cost of another $O(k^{1/2})$ correction. In the remaining terms we note that

$$\prod_{j=\ell+1}^{k}(b\theta_j)^{-2} = \prod_{j=\ell+1}^{k} b^{2(\sigma_j^2-1)} = 1 + O\biggl(\sum_{j\geqslant\sqrt{k}}\mathfrak{d}_j\biggr) + O\biggl(\sum_{j\geqslant n-k}\mathfrak{d}_j\biggr), \tag{3.84}$$

where $\{\mathfrak{d}_j\}_{j\geqslant 0}$ is the sequence from Assumption 1.1. Using this in (3.83) and inverting the two negative powers then proves (3.27). Plugging this in (3.70) gives also (3.28). $\square$

Unlike Theorems 3.4–3.5 whose proofs borrowed from results proved in our earlier work, Theorem 3.6 will have to be proved from "scratch" using different ideas. The details will be given in Section 6.

4. Asymptotic covariance structure

We are now ready to commence the actual proof of Theorem 1.3. We rely heavily on the convergence of the renormalization-group iterations established in the earlier sections along with the representation of the field as a tree-indexed Markov chain. We again suppose that Assumptions 1.1–1.2 hold throughout.

4.1 Markov-chain representation.

In Lemma 3.3 we showed that $P_{n,\beta}$ is the law on the leaves of a tree-indexed Markov chain. Along any branch of the tree, that tree-indexed Markov chain is just an ordinary Markov chain with transition probabilities (3.15). As it turns out, the proof of Theorem 1.3 can be reduced to properties of this chain.

In order to match the labeling of the v_k's, we will label the Markov chain backwards; i.e., from the leaves to the root. A run of the chain is thus a sequence of real-valued random variables $\{\phi_k\}_{k=n,\dots,0}$ such that, for all Borel $A \subseteq \mathbb{R}$,

$$P(\phi_n \in A) = \mathfrak{p}_n(A\,|\,0) \tag{4.1}$$

and

$$P(\phi_k \in A\,|\mathcal{F}_{k+1}) = \mathfrak{p}_k(A|\phi_{k+1}), \tag{4.2}$$

where $\mathfrak{p}_k(\cdot|\cdot)$ is as in (3.15) and

$$\mathcal{F}_k := \sigma(\phi_i\colon i = k,\dots,n). \tag{4.3}$$

The above is consistent with setting the "initial" state of the chain to $\phi_{n+1} := 0$ and using $\mathcal{F}_{n+1}$ to denote the trivial σ-algebra.

We will use E to denote expectation with respect to P and, for $x,y \in \Lambda_n$, write $k(x,y)$ for the depth of the nearest common ancestor of x and y in the underlying b-ary tree structure on $\mathbb{T}_n$. Explicitly, for distinct $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ in Λ_n we set

$$k(x,y) := n - \min\{j = 0,\dots,n\colon x_j \neq y_j\} = \log_{b^{1/2}} d(x,y) \tag{4.4}$$

with the convention $x_0 = y_0$, and put $k(x,y) := 0$ when $x = y$. To reduce clutter we sometimes use the same letter for both the field values and the states of the chain as the precise meaning will always be clear from context.

A key step in the proof of Theorem 1.3 then comes in:

Proposition 4.1 *Let $\beta > 0$ and assume that $\{v_k'\}_{k=0}^n$ and $\{v_k''\}_{k=0}^n$ are bounded uniformly in $n \geqslant 1$. Then for all $n \geqslant 1$ and all $x,y \in \Lambda_n$,*

$$\langle \phi_x \phi_y \rangle_{n,\beta} = \sum_{i=k(x,y)}^{n} \left(\frac{1}{\beta} + \frac{1}{\beta^2}\frac{b+1}{b-1} E\bigl[v_i'(\phi_{i+1})^2 - v_i''(\phi_{i+1})\bigr] \right) + O(1) \tag{4.5}$$

holds with $O(1)$ that is bounded uniformly in $n \geqslant 1$ and $x,y \in \Lambda_n$.

To prove this proposition note that, for $k := k(x,y)$, the tree-indexed Markov chain representation gives

$$\langle \phi_x \phi_y \rangle_{n,\beta} = E\bigl(E(\phi_0|\mathcal{F}_k)^2\bigr) \tag{4.6}$$

and so all we need to do is to extract the asymptotic form of $E(\phi_0|\mathcal{F}_k)$, uniformly in $n \geqslant k \geqslant 0$. This leads to somewhat lengthy calculations for the underlying Markov chain, part of which we relegate to:

Lemma 4.2 *Let $\beta > 0$ and $n \geqslant 1$. Then for all $k = 0,\dots,n$,*

$$E(\phi_k|\mathcal{F}_{k+1}) = \phi_{k+1} - \frac{\sigma_k^2}{\beta} v_k'(\phi_{k+1}) \tag{4.7}$$

and, for all $k = 1,\dots,n$,

$$E\bigl(v_{k-1}'(\phi_k)\big|\,\mathcal{F}_{k+1}\bigr) = \frac{1}{b} v_k'(\phi_{k+1}). \tag{4.8}$$

Proof. Abbreviate $\beta_k := \beta\sigma_k^{-2}$ and recall our notation μ_{σ^2} for the law of $\mathcal{N}(0,\sigma^2)$. We start by computing some relevant Gaussian integrals. The first one of these is

$$\begin{aligned}\int \mathrm{e}^{-bv_{k-1}(\phi+\zeta)}\zeta\,\mu_{1/\beta_k}(\mathrm{d}\zeta) &= \frac{1}{\beta_k}\frac{\mathrm{d}}{\mathrm{d}\phi}\int \mathrm{e}^{-bv_{k-1}(\phi+\zeta)}\mu_{1/\beta_k}(\mathrm{d}\zeta)\\ &= \frac{1}{\beta_k}\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{e}^{-v_k(\phi)} = -\frac{1}{\beta_k}v_k'(\phi)\mathrm{e}^{-v_k(\phi)}\end{aligned} \tag{4.9}$$

for $k = 1,\dots,n$. Here we first performed a Gaussian integration by parts, then swapped differentiation with respect to ζ by that with respect to ϕ and finally applied the recursive relation between v_k and v_{k-1}. The second integral to compute uses the relation between v_k and v_{k-1} with the result

$$\int \mathrm{e}^{-bv_{k-1}(\phi+\zeta)}v_{k-1}'(\phi+\zeta)\,\mu_{1/\beta_k}(\mathrm{d}\zeta) = -\frac{1}{b}\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{e}^{-v_k(\phi)} = \frac{1}{b}v_k'(\phi)\mathrm{e}^{-v_k(\phi)} \tag{4.10}$$

for all $k = 1, \dots, n$. The above manipulations are justified by the fact that the v_k's are periodic C^∞-functions and so no issues arise from swapping derivatives and integrals and no boundary terms pop up during integration by parts.

Moving to statements above, for (4.8) we only need to use the top line in the definition (3.15) of the transition probability to get

$$E\big(v'_{k-1}(\phi_k)\,\big|\,\mathcal{F}_{k+1}\big) = \mathrm{e}^{v_k(\phi_{k+1})}\int \mathrm{e}^{-bv_{k-1}(\phi_{k+1}+\zeta)}v'_{k-1}(\phi_{k+1}+\zeta)\mu_{1/\beta_k}(\mathrm{d}\zeta) \tag{4.11}$$

Using (4.10) we now obtain (4.8). For (4.7) we first treat $k \geqslant 1$ where (4.9) combined with the top line in (3.15) show

$$\begin{aligned} E(\phi_k|\mathcal{F}_{k+1}) &= \mathrm{e}^{v_k(\phi_{k+1})}\int \mathrm{e}^{-bv_{k-1}(\phi_{k+1}+\zeta)}(\phi_{k+1}+\zeta)\mu_{1/\beta_k}(\mathrm{d}\zeta) \\ &= \phi_{k+1} - \frac{1}{\beta_k}v'_k(\phi_{k+1}). \end{aligned} \tag{4.12}$$

For $k = 0$ we instead use the bottom line in (3.15) to get

$$\begin{aligned} E(\phi_0|\mathcal{F}_1) &= \phi_1 + \mathrm{e}^{v_0(\phi_1)}\int \mathrm{e}^{-\frac{\beta_0}{2}(\phi-\phi_1)^2}(\phi-\phi_1)\nu(\mathrm{d}\phi) \\ &= \phi_1 + \mathrm{e}^{v_0(\phi_1)}\frac{1}{\beta_0}\frac{\mathrm{d}}{\mathrm{d}\phi_1}\int \mathrm{e}^{-\frac{\beta_0}{2}(\phi-\phi_1)^2}\nu(\mathrm{d}\phi) \\ &= \phi_1 - \frac{1}{\beta_0}v'_0(\phi_1) \end{aligned} \tag{4.13}$$

by invoking the definition of v_0 in the last step. □

Remark 4.3 We note that (4.8) shows that, for the model corresponding to the renormalization fixed point, the associated potential is an eigenvector of the Markov transition kernel restricted to the space of 1-periodic functions. This fact was key for the derivations in Benfatto and Renn; see [12, Eq. 4.22] and thereafter.

As a consequence of the identities (4.7–4.8) we then get:

Corollary 4.4 *The sequence $\{M_k\}_{k=0}^{n+1}$ defined by $M_0 := \phi_0$ and by*

$$M_k := \phi_k - \frac{1}{\beta}\bigg(\sum_{i=1}^k b^{1-i}\sigma_{k-i}^2\bigg)v'_{k-1}(\phi_k), \quad k = 1, \dots, n+1, \tag{4.14}$$

is a reverse martingale; i.e., $E(M_k|\mathcal{F}_{k+1}) = M_{k+1}$ holds true a.s. for all $k = 0, \dots, n$.

Proof. Write (4.14) as $M_k := \phi_k - s_k\beta^{-1}v'_{k-1}(\phi_k)$ where $s_0 := 0$ to avoid having to interpret v'_{-1}. For $\{M_k\}_{k=0}^{n+1}$ to be a reverse martingale, the identities (4.7–4.8) dictate that $s_{k+1} = b^{-1}s_k + \sigma_k^2$ for all $k = 0, \dots, n$. This is satisfied by $s_k := \sum_{i=1}^k b^{1-i}\sigma_{k-i}^2$. □

Let us keep writing $s_k := \sum_{i=1}^k b^{1-i}\sigma_{k-i}^2$. The fact that $\{M_k\}_{k=0}^{n+1}$ is a martingale with $M_0 = \phi_0$ allows us to continue (4.6) as

$$\begin{aligned}\langle\phi_x\phi_y\rangle_{n,\beta} &= E\bigl(E(\phi_0|\mathcal{F}_k)^2\bigr)\\ &= E(M_k^2) = E(\phi_k^2) - \frac{2s_k}{\beta}E\bigl(\phi_k v_{k-1}'(\phi_k)\bigr) + \frac{s_k^2}{\beta^2}E\bigl(v_{k-1}'(\phi_k)^2\bigr).\end{aligned}\tag{4.15}$$

To compute the expectation on the left, we need to iteratively compute the expectations arising on the right. This is done in:

Lemma 4.5 *Abbreviate $\beta_k := \beta\sigma_k^{-2}$. For all $k = 0,\dots,n$,*

$$E\bigl(\phi_k^2\,\big|\,\mathcal{F}_{k+1}\bigr) = \phi_{k+1}^2 + \frac{1}{\beta_k} + \frac{1}{\beta_k^2}\bigl[v_k'(\phi_{k+1})^2 - v_k''(\phi_{k+1})\bigr] - \frac{2}{\beta_k}\phi_{k+1}v_k'(\phi_{k+1})\tag{4.16}$$

and, for all $k = 1,\dots,n$, also

$$E\bigl(\phi_k v_{k-1}'(\phi_k)\,\big|\,\mathcal{F}_{k+1}\bigr) = \frac{1}{b}\phi_{k+1}v_k'(\phi_{k+1}) - \frac{1}{\beta_k}\frac{1}{b}\bigl[v_k'(\phi_{k+1})^2 - v_k''(\phi_{k+1})\bigr].\tag{4.17}$$

Proof. We again start by computing some relevant integrals. The first of these uses similar ideas as (4.9) with the result

$$\begin{aligned}\int \mathrm{e}^{-bv_{k-1}(\phi+\zeta)}\zeta^2\,\mu_{1/\beta_k}(\mathrm{d}\zeta) &= \frac{1}{\beta_k}\mathrm{e}^{-v_k(\phi)} + \frac{1}{\beta_k}\frac{\mathrm{d}}{\mathrm{d}\phi}\int \mathrm{e}^{-bv_{k-1}(\phi+\zeta)}\zeta\,\mu_{1/\beta_k}(\mathrm{d}\zeta)\\ &= \frac{1}{\beta_k}\mathrm{e}^{-v_k(\phi)} + \frac{1}{\beta_k^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\mathrm{e}^{-v_k(\phi)}\\ &= \Bigl(\frac{1}{\beta_k} + \frac{1}{\beta_k^2}\bigl[v_k'(\phi)^2 - v_k''(\phi)\bigr]\Bigr)\mathrm{e}^{-v_k(\phi)}\end{aligned}\tag{4.18}$$

for all $k = 1,\dots,n$. Here we first interpreted one of the ζ's as a term coming from the derivative of the probability density of μ_{1/β_k} and then used that to integrate by parts, which reduces the computation to the integral in (4.9). Differentiating twice the formula for e^{-v_0} in turn shows

$$\begin{aligned}&\int \mathrm{e}^{-\frac{\beta_0}{2}(\phi-\phi_1)^2}(\phi-\phi_1)^2\nu(\mathrm{d}\phi)\\ &\qquad= \frac{1}{\beta_0}\int \mathrm{e}^{-\frac{\beta_0}{2}(\phi-\phi_1)^2}\nu(\mathrm{d}\phi) + \frac{1}{\beta_0^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\int \mathrm{e}^{-\frac{\beta_0}{2}(\phi-\phi_1)^2}\nu(\mathrm{d}\phi)\\ &\qquad= \Bigl(\frac{1}{\beta_0} + \frac{1}{\beta_0^2}\bigl[v_0'(\phi)^2 - v_0''(\phi)\bigr]\Bigr)\mathrm{e}^{-v_0(\phi)}.\end{aligned}\tag{4.19}$$

To get (4.16) we now invoke $\phi_k^2 = \phi_{k+1}^2 + (\phi_k-\phi_{k+1})^2 + 2\phi_{k+1}(\phi_k-\phi_{k+1})$ and then apply (4.18–4.19) to the second term and (4.7) to the third term.

The proof of (4.17) again starts by computing an integral; namely,

$$\begin{aligned}&\int \mathrm{e}^{-bv_{k-1}(\phi+\zeta)}v_{k-1}'(\phi+\zeta)\zeta\,\mu_{1/\beta_k}(\mathrm{d}\zeta)\\ &\quad= -\frac{1}{b}\frac{\mathrm{d}}{\mathrm{d}\phi}\int \mathrm{e}^{-bv_{k-1}(\phi+\zeta)}\zeta\,\mu_{1/\beta_k}(\mathrm{d}\zeta) = -\frac{1}{\beta_k}\frac{1}{b}\bigl[v_k'(\phi)^2 - v_k''(\phi)\bigr]\mathrm{e}^{-v_k(\phi)},\end{aligned}\tag{4.20}$$

where the last equality follows by plugging in an intermediate step from (4.18). This shows that, for all $k = 1, \dots, n$,

$$E\big(\phi_k v_{k-1}'(\phi_k)\,\big|\,\mathcal{F}_{k+1}\big) = \mathrm{e}^{v_k(\phi_{k+1})}\int \mathrm{e}^{-bv_{k-1}(\phi_{k+1}+\zeta)}\,(\phi_{k+1}+\zeta)\,v_{k-1}'(\phi_{k+1}+\zeta)\,\mu_{1/\beta_k}(\mathrm{d}\zeta)$$
$$= \frac{1}{b}\phi_{k+1}v_k'(\phi_{k+1}) - \frac{1}{\beta_k}\,\frac{1}{b}\big[v_k'(\phi_{k+1})^2 - v_k''(\phi_{k+1})\big], \tag{4.21}$$

where the first term arises via (4.10) and the second term via (4.20). □

We are now ready to give:

Proof of Proposition 4.1. We will keep using the above shorthands s_k and β_k whenever convenient. The argument aims directly at an iterative computation of $E(M_k^2)$. Here the identities (4.16–4.17) give

$$\begin{aligned} E\big(M_k^2\,\big|\,\mathcal{F}_{k+1}\big) &= E\bigg(\Big[\phi_k - \frac{s_k}{\beta}v_{k-1}'(\phi_k)\Big]^2\,\bigg|\,\mathcal{F}_{k+1}\bigg) \\ &= \phi_{k+1}^2 + \frac{1}{\beta_k} + \frac{1}{\beta_k^2}\big[v_k'(\phi_{k+1})^2 - v_k''(\phi_{k+1})\big] - \frac{2}{\beta_k}\phi_{k+1}v_k'(\phi_{k+1}) \\ &\quad - \frac{2s_k}{\beta}\bigg(\frac{1}{b}\phi_{k+1}v_k'(\phi_{k+1}) - \frac{1}{\beta_k}\,\frac{1}{b}\big[v_k'(\phi_{k+1})^2 - v_k''(\phi_{k+1})\big]\bigg) \\ &\quad + \frac{s_k^2}{\beta^2}E\big(v_{k-1}'(\phi_k)^2\,\big|\,\mathcal{F}_{k+1}\big). \end{aligned} \tag{4.22}$$

Next note that, by the definition of β_k and the recursion $s_{k+1} = b^{-1}s_k + \sigma_k^2$, the terms containing $\phi_{k+1}v_k'(\phi_{k+1})$ combine into the cross-term that arises from squaring the expression $M_{k+1} = \phi_{k+1} - s_{k+1}\beta^{-1}v_k'(\phi_{k+1})$. This wraps (4.22) into

$$\begin{aligned} E\big(M_k^2\,\big|\,\mathcal{F}_{k+1}\big) &= M_{k+1}^2 + \frac{\sigma_k^2}{\beta} + \frac{\sigma_k^2}{\beta^2}\Big(\sigma_k^2 + \frac{2s_k}{b}\Big)\big[v_k'(\phi_{k+1})^2 - v_k''(\phi_{k+1})\big] \\ &\quad + \frac{1}{\beta^2}\Big(s_k^2 E\big(v_{k-1}'(\phi_k)^2\,\big|\,\mathcal{F}_{k+1}\big) - s_{k+1}^2 v_k'(\phi_{k+1})^2\Big). \end{aligned} \tag{4.23}$$

Taking expectation and invoking the assumed uniform boundedness of $\{v_k'\}_{k=0}^n$ and $\{v_k''\}_{k=0}^n$, we may replace σ_k^2 by 1 and s_k by $\frac{b}{b-1}$ to get

$$\begin{aligned} E(M_k^2) &= E(M_{k+1}^2) + \frac{1}{\beta} + \frac{1}{\beta^2}\frac{b+1}{b-1}E\big[v_k'(\phi_{k+1})^2 - v_k''(\phi_{k+1})\big] \\ &\quad + \frac{1}{\beta^2}\Big(\frac{b}{b-1}\Big)^2\Big(E\big(v_{k-1}'(\phi_k)^2\big) - E\big(v_k'(\phi_{k+1})^2\big)\Big) \\ &\quad + O\bigg(\sum_{i=0}^k b^{-i}|\sigma_{k-i}^2 - 1|\bigg) + O(b^{-k}), \end{aligned} \tag{4.24}$$

where $O(b^{-k})$ arises from the bound on the tail of the infinite series for the asymptotic value of s_k. Under Assumption 1.1, adding up the error terms over $k = 0, \dots, n$ shows that these produce an $O(1)$ correction under iteration. The same applies to the middle term as it leads to a telescoping sum which is bounded by the assumed uniform boundedness of $\{v_k'\}_{k=0}^n$. Iterations of (4.24) then prove the desired claim. □

4.2 Technical lemmas.

In order to process the formula in Proposition 4.1 further we need a couple of technical lemmas. First we note a way to simplify the right-hand side of (4.5).

Lemma 4.6 *Let $\beta > 0$ and assume that $\{v_k'\}_{k=0}^n$ and $\{v_k''\}_{k=0}^n$ are bounded uniformly in $n \geqslant 1$. Then for all $n \geqslant 1$ and all $k = 0, \dots, n$,*

$$\sum_{i=k}^{n} E\bigl[v_i'(\phi_{i+1})^2 - v_i''(\phi_{i+1})\bigr] = -b\sum_{i=k}^{n} E\bigl(v_i'(\phi_{i+1})^2\bigr) + O(1), \tag{4.25}$$

where $O(1)$ bounded uniformly in n and k subject to $n \geqslant k$.

Proof. By the assumed boundedness of $\{v_k'\}_{k=0}^n$ and $\{v_k''\}_{k=0}^n$ it suffices to prove this for $k \geqslant 1$. Here the definition of v_k from v_{k-1} gives

$$\begin{aligned} &E\bigl(b^2 v_{k-1}'(\phi_k)^2 - b v_{k-1}''(\phi_k)\,\big|\,\mathcal{F}_{k+1}\bigr) \\ &\quad= \mathrm{e}^{v_k(\phi_{k+1})}\int \mathrm{e}^{-bv_{k-1}(\phi_{k+1}+\zeta)}\bigl[b^2 v_{k-1}'(\phi_{k+1}+\zeta)^2 - b v_{k-1}''(\phi_{k+1}+\zeta)\bigr]\mu_{1/\beta_k}(\mathrm{d}\zeta) \\ &\quad= \mathrm{e}^{v_k(\phi_{k+1})}\frac{\mathrm{d}^2}{\mathrm{d}\phi_{k+1}^2}\int \mathrm{e}^{-bv_{k-1}(\phi_{k+1}+\zeta)}\mu_{1/\beta_k}(\mathrm{d}\zeta) = v_k'(\phi_{k+1})^2 - v_k''(\phi_{k+1}). \end{aligned} \tag{4.26}$$

Taking expectation then shows

$$\sum_{i=k+1}^{n} E\bigl[v_i'(\phi_{i+1})^2 - v_i''(\phi_{i+1})\bigr] = \sum_{i=k}^{n-1} E\bigl[b^2 v_i'(\phi_{i+1})^2 - b v_i''(\phi_{i+1})\bigr]. \tag{4.27}$$

Relying again on the boundedness of $\{v_k'\}_{k=0}^n$ and $\{v_k''\}_{k=0}^n$, rearranging terms yields

$$(b^2-1)\sum_{i=k}^{n} E\bigl(v_i'(\phi_{i+1})^2\bigr) = (b-1)\sum_{i=k}^{n} E\bigl(v_i''(\phi_{i+1})\bigr) + O(1). \tag{4.28}$$

Canceling $b-1$ on both sides then shows the claim. □

Next, in order to determine the asymptotic behavior of the sum on the right of (4.25), we need to control the law of ϕ_k for large k. While this law does not converge by itself due to the fact that the variance of ϕ_k stays of order k, the law of its fractional part (which is all what we need to compute expectations of 1-periodic functions) does converge as long as v_k tends to a limit. In quantitative form, this is the subject of:

Lemma 4.7 *Let $\beta > 0$ and let $v_\star$ be a 1-periodic continuous function such that (3.36) holds. Let $\nu_\star$ be the Borel measure on $[0,1)$ defined by*

$$\nu_\star(\mathrm{d}z) := \Bigl[\int_{[0,1]} \mathrm{e}^{-(b+1)v_\star(z')}\mathrm{d}z'\Bigr]^{-1}\mathrm{e}^{-(b+1)v_\star(z)}1_{[0,1)}(z)\mathrm{d}z. \tag{4.29}$$

For $k = 1,\dots,n$, abbreviate

$$\begin{aligned} A_{n,k} := (b+1)\sup_{z\in\mathbb{R}}\bigl|v_\star(z)\bigr| + \sup_{z,z'\in\mathbb{R}}\bigl|v_k(z) - bv_{k-1}(z')\bigr| \\ + \frac{\beta}{2}\max\{\sigma_k^{-2},1\} + \frac{1}{2}\log(2\pi/\beta) + \log\max\{\sigma_k,1\} \end{aligned} \tag{4.30}$$

and

$$\delta_k := \sup_{z,z'\in\mathbb{R}}\Bigl|\bigl[v_k(z) - bv_{k-1}(z')\bigr] - \bigl[v_\star(z) - bv_\star(z')\bigr]\Bigr| + \log\max\{\sigma_k,\sigma_k^{-1}\}. \tag{4.31}$$

Then for all $k = 1,\dots,n$,

$$\bigl\|P(\phi_k \bmod 1 \in \cdot) - \nu_\star\bigr\|_{\mathrm{TV}} \leqslant \prod_{i=k}^{n}(1-\mathrm{e}^{-A_{n,i}}) + \sum_{j=k+1}^{n}(1-\mathrm{e}^{-\delta_{j-1}})\prod_{i=k}^{j-2}(1-\mathrm{e}^{-A_{n,i}}), \tag{4.32}$$

where the last product is interpreted as 1 *when $j = k+1$.*

Proof. The proof proceeds by a coupling argument. First note that the assumptions about $v_\star$ ensure that

$$\mathsf{P}_\star(B\,|\,\phi) = \mathrm{e}^{v_\star(\phi)}\int \mathrm{e}^{-bv_\star(\phi+\zeta)}1_B(\phi+\zeta \bmod 1)\mu_{1/\beta}(\mathrm{d}\zeta) \tag{4.33}$$

is a transition kernel on $[0,1)$. Let $(\phi_n^\star,\dots,\phi_0^\star)$ denote a run of Markov chain with transition probability $\mathsf{P}_\star$ and $\phi_n^\star$ drawn from $\nu_\star$ above. Write $P_\star$ for the distribution of the chain. Using that $v_\star$ obeys (3.36) we now check that $\nu_\star$ is stationary for $\mathsf{P}_\star$. In particular, we have $P_\star(\phi_k^\star\in\cdot) = \nu_\star$ for all $k = 1,\dots,n$.

Recall the following standard coupling of random variables X and Y taking values in $[0,1)$ with probability densities denoted as f, resp., g:

$$\begin{aligned} P\bigl((X,Y)\in B\bigr) = \int 1_B(x,x) f\wedge g(x)\mathrm{d}x \\ + \int_B \frac{[f(x) - f\wedge g(x)][g(y) - f\wedge g(y)]}{1-\int f\wedge g(z)\,\mathrm{d}z}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{4.34}$$

where $f\wedge g(x) := \min\{f(x),g(x)\}$. If the pair is drawn using the first term, we will say that X and Y get "coupled," while if the pair is drawn using the second term, we say that they get "uncoupled."

We will now apply the above coupling recursively to generate a sequence

$$({\phi_{n+1}}',{\phi_{n+1}}^\star),\dots,(\phi_1',\phi_1^\star) \tag{4.35}$$

of $[0,1)\times[0,1)$-valued pairs of random variables as follows: Draw $({\phi_{n+1}}',{\phi_{n+1}}^\star)$ from $\delta_0\otimes\nu_\star$. Then, given a sample of $(\phi_{k+1}',\phi_{k+1}^\star)$ for some $k = 1,\dots,n$, draw $(\phi_k',\phi_k^\star)$ from the above coupling measure with

$$f(z) := \mathrm{e}^{v_k(\phi_{k+1}) - bv_{k-1}(z)}\frac{1}{\sqrt{2\pi/\beta_k}}\sum_{j\in\mathbb{Z}}\mathrm{e}^{-\frac{\beta_k}{2}(z-\phi_{k+1}'+j)^2}, \tag{4.36}$$

where $\beta_k := \beta\sigma_k^{-2}$, and

$$g(z) := \mathrm{e}^{v_\star(\phi^\star_{k+1})-bv_\star(z)}\frac{1}{\sqrt{2\pi/\beta}}\sum_{j\in\mathbb{Z}}\mathrm{e}^{-\frac{\beta}{2}(z-\phi^\star_{k+1}+j)^2}. \tag{4.37}$$

As f is the probability density of $\mathfrak{p}_k(\cdot \bmod 1|\phi'_{k+1})$ and g the probability density $\mathsf{P}_\star(\cdot|\phi^\star_k)$, this gives us a (Markovian) coupling of $(\phi_{n+1} \bmod 1,\dots,\phi_1 \bmod 1)$ and a run of the Markov chain with transition probability $\mathsf{P}_\star$ with initial law $\nu_\star$.

We will now use the explicit expression (4.34) to control the probabilities that $(\phi'_k,\phi^\star_k)$ get "coupled" or get "uncoupled." We start by deriving bounds on the terms entering on the right of (4.34). First note that, for f and g as in (4.36–4.37), retaining only the $j=0$ term in the sums shows

$$f\wedge g(z)\geqslant \mathrm{e}^{-A_{n,k}} \tag{4.38}$$

for all $z\in[0,1)$, regardless of the values of ϕ'_{k+1} and $\phi^\star_{k+1}$. On the other hand, assuming that $\phi'_{k+1}=\phi^\star_{k+1}$, if $\sigma_k^2\geqslant 1$, then $\beta_k\leqslant\beta$ and from (4.31) we get $f(z)\geqslant \mathrm{e}^{-\delta_k}g(z)$ for all $z\in[0,1)$. This implies $f\wedge g(z)\geqslant \mathrm{e}^{-\delta_k}g(z)$ leading to

$$g(z)-f\wedge g(z)\leqslant (1-\mathrm{e}^{-\delta_k})g(z) \tag{4.39}$$

for all $z\in[0,1)$. Similarly, still under $\phi'_{k+1}=\phi^\star_{k+1}$, if $\sigma_k^2\leqslant 1$ then $\beta\leqslant\beta_k$ and so $f(z)\leqslant \mathrm{e}^{\delta_k}g(z)$ for all $z\in[0,1)$. This now gives $f\wedge g(z)\geqslant \mathrm{e}^{-\delta_k}f(z)$ and so

$$f(z)-f\wedge g(z)\leqslant (1-\mathrm{e}^{-\delta_k})f(z) \tag{4.40}$$

holds for all $z\in[0,1)$. Using the fact that $(\phi'_k,\phi^\star_k)$ get "coupled" with probability equal the the total "mass" of the first term on the right of (4.34) and get "uncoupled" with probability equal to the total "mass" of the second part, the above observations readily translate into the inequalities

$$\widetilde{P}\big(\phi'_k=\phi^\star_k\,\big|\mathcal{F}^\star_{k+1}\big)\geqslant \mathrm{e}^{-A_{n,k}}\quad\text{a.s.} \tag{4.41}$$

and

$$\widetilde{P}\big(\phi'_k\neq\phi^\star_k\,\big|\mathcal{F}^\star_{k+1}\big)\leqslant 1-\mathrm{e}^{-\delta_k}\quad\text{a.s. on }\{\phi'_{k+1}=\phi^\star_{k+1}\}, \tag{4.42}$$

where we set $\mathcal{F}^\star_k:=\sigma(\phi'_i,\phi^\star_i\colon i=k,\dots,n)$ wrote $\widetilde{P}$ for the coupling measure.

We now observe that the event that $\phi'_k\neq\phi^\star_k$ entails one of two possibilities: either the chains never got "coupled" up to and including the state indexed by k, or they did get "coupled" at some index $j=k+1,\dots,n$ but then got "uncoupled" and stayed so until and including the state indexed by k. This means

$$\widetilde{P}(\phi'_k\neq\phi^\star_k)\leqslant \widetilde{P}\bigg(\bigcap_{i=k}^n\{\phi'_i\neq\phi^\star_i\}\bigg)+\sum_{j=k+1}^n\widetilde{P}\bigg(\{\phi'_j=\phi^\star_j\}\cap\bigcap_{i=k}^{j-1}\{\phi'_i\neq\phi^\star_i\}\bigg). \tag{4.43}$$

Using (4.41), the first probability is bounded inductively by the product of $1-\mathrm{e}^{-A_{n,i}}$ for i ranging from k to n while (4.41–4.42) bounds the second probability by $1-\mathrm{e}^{-\delta_{j-1}}$ times the product of $1-\mathrm{e}^{-A_{n,i}}$ for i ranging from k to $j-2$. Since $P(\phi'_k\neq\phi^\star_k)$ dominates the total variation on the left (4.32), the claim follows. $\square$

4.3 Proof of Theorem 1.3.

We are now in a position to prove the conclusion concerning the covariance of the field. We start with sub and supercritical cases that can be handled concurrently:

Proof of Theorem 1.3, $\beta \neq \beta_c$. Suppose $b \geqslant 2$ and $\beta > 0$ are such that either Theorem 3.4 or Theorem 3.6 applies, whichever is relevant. This in particular means existence of a 1-periodic continuously differentiable $v_\star\colon \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ such that (3.34–3.35) hold with some some $C', \eta' > 0$. (For $\beta < \beta_c$ we have $v_\star = 0$ for which this follows from (3.22–3.23).) Morever, $\{v_k'\}_{k=0}^n$ and $\{v_k''\}_{k=0}^n$ are bounded uniformly in $n \geqslant 1$.

The bounds (3.35) enable Proposition 4.1 and Lemma 4.6. In the notation of Lemma 4.7 the bound (3.32) implies

$$\delta_k \leqslant C' \mathrm{e}^{-\eta' \min\{k,n-k\}} + \frac{1}{2}\max\big\{|\sigma_k^2 - 1|, |\sigma_k^{-2} - 1|\big\} \tag{4.44}$$

while the uniform boundedness of $\{v_k'\}_{k=0}^n$ and $\{v_k''\}_{k=0}^n$ gives $\sup_{n \geqslant k \geqslant 0} A_{n,k} < \infty$. From (3.35), the coupling inequality (4.32) and Assumption 1.1 we then get

$$\sup_{n\geqslant 1} \sum_{k=0}^{n} \Big| E\big(v_k'(\phi_{k+1})^2\big) - E_\star\big(v_\star'(\phi)^2\big)\Big| < \infty, \tag{4.45}$$

where $E_\star$ denotes expectation with respect to $\nu_\star$ and where the $k = 0$ term is handled using the uniform bound on v_0' and $v_\star'$. Since $E_\star(\tilde{v}_\star'(\phi)^2)$ does not depend on k, it follows that the sum on the right of (4.25) equals

$$(n-k)E_\star(v_\star'(\phi)^2) + O(1). \tag{4.46}$$

This implies the claim with

$$\sigma^2(\beta) := \frac{1}{\beta} - b\frac{1}{\beta^2}\frac{b+1}{b-1}\frac{\int_0^1 \mathrm{e}^{-(b+1)v_\star(\phi)} v_\star'(\phi)^2\,\mathrm{d}\phi}{\int_0^1 \mathrm{e}^{-(b+1)v_\star(\phi)}\,\mathrm{d}\phi} \tag{4.47}$$

which equals $1/\beta$ when $\beta \leqslant \beta_c$ as $v_\star = 0$ but is strictly less than that whenever $v_\star$ is non-trivial, as is the case for $\beta > \beta_c$. □

Remark 4.8 The computations in Section 6 (see, e.g., Lemma 6.8) show that

$$\lambda_\star(1) = \sqrt{\frac{2(b^3-1)}{(b+1)^2(b+1)^3}}\sqrt{b\theta - 1} + O(b\theta - 1). \tag{4.48}$$

This, along with the bounds (3.31) implies

$$v_\star'(\phi) = 4\pi\sqrt{\frac{2(b^3-1)}{(b-1)^2(b+1)^3}}\sqrt{b\theta-1}\,\sin(2\pi\phi) + O(b\theta - 1). \tag{4.49}$$

It follows that, up to corrections of order $(b\theta - 1)^{3/2}$, the integrals in (4.47) reduce to integrals with respect to the Lebesgue measure only. As $\int_0^1 \sin(2\pi\phi)^2 \mathrm{d}\phi = 1/2$ we get

$$\sigma^2(\beta) = \frac{1}{\beta} - (4\pi)^2\frac{1}{\beta_c^2}b\frac{b+1}{b-1}\frac{2(b^3-1)}{(b-1)^2(b+1)^3}(b\theta-1)\frac{1}{2} + O(|b\theta-1|^{3/2}). \tag{4.50}$$

A computation gives $b\theta - 1 = \frac{2\pi^2}{\beta_c^2}(\beta - \beta_c) + O((\beta-\beta_c)^2)$. This now gives (1.16).

With the above stated, the critical case requires only minor changes:

Proof of Theorem 1.3, $\beta = \beta_c$. Note that $v_\star = 0$ in this case. Since $(v_k')^2$ decays as $1/k$ while $\|v_k\|_\infty$ decays as $1/\sqrt{k}$, we have

$$E\big(v_k(\phi_{k+1})^2\big) = \int_{[0,1)} v_k'(\phi)^2 \mathrm{d}\phi + O(k^{-3/2}). \tag{4.51}$$

In particular, in this case the finite-k corrections to the law yield only $O(1)$ term. Writing the right-hand side of (3.28) as $4\pi A \sin(2\pi\phi)$ and using again that $\int_0^1 \sin(2\pi\phi)^2 \mathrm{d}\phi = 1/2$, the sum in (4.5) thus equals

$$\frac{1}{\beta_c}(n-k) + \left[\frac{1}{\beta_c^2} b \frac{b+1}{b-1} 8\pi^2 A^2\right] \log \frac{n}{k} + O(1). \tag{4.52}$$

Plugging $n := \log_{b^{1/2}}(\mathrm{diam}(\Lambda_n))$ and $k := \log_{b^{1/2}}(2 + d(x,y)) + O(1)$ and substituting for A yields the claim. $\square$

Remark 4.9 The previous proofs demonstrate the reason for the numerical closeness of the coefficients in front of the second order term at β_c and the near-critical expansion of $\sigma^2(\beta)$ in (1.16). Indeed, both rely on the expansion $v_k'(\phi) = (4\pi A\epsilon)\sin(2\pi\phi) + O(\epsilon^2)$, where $\epsilon := k^{-1/2}$ in the critical case and $\epsilon := \sqrt{2(b\theta - 1)}$ in the near-critical case and A is the constant on the right of (3.27). Plugging this into (4.5) with the help of (4.25), we also need that $E(\sin(2\pi\phi_k)^2) = 1/2 + O(\epsilon)$ once k and $n-k$ are sufficiently large.

Remark 4.10 The covariance computation reveals a log-correlated structure that has, in recent years, allowed control of the limit law of the maximum and the full extremal process of the underlying field in a number of specific models of interest. Two examples most relevant for the present work where this has been done are the Branching Random Walk (Aïdekon [3], Madaule [47]) and the GFF on subdomains of $\mathbb{Z}^2$ (Bramson, Ding and Zeitouni [22], Biskup and Louidor [17–19]; see also [15]). In [16] the present authors used the tree-indexed Markov chain representation of the hierarchical DG-model to control the maximum and the extremal process throughout the subcritical regime. This leaves the question of what happens at, and beyond, β_c.

Unfortunately, the covariance calculation is not sufficient to make reliable predictions about the law of the maximum. Indeed, we need a sharp asymptotic of the probability that ϕ_x exceeds quantities of order n, and likely quite a bit more. While we presently do not see how to extract this information from our calculations, we do believe that the maximum at $\beta > \beta_c$ scales as for the Branching Random Walk with step distribution $\mathcal{N}(0, \sigma^2(\beta))$ while, at $\beta = \beta_c$, the variance of the steps should be taken as $1/\beta_c - \bar{c}\frac{\log n}{n}$. In short, we conjecture that the model is well approximated by the Branching Random Walk with step distribution adjusted to match (1.14).

5. Fractional charge asymptotic

We now move to the computation of the asymptotic of the fractional charge correlation stated in Theorem 1.4. We start with some general observations that apply to all $\beta > 0$ and then prove the statement for subcritical, critical and supercritical β.

5.1 General considerations.

The proof of Theorem 1.4 again relies on the Markov chain representation of the field; see Section 3.1. As before, we will write $\phi_{n+1}, \phi_n, \dots, \phi_0$ for a run of the chain along the path from the root to a generic leaf-vertex indexed so that ϕ_n is the value at the root and $\phi_{n+1} = 0$. We start by introducing the key iteration for the fractional-charge setting:

Lemma 5.1 *Let $\alpha \in \mathbb{R}$ and, given $\{w_1(q)\}_{q\in\mathbb{N}} \in \ell^1(\mathbb{Z})$, define $\{w_k(\cdot)\}_{k=1}^{n+1}$ by the recursion*

$$w_{k+1}(q) := \frac{a_{k-1}(0)^b}{a_k(0)} \sum_{\substack{\ell_1,\dots,\ell_b\in\mathbb{Z}\\ \ell_1+\dots+\ell_b=q}} \left(\prod_{i=1}^{b-1} \frac{a_{k-1}(\ell_i)}{a_{k-1}(0)}\right) w_k(\ell_b)\, \theta_k^{(q+\alpha)^2}. \tag{5.1}$$

Then $\{w_k(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ for all $k = 1, \dots, n+1$ and so we can set

$$f_k(z) := a_{k-1}(0)\mathrm{e}^{v_{k-1}(z)} \sum_{q\in\mathbb{Z}} w_k(q)\mathrm{e}^{2\pi\mathrm{i}qz}. \tag{5.2}$$

Moreover, with $\mathcal{F}_k$ as in (4.3)*,*

$$E\big(\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_k} f_k(\phi_k) \,\big|\, \mathcal{F}_{k+1}\big) = \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_{k+1}} f_{k+1}(\phi_{k+1}) \tag{5.3}$$

then holds for all $k = 1, \dots, n$. In short, $\{\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_k} f_k(\phi_k)\}_{k=1}^{n+1}$ is a reverse martingale.

Proof. Using that $\{a_k(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ for all $k = 0, \dots, n$ we check that $\{w_{k+1}(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ whenever $\{w_k(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, so the main part to prove is (5.3). Continuing to abbreviate $\beta_k := \beta\sigma_k^{-2}$, let $k = 1, \dots, n$ and observe

$$\begin{aligned}
&E\big(\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_k} f_k(\phi_k) \,\big|\, \mathcal{F}_{k+1}\big)\\
&\qquad= \mathrm{e}^{v_k(\phi_{k+1})} \int \mathrm{e}^{-bv_{k-1}(\phi_{k+1}+\zeta)} \mathrm{e}^{2\pi\mathrm{i}\alpha(\phi_{k+1}+\zeta)} f_k(\phi_{k+1}+\zeta)\mu_{1/\beta_k}(\mathrm{d}\zeta)\\
&\qquad= a_{k-1}(0)\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_{k+1}}\mathrm{e}^{v_k(\phi_{k+1})}\\
&\qquad\qquad\times \int \mathrm{e}^{-(b-1)v_{k-1}(\phi_{k+1}+\zeta)}\mathrm{e}^{2\pi\mathrm{i}\alpha\zeta} \sum_{\ell_b\in\mathbb{N}} w_k(\ell_b)\,\mathrm{e}^{2\pi\mathrm{i}\ell_b(\phi_{k+1}+\zeta)}\mu_{1/\beta_k}(\mathrm{d}\zeta).
\end{aligned} \tag{5.4}$$

The sum over ℓ_b can be exchanged with the integral thanks to $\{w_k(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Invoking $\mathrm{e}^{-v_{k-1}(z)} = \sum_{\ell\in\mathbb{Z}} a_{k-1}(\ell)\mathrm{e}^{2\pi\mathrm{i}\ell z}$ along with the fact that $\int \mathrm{e}^{2\pi\mathrm{i}r\zeta}\mu_{1/\beta_k}(\mathrm{d}\zeta) = \theta_k^{r^2}$ for any $r \in \mathbb{R}$, we then compute the resulting integral to be

$$\begin{aligned}
&\sum_{\ell_b\in\mathbb{Z}} w_k(\ell_b) \int \mathrm{e}^{-(b-1)v_{k-1}(\phi_{k+1}+\zeta)}\mathrm{e}^{2\pi\mathrm{i}\alpha\zeta}\mathrm{e}^{2\pi\mathrm{i}\ell_b(\phi_{k+1}+\zeta)}\mu_{1/\beta_k}(\mathrm{d}\zeta)\\
&\qquad= \sum_{\ell_1,\dots,\ell_b\in\mathbb{Z}} \left(\prod_{i=1}^{b-1} a_{k-1}(\ell_i)\right) w_k(\ell_b)\mathrm{e}^{2\pi\mathrm{i}(\ell_1+\dots+\ell_b)\phi_{k+1}}\, \theta_k^{(\ell_1+\dots+\ell_b+\alpha)^2}.
\end{aligned} \tag{5.5}$$

Writing the sum as two sums, one over $q \in \mathbb{Z}$ and the other over $\ell_1, \dots, \ell_b \in \mathbb{Z}$ subject to $\ell_1 + \dots + \ell_b = q$, the definition of w_{k+1} reduces the expectation of interest to

$$E\big(\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_k} f_k(\phi_k) \,\big|\, \mathcal{F}_{k+1}\big) = \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_{k+1}} a_k(0)\mathrm{e}^{v_k(\phi_{k+1})} \sum_{q\in\mathbb{Z}} w_{k+1}(q)\mathrm{e}^{2\pi\mathrm{i}q\phi_{k+1}}. \tag{5.6}$$

This equals $\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_{k+1}} f_{k+1}(\phi_{k+1})$ thus proving (5.3) as desired. $\square$

Note that the case $k = 0$ is excluded from the previous lemma in light of the conditional law of ϕ_0 given $\mathcal{F}_1$ being of a different form than the law of ϕ_k given $\mathcal{F}_{k+1}$ for $k \geqslant 1$. To get the iteration started, we thus need:

Lemma 5.2 *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ admits the Fourier representation*

$$f(z) = \sum_{q\in\mathbb{Z}} w(q)\mathrm{e}^{2\pi\mathrm{i}qz} \tag{5.7}$$

such that $\{w(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Then for all $\alpha \in \mathbb{R}$,

$$E\bigl(\,\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_0} f(\phi_0)\,\big|\,\mathcal{F}_1\bigr) = \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_1 + v_0(\phi_1)}\sqrt{\frac{2\pi\sigma_0^2}{\beta}}\sum_{q\in\mathbb{Z}} w * a(q)\,\theta_0^{(q+\alpha)^2}\mathrm{e}^{2\pi\mathrm{i}q\phi_1}, \tag{5.8}$$

where $\{a(q)\}_{q\in\mathbb{Z}}$ are the Fourier coefficients of ν; i.e., $a(q) := \int_{[0,1)} \mathrm{e}^{-2\pi\mathrm{i}qz}\nu(\mathrm{d}z)$, and $w * a$ is the usual convolution,

$$w * a(q) := \sum_{\ell\in\mathbb{Z}} w(q-\ell)a(\ell), \quad q \in \mathbb{Z}. \tag{5.9}$$

Proof. In light of $\{w(q)\}_{q\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ it suffices to focus on $f(z) := \mathrm{e}^{2\pi\mathrm{i}qz}$ for some $q \in \mathbb{Z}$. With $\beta_0 := \beta\sigma_0^{-2}$, the bottom line in (3.15) then gives

$$E\bigl(\,\mathrm{e}^{2\pi\mathrm{i}(q+\alpha)\phi_0}\,\big|\,\mathcal{F}_1\bigr) = \mathrm{e}^{v_0(\phi_1)+2\pi\mathrm{i}(q+\alpha)\phi_1}\int \mathrm{e}^{2\pi\mathrm{i}(q+\alpha)(\phi-\phi_1)-\frac{\beta_0}{2}(\phi-\phi_1)^2}\nu(\mathrm{d}\phi). \tag{5.10}$$

Using the 1-periodicity of ν we now check that the integral is a 1-periodic function of ϕ_1. The Gaussian decay of the integrand in turn permits us to swap any number of derivatives with respect to ϕ_1 with the integral which means that the integral is actually a C^∞-function of ϕ_1. It follows that one can express the integral as a uniformly convergent Fourier series. Comparing the Fourier coefficients we get (5.8). $\square$

To connect the above to the main objective of Theorem 1.4, we note:

Corollary 5.3 *Let $\beta > 0$ and $\alpha \in \mathbb{R}$ and let $\{w_k\}_{k=1}^{n+1}$ be defined by* (5.1) *with*

$$w_1(q) := \sqrt{\frac{2\pi\sigma_0^2}{\beta}}\,\frac{a(q)}{a_0(0)}\,\theta_0^{(q+\alpha)^2}, \quad q \in \mathbb{Z}, \tag{5.11}$$

where $\{a(q)\}_{q\in\mathbb{Z}}$ denote for the Fourier coefficients of ν. Let $\{f_k\}_{k=1}^{n+1}$ be defined from $\{w_k\}_{k=1}^{n+1}$ as in (5.2). *Then $w_k(\cdot)$ is strictly positive for all $k = 1, \dots, n+1$ and*

$$\langle \mathrm{e}^{2\pi\mathrm{i}\alpha(\phi_x-\phi_y)}\rangle_{n,\beta} = E\bigl(\,|f_k(\phi_k)|^2\bigr) \tag{5.12}$$

holds with $k := k(x,y)$ for all distinct $x, y \in \Lambda_n$. (Here $k(x,y)$ is as in (4.4).*)*

Proof. The strict positivity of $w_k(\cdot)$ is immediate from (5.11), (5.1) and the positivity of a_k's so we only need to prove (5.12). Let $x, y \in \Lambda_n$ be distinct and set $k := k(x,y)$. The tree-indexed Markov representation of the field $\{\phi_x\}_{x\in\Lambda_n}$ from Section 3.1 yields

$$\langle \mathrm{e}^{2\pi\mathrm{i}\alpha(\phi_x-\phi_y)}\rangle_{n,\beta} = E\Bigl(E(\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_0}\,|\,\mathcal{F}_k)E(\mathrm{e}^{-2\pi\mathrm{i}\alpha\phi_0}\,|\,\mathcal{F}_k)\Bigr). \tag{5.13}$$

The first conditional expectation corresponds to the choice $f := 1$ in (5.8) for which $w(q) := \delta_{q,0}$. The identity (5.8) then gives $E(\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_0} \mid \mathcal{F}_1) = \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_1} f_1(\phi_1)$ for f_1 corresponding to w_1 from (5.11) via (5.2). Using (5.3) we obtain $E(\mathrm{e}^{2\pi\mathrm{i}\alpha\phi_0} \mid \mathcal{F}_k) = \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_k} f_k(\phi_k)$. Since the second expectation in (5.13) is the complex conjugate of the first, plugging this in (5.13) yields (5.12). $\square$

We remark that, besides k, the sequence $\{w_k(q)\}_{q\in\mathbb{Z}}$ depends also on n through the n-dependence of the sequence $\{\sigma_k^2\}_{k=0}^n$. While we keep that dependence implicit, it may need to be noted in statements where uniformity in n is required.

5.2 Below criticality.

Corollary 5.3 tells us that, for the asymptotic of the fractional charge at large separations of x and y, we need to track the large-k asymptotic form of f_k initiated from (5.11). As it turns out, the cases of $\beta \leqslant \beta_{\mathrm{c}}$ are united by the fact that f_k is completely dominated by the coefficient $w_k(0)$.

Lemma 5.4 *Let $\beta \in (0, \beta_{\mathrm{c}}]$, $\alpha \in (-1/2, 1/2)$ and let w_1 be as in* (5.11). *Then*

$$\lim_{k\to\infty} \sup_{n\geqslant k} \sum_{q\neq 0} \frac{w_k(q)}{w_k(0)} = 0. \tag{5.14}$$

Proof. We will prove that, under the stated conditions on β and α, there exist $\widetilde{C} > 0$, $\widetilde{\eta} > 0$ and $k_0 \geqslant 2$ such that

$$\sum_{q\neq 0} \frac{w_k(q)}{w_k(0)} \leqslant \widetilde{C} \sum_{j=2}^{k-k_0} \mathrm{e}^{-\widetilde{\eta} j} \sum_{q\neq 0} \frac{a_{k-j}(q)}{a_{k-j}(0)} \tag{5.15}$$

holds for all $k \geqslant k_0 + 2$, uniformly in n. For this we note that, reducing the sum in (5.1) to the term with $\ell_1 = \dots = \ell_{b-1} = 0$ shows

$$w_{k+1}(0) \geqslant \frac{a_{k-1}(0)^b}{a_k(0)}\, \theta_k^{\alpha^2}\, w_k(0). \tag{5.16}$$

Dividing each side of (5.1) by the corresponding side of this bound then gives

$$\frac{w_{k+1}(q)}{w_{k+1}(0)} \leqslant \sum_{\substack{\ell_1,\dots,\ell_b\in\mathbb{Z} \\ \ell_1+\dots+\ell_b=q}} \left(\prod_{i=1}^{b-1} \frac{a_{k-1}(\ell_i)}{a_{k-1}(0)}\right) \frac{w_k(\ell_b)}{w_k(0)}\, \theta_k^{(q+\alpha)^2-\alpha^2}. \tag{5.17}$$

Now observe that for $q \neq 0$ we have $(q+\alpha)^2 - \alpha^2 = q^2 + 2q\alpha \geqslant 1 - 2|\alpha|$ and abbreviate

$$u_k := \sum_{q\neq 0} \frac{a_{k-1}(q)}{a_{k-1}(0)}. \tag{5.18}$$

Summing (5.17) over $q \neq 0$, we have two cases to consider on the right-hand side: either $\ell_b = 0$, in which case at least one of $\ell_1, \dots, \ell_{b-1}$ must be non-zero, or $\ell_b \neq 0$ in which case the remaining indices can, as an upper bound, be summed over as free. This yields

$$\sum_{q\neq 0} \frac{w_{k+1}(q)}{w_{k+1}(0)} \leqslant (1+u_k)^{b-2} b\theta_k^{1-2|\alpha|} u_k + (1+u_k)^{b-1} \theta_k^{1-2|\alpha|} \sum_{q\neq 0} \frac{w_k(q)}{w_k(0)}, \tag{5.19}$$

where the factor b in the first term dominates the number of choices of the first non-zero term among $\ell_1,\dots,\ell_{b-1}$.

Now observe that Theorems 3.4 and 3.5 imply $\sup_{n\geqslant k} u_k \to 0$ as $k\to\infty$. In light of $|\alpha|<1/2$, Assumption 1.1 and $\sup_{k\leqslant n}\theta_k<1$, there exists $k_0\geqslant 1$ such that

$$\mathrm{e}^{-\tilde\eta} := \sup_{n\geqslant k\geqslant k_0} (1+u_k)^{b-1}\theta_k^{1-2|\alpha|} < 1. \tag{5.20}$$

Using this in (5.19) and iterating we get (5.15) by invoking $\{w_{k_0}(q)\}_{q\in\mathbb{Z}}\in\ell^1(\mathbb{Z})$ which is checked from (5.1), (5.11) and the boundedness of $\{a(q)\}_{q\in\mathbb{Z}}$. With (5.15) in hand, we then get (5.14) by using again $\sup_{n\geqslant k} u_k\to 0$ as $k\to\infty$. $\square$

Using very similar arguments, we also get:

Lemma 5.5 *Let $\beta\in(0,\beta_c]$, $\alpha\in(-1/2,1/2)$ and let w_1 be as in* (5.11). *Then*

$$0<\inf_{n\geqslant k\geqslant 1}\frac{w_{k+1}(0)}{w_k(0)\theta_k^{\alpha^2}} \leqslant \sup_{n\geqslant k\geqslant 1}\frac{w_{k+1}(0)}{w_k(0)\theta_k^{\alpha^2}}<\infty. \tag{5.21}$$

Proof. For the lower bound in (5.21) we use (5.16) to get

$$\frac{w_{k+1}(0)}{w_k(0)\theta_k^{\alpha^2}} \geqslant \frac{a_{k-1}(0)^b}{a_k(0)}. \tag{5.22}$$

The quantity on the right is positive uniformly in $n\geqslant k\geqslant 1$ thanks to (3.21) (for $\beta<\beta_c$) and (3.24) (for $\beta=\beta_c$) and the bound $a_k(0)\leqslant(\sum_{q\in\mathbb{Z}}a_{k-1}(q))^b$. For the upper bound in (5.21) we relax the condition $\ell_1+\dots+\ell_b=q$ in (5.17) to get

$$\frac{w_{k+1}(0)}{w_k(0)\theta_k^{\alpha^2}} \leqslant \biggl(\sum_{\ell\in\mathbb{Z}}\frac{a_{k-1}(\ell)}{a_{k-1}(0)}\biggr)^{b-1}\sum_{q\in\mathbb{Z}}\frac{w_k(q)}{w_k(0)}. \tag{5.23}$$

The first sum is again bounded uniformly in $n+1\geqslant k\geqslant 1$ thanks to (3.21) and (3.24) while the second sum is bounded thanks to (5.19). $\square$

We are now ready for:

Proof of Theorem 1.4, $\beta<\beta_c$. We will derive an approximate recursion equation for the sequence $\{w_k(0)\}_{k=1}^{n+1}$. For this we separate the term with $\ell_1=\dots=\ell_b=0$ in (5.1) and use arguments similar to those underlying the proof of (5.23) to bound the remaining terms. This yields

$$\begin{aligned}\biggl|w_{k+1}(0)-&\frac{a_{k-1}(0)^b}{a_k(0)}\theta_k^{\alpha^2}w_k(0)\biggr|\\ &\leqslant\Biggl[b\biggl(\sum_{\substack{\ell\in\mathbb{Z}\\ \ell\neq0}}\frac{a_{k-1}(\ell)}{a_{k-1}(0)}\biggr)\biggl(\sum_{\ell\in\mathbb{Z}}\frac{a_{k-1}(\ell)}{a_{k-1}(0)}\biggr)^{b-2}\sum_{\ell\in\mathbb{Z}}\frac{w_k(\ell)}{w_k(0)}\Biggr]\theta_k^{\alpha^2}w_k(0),\end{aligned} \tag{5.24}$$

where the prefactor b accounts for the choice of the smallest index with $\ell_i\neq 0$. The last two sum on the right are bounded uniformly in $n\geqslant k\geqslant 1$ as above while the exponential decay (3.21) in Theorem 3.4 shows that the first sum decays exponentially in k. By the same reasoning, $a_{k-1}(0)^b/a_k(0)$ differs from 1 by a factor that decays exponentially with k. It follows that for each $\beta \in (0, \beta_c)$ there exists $\eta > 0$ such that

$$w_{k+1}(0) = e^{O(e^{-\eta k})} \theta_k^{\alpha^2} w_k(0), \tag{5.25}$$

with the implicit constant in $O(e^{-\eta k})$ bounded uniformly in $n \geqslant k \geqslant 1$. Here we relied on Lemma 5.5 in moving the error to the exponent for small k.

We now set $n' := \lfloor n/2 \rfloor$ and let $C_n := (w_{n'}(0)\theta^{-\alpha^2 n'})^2$. In light of $\theta_i^{\alpha^2} = \theta^{\alpha^2}\theta^{\alpha^2(\sigma_i^2-1)}$, for $k \leqslant n'$ we then have

$$w_k(0)\theta^{-\alpha^2 k} = \sqrt{C_n}\, \exp\left\{ -\sum_{i=k}^{n'-1} \log\Bigl(\frac{w_{i+1}(0)}{w_i(0)\theta_i^{\alpha^2}}\Bigr) + \alpha^2 \log(1/\theta) \sum_{i=k}^{n'-1} (\sigma_i^2 - 1) \right\} \tag{5.26}$$

The same identity with the limits of the sums interchanged applies also to $k \geqslant n'$ (provided $k \leqslant n$). Invoking (5.25) and Assumption 1.1, we get $w_k(0)\theta^{-\alpha^2 k} = \sqrt{C_n} + o(1)$ where $o(1) \to 0$ as $\min\{k, n-k\} \to \infty$. To see that $\{C_n\}_{n\geqslant 1}$ is bounded and uniformly positive, take $k := 1$ in (5.26) and note that, by Assumption 1.1, $w_1(0)$ is positive and bounded uniformly in $n \geqslant 1$. With the help of Lemma 5.4 and some elementary facts from Fourier analysis, we now conclude that

$$\bigl|f_k(z)\bigr|^2 = \bigl[C_n + o(1)\bigr]\theta^{2\alpha^2 k} \tag{5.27}$$

holds with $o(1) \to 0$ as $\min\{k, n-k\} \to \infty$, uniformly in $z \in \mathbb{R}$. Plugging this in (5.12) then proves (1.18) for $\beta < \beta_c$. $\square$

5.3 At criticality.

In the critical and supercritical situations, a simple (albeit approximate) recursion linking $w_k(0)$ to $w_{k-1}(0)$ is not enough to capture the actual decay. Indeed, as we will show, $w_k(0)$ will receive non-trivial contributions from $w_{k-j}(0)$ with $j \geqslant 2$ as well. To prepare the needed formulas, we first condense (5.1) as

$$w_{k+1}(q) = \theta_k^{(q+\alpha)^2} \sum_{\ell \in \mathbb{N}} \gamma_k(q-\ell) w_k(\ell), \tag{5.28}$$

where, abusing our earlier notation, we set

$$\gamma_k(q) := \frac{a_{k-1}(0)^b}{a_k(0)} \sum_{\substack{\ell_1,\dots,\ell_{b-1}\in\mathbb{Z} \\ \ell_1+\dots+\ell_{b-1}=q}} \prod_{i=1}^{b-1} \frac{a_{k-1}(\ell_i)}{a_{k-1}(0)}. \tag{5.29}$$

Note that, as many of the above objects, γ_k depends also on n but we keep that dependence implicit. We now rewrite (5.28) as follows:

Lemma 5.6 *For each $n+1 \geqslant k \geqslant 2$ and $p, q \in \mathbb{Z}$ let*

$$\Gamma_{k,1}(p,q) := \theta_{k-1}^{(p+\alpha)^2} \theta^{-\alpha^2} \gamma_{k-1}(p-q) \tag{5.30}$$

and for $n+1 \geqslant k \geqslant 3$ and $j = 2, \dots, k-1$, set

$$\Gamma_{k,j}(p,q) := \sum_{\substack{q_1,\dots,q_{j-1}\in\mathbb{Z}\smallsetminus\{0\} \\ q_0=p,\, q_j=q}} \left(\prod_{i=0}^{j-1} \theta_{k-i-1}^{(q_i+\alpha)^2} \theta^{-\alpha^2} \right) \prod_{i=1}^{j} \gamma_{k-i}(q_{i-1}-q_i). \tag{5.31}$$

Then for all $n+1 \geqslant k \geqslant 2$,

$$w_k(p) = \sum_{j=1}^{k-1} \Gamma_{k,j}(p,0)\theta^{\alpha^2 j} w_{k-j}(0) + \sum_{q\in\mathbb{Z}\smallsetminus\{0\}} \Gamma_{k,k-1}(p,q)\theta^{\alpha^2(k-1)} w_1(q). \tag{5.32}$$

Moreover, for all $n+1 \geqslant k \geqslant 2$, all $1 \leqslant j < k$ and all $p \in \mathbb{Z}$,

$$\sum_{q\in\mathbb{Z}} \Gamma_{k,j}(p,q) \leqslant \theta^{-\tilde{c}+|p|(|p|-1)\sigma^2_{\min}+(1-2|\alpha|)(j-1)} \prod_{i=1}^{j} \sum_{\ell\in\mathbb{Z}} \gamma_{k-i}(\ell) \tag{5.33}$$

holds with $\tilde{c} := \max\{\alpha^2, (1-|\alpha|)^2\} \sum_{i=0}^{n} |\sigma_i^2 - 1|$ and $\sigma^2_{\min} := \inf_{i=0,\dots,n} \sigma_i^2$.

Proof. We start by proving (5.32). As is checked directly from (5.28), this holds for $k = 2$. For general $k \geqslant 3$, we plug (5.32) for $w_{k-1}(\ell)$ in

$$w_k(p) = \theta_{k-1}^{(p+\alpha)^2} \gamma_{k-1}(p) w_{k-1}(0) + \sum_{\ell\neq 0} \theta_{k-1}^{(p+\alpha)^2} \gamma_{k-1}(p-\ell) w_{k-1}(\ell) \tag{5.34}$$

which itself follows from (5.28). With the help of

$$\theta_{k-1}^{(p+\alpha)^2} \gamma_{k-1}(p) = \Gamma_{k,1}(p,0)\theta^{\alpha^2} \tag{5.35}$$

and

$$\sum_{\ell\neq 0} \theta_{k-1}^{(p+\alpha)^2} \gamma_{k-1}(p-\ell) \Gamma_{k-1,j}(\ell,q) = \theta^{\alpha^2} \Gamma_{k,j+1}(p,q) \tag{5.36}$$

we now prove (5.32) for k from (5.32) for $k-1$ and thus show (5.32) to be valid for all $n+1 \geqslant k \geqslant 2$ by induction.

For the bound (5.33) recall that

$$(q+\alpha)^2 \geqslant |q|^2 - 2|\alpha||q| + \alpha^2 = |q|(|q|-1) + |q|(1-2|\alpha|) + \alpha^2 \tag{5.37}$$

For $q_0 := p$ and any $q_1, \dots, q_{j-1} \neq 0$ we thus get

$$\begin{aligned} &\prod_{i=0}^{j-1} \theta_{k-i-1}^{(q_i+\alpha)^2} \theta^{-\alpha^2} \\ &\qquad \leqslant \left(\theta^{\alpha^2(\sigma^2_{k-1}-1)} \prod_{i=1}^{j-1} \theta^{(\sigma^2_{k-i-1}-1)(1-2|\alpha|+\alpha^2)} \right) \theta^{|p|(|p|-1)\sigma^2_{k-1}+(1-2|\alpha|)(j-1)}. \end{aligned} \tag{5.38}$$

Plugging this in (5.31), using $\sigma^2_{k-1} \geqslant \sigma^2_{\min}$ and noting that the quantity in the large parentheses is bounded by $\theta^{-\tilde{c}}$, we get (5.33) by performing the sums over $q_1, \dots, q_j$. $\square$

The main point of the bound (5.33) is that, for $\alpha \in (-1/2, 1/2)$ and with Assumption 1.2 in force, the second sum on the right of (5.32) decays exponentially faster than the first and can thus be regarded as an error whenever

$$\theta^{(1-2|\alpha|)} \sum_{\ell\in\mathbb{Z}} \gamma_k(\ell) < 1. \tag{5.39}$$

The latter is true for $\min\{k, n-k\}$ large up to, and even slightly above β_c.

Formula (5.32) reduces the iterations to those of the sequence $\{w_k(0)\}_{k=1}^{n+1}$. (That this suffices at criticality has been shown in Lemma 5.4. We wrote (5.32) in more generality to prepare for supercritical situations.) Next we need good control of the asymptotic behavior of the coefficients $\Gamma_{k,j}(0,0)$. This comes in:

Lemma 5.7 *Let $\beta = \beta_c$, $\alpha \in (-1/2, 1/2)$ and let $A := [\frac{b^3-1}{(b-1)^2(b+1)^3}]^{1/2}$ be the constant on the right of* (3.27). *For $0 \leqslant k \leqslant n$ set*

$$\mathfrak{e}_k := \sum_{j \geqslant \min\{\sqrt{k}, n-\sqrt{k}\}} \mathfrak{d}_j, \tag{5.40}$$

where $\{\mathfrak{d}_j\}_{j\geqslant 0}$ is the sequence from Assumption 1.1. Then

$$\Gamma_{k,1}(0,0) = \exp\bigl\{-2(b-1)A^2k^{-1} + O(k^{-2}) + O(k^{-1}\mathfrak{e}_k) + O(\mathfrak{d}_{\min\{k-1,n-k+1\}})\bigr\} \tag{5.41}$$

and, for $j = 2, \dots, k-1$ and η defined by $\mathrm{e}^{-\eta} := b^{-\frac{1}{2}(1-2|\alpha|)}$,

$$\begin{aligned}\Gamma_{k,j}(0,0) = \bigl(b^{-(1+2\alpha)(j-1)} + b^{-(1-2\alpha)(j-1)}\bigr)(b-1)^2A^2\frac{1}{k} \\ + O(\mathrm{e}^{-\eta j}k^{-2}) + O\bigl(\mathrm{e}^{-\eta j}k^{-1}(\mathfrak{e}_{k-1} + \mathfrak{e}_{k-j})\bigr)\end{aligned} \tag{5.42}$$

where the error bounds are uniform in j, k and n such that $2 \leqslant j < k \leqslant n+1$ and in α on compact subsets of $(-1/2, 1/2)$.

Proof. The proof will require the results in Theorem 3.5 to extract the asymptotic behavior of the relevant coefficients γ_k. For $\Gamma_{k,1}(0,0)$ we only need $\gamma_k(0)$. Here we treat explicitly the term corresponding to permutations of $(1,-1,0,\dots,0)$ in the sum in (5.29) as well as in the sum representing $a_k(0)$ via $a_{k-1}(\cdot)$ and note that, by (3.24), the remaining terms give contributions of order k^{-2}. Applying also (3.27), Theorem 3.5 gives

$$\begin{aligned}\gamma_k(0) &= \frac{1 + (b-1)(b-2)\lambda_k^2 + O(\lambda_k^4)}{1 + b(b-1)\lambda_k^2 + O(\lambda_k^4)} \\ &= 1 - 2(b-1)\lambda_k^2 + O(\lambda_k^4) \\ &= 1 - 2(b-1)A^2k^{-1} + O(k^{-2}) + O(k^{-1}\mathfrak{e}_k) \\ &= \exp\bigl\{-2(b-1)A^2k^{-1} + O(k^{-2}) + O(k^{-1}\mathfrak{e}_k)\bigr\},\end{aligned} \tag{5.43}$$

where $\lambda_k := a_k(1)/a_k(0)$ and where we used that $\mathfrak{e}_k^2 = O(\mathfrak{e}_k)$ and noted that the exponential form is legit thanks to $\gamma_{k-1}(0)$ being positive uniformly in $n \geqslant k \geqslant 1$. From (5.43) and $\theta_{k-1} = \theta^{\sigma_{k-1}^2}$ we then readily get

$$\Gamma_{k,1}(0,0) = \mathrm{e}^{O(\sigma_{k-1}^2 - 1)} \exp\bigl\{-2(b-1)A^2k^{-1} + O(k^{-2}) + O(k^{-1}\mathfrak{e}_k)\bigr\}. \tag{5.44}$$

With the help of (1.6) this shows (5.41).

For $\Gamma_{k,j}(0,0)$ with $j = 2, \dots, k-1$ we will need the leading-order asymptotic of $\gamma_k(1)$ and suitable bounds on $\gamma_k(q)$ with $|q| > 1$. For $\gamma_k(1)$ we treat explicitly the terms corresponding to permutations of $(1,0,\dots,0)$ in (5.29) and apply (3.27) to get

$$\begin{aligned}\gamma_k(1) &= (b-1)A\frac{1}{\sqrt{k}} + O(k^{-1}) + O(k^{-1/2}\mathfrak{e}_k) \\ &= (b-1)A\frac{1}{\sqrt{k}}\,\mathrm{e}^{O(k^{-1/2}) + O(\mathfrak{e}_k)}.\end{aligned} \tag{5.45}$$

For the remaining coefficients we invoke (3.24) with the result

$$\gamma_k(q) \leqslant c\Bigl(\frac{1}{1+\sqrt{k}}\Bigr)^{|q|} \tag{5.46}$$

where c is a constant independent of $q \in \mathbb{Z}$ and $n \geqslant k \geqslant 1$.

With (5.45–5.46) in hand, we now treat the $(j-1)$-tuples with $q_1 = \dots = q_{j-1} = \pm 1$ in (5.31) (with $p = q = 0$) and, for the remaining terms, bound the prefactors using (5.38) while noting that the $q_0, \dots, q_j$ under the sum necessarily satisfy $|q_1 - q_0| + \dots + |q_j - q_{j-1}| \geqslant 4$. This yields

$$\begin{aligned}\Gamma_{k,j}(0,0) = \bigl[h_{k,j}(\alpha)+h_{k,j}(-\alpha)\bigr]\gamma_{k-1}(1)\gamma_{k-j}(1)\prod_{i=2}^{j-1}\gamma_{k-i}(0)\\ + O(1)b^{-(1-2|\alpha|)j}\sum_{\substack{p_1,\dots,p_{j-1}\in\mathbb{Z}\\ |p_1|+\dots+|p_{j-1}|\geqslant 4}}\prod_{i=1}^{j}\gamma_{k-i}(p_i),\end{aligned} \tag{5.47}$$

where p_i represents $q_i - q_{i-1}$,

$$h_{k,j}(\alpha) := b^{-(1+2\alpha)(j-1)-\alpha^2(\sigma_{k-1}^2-1)}\prod_{i=1}^{j-1} b^{-(1+\alpha)^2(\sigma_{k-i-1}^2-1)} \tag{5.48}$$

and where the symmetry $\gamma_k(-\ell) = \gamma_k(\ell)$ was invoked to simplify the first term.

Let $\eta > 0$ be as in the statement. Using (5.46) we now check that the second term on the right of (5.47) is at most $O(\mathrm{e}^{-\eta j}k^{-2})$, uniformly in $j = 2, \dots, k-1$. As to the first term, collecting the error terms in (5.43), (5.45) and (5.48) shows

$$h_{k,j}(\alpha)\gamma_{k-1}(1)\gamma_{k-j}(1)\prod_{i=2}^{j-1}\gamma_{k-i}(0) = b^{-(1+2\alpha)(j-1)}(b-1)^2\frac{A^2}{k}\mathrm{e}^{O(\tilde{\mathfrak{u}}_{k,j})}, \tag{5.49}$$

where

$$\begin{aligned}\tilde{\mathfrak{u}}_{k,j} := \log\frac{k}{k-j} + \sum_{i=1}^{j}|\sigma_{k-i}^2 - 1| + \mathfrak{e}_{k-1} + \mathfrak{e}_{k-j} + (k-1)^{-1/2} + (k-j)^{-1/2}\\ + \sum_{i=1}^{j}\frac{1}{k-i} + \sum_{i=1}^{j}\frac{1}{(k-i)^2} + \sum_{i=1}^{j}\frac{\mathfrak{e}_{k-i}}{k-i}.\end{aligned} \tag{5.50}$$

Now observe that the first, fifth, sixth, seventh and eighth term on the right are bounded by a constant times j/k, while the second and the last term are bounded by $\mathfrak{e}_{k-j} + \mathfrak{e}_{k-1}$. It follows that

$$\tilde{\mathfrak{u}}_{k,j} = O(\mathfrak{e}_{k-1} + \mathfrak{e}_{k-j}) + O(j/k). \tag{5.51}$$

Borrowing part of the exponential decay from the prefactor $b^{-(1+2\alpha)(j-1)}$ to absorb the $O(j/k)$-term shows that the quantity in (5.49) equals

$$b^{-(1+2\alpha)(j-1)}(b-1)^2\frac{A^2}{k} + O(\mathfrak{e}_{k-1} + \mathfrak{e}_{k-j})k^{-1}\mathrm{e}^{-\eta j}. \tag{5.52}$$

Plugging this in (5.47), we get (5.42). $\square$

We will now show how to use the above to control the fractional-charge asymptotic when $\beta = \beta_c$. Set $\tilde{\tau} := \frac{1}{2}\tau(\alpha)$ for $\tau(\alpha)$ from (1.19) and observe that we then have

$$\tilde{\tau} + 2(b-1)A^2 = \sum_{j=2}^{\infty}\bigl(b^{-(1+2\alpha)(j-1)} + b^{-(1-2\alpha)(j-1)}\bigr)(b-1)^2A^2 \tag{5.53}$$

for A as above. Setting

$$r_k := k^{-\tilde{\tau}}\theta^{-\alpha^2 k}w_k(0), \tag{5.54}$$

the proof of Theorem 3.5 will boil down to showing that r_k is close to a positive and finite n-dependent constant once k and $n-k$ are large. For this we first prove:

Lemma 5.8 *Let $\beta = \beta_c$ and suppose that the sequence $\{\mathfrak{d}_j\}_{j\geqslant 0}$ from Assumption 1.1 obeys $\sum_{j\geqslant 1}\mathfrak{d}_j\log(j) < \infty$. Then*

$$0 < \inf_{n\geqslant k\geqslant 1} r_k \leqslant \sup_{n\geqslant k\geqslant 1} r_k < \infty. \tag{5.55}$$

Proof. For A as above, denote

$$h_j := (b^{-(1+2\alpha)(j-1)} + b^{-(1-2\alpha)(j-1)})(b-1)^2A^2 \tag{5.56}$$

and, recalling $\mathrm{e}^{-\eta} := b^{-\frac{1}{2}(1-2|\alpha|)}$, abbreviate

$$\begin{aligned}\mathfrak{s}_k &:= k^{-2} + k^{-1}\mathfrak{e}_k + \mathfrak{d}_{\min\{k-1,n-k+1\}},\\ \mathfrak{u}_{k,j} &:= \mathrm{e}^{-\eta j}\bigl[k^{-2} + k^{-1}(\mathfrak{e}_{k-1} + \mathfrak{e}_{k-j})\bigr].\end{aligned} \tag{5.57}$$

Then use the asymptotic forms from Lemma 5.7 to cast (5.32) as

$$\begin{aligned} r_k = \Bigl(1-\frac{1}{k}\Bigr)^{\tilde{\tau}}\mathrm{e}^{-2(b-1)A^2k^{-1}+O(\mathfrak{s}_k)}r_{k-1}\\ + \sum_{j=2}^{k-1}h_j\frac{1}{k}\Bigl(1-\frac{j}{k}\Bigr)^{\tilde{\tau}}r_{k-j} + O\Bigl(\sum_{j=2}^{k-1}\mathfrak{u}_{k,j}r_{k-j}\Bigr) + O(\mathrm{e}^{-\eta k}),\end{aligned} \tag{5.58}$$

where the last error term arises from the aforementioned bound on the second sum in (5.32). Now set $M_k := \max_{j=1,\dots,k} r_j$ and invoke $1-\frac{1}{k} \leqslant \mathrm{e}^{-1/k}$ and $1-\frac{j}{k} \leqslant 1$ to get

$$M_k \leqslant \Biggl[\mathrm{e}^{-[\tilde{\tau}+2(b-1)A^2]k^{-1}} + \frac{1}{k}\sum_{j\geqslant 2}h_j + c\Bigl(\mathfrak{s}_k + \sum_{j=2}^{k-1}\mathfrak{u}_{k,j}\Bigr)\Biggr]M_{k-1} + c\mathrm{e}^{-\eta k} \tag{5.59}$$

for some constant $c > 0$ independent of $n \geqslant k > j \geqslant 2$. The choice of $\tilde{\tau}$ ensures (via (5.53)) that the sum of the first two terms in the square brackets equals $1 + O(k^{-2})$. To prove uniform boundedness of $\{M_k\}_{k=1}^n$ it thus suffices to show that $\sum_{k=1}^n \mathfrak{s}_k$ and $\sum_{k=1}^n\sum_{j=2}^{k-1}\mathfrak{u}_{k,j}$ are bounded uniformly in n. In light of Assumption 1.1, this reduces to uniform boundedness of $\sum_{k=1}^n k^{-1}\mathfrak{e}_k$. Here we compute

$$\sum_{k=1}^n k^{-1}\mathfrak{e}_k \leqslant 2\sum_{k\geqslant 1}\frac{1}{k}\sum_{j\geqslant\sqrt{k}}\mathfrak{d}_j \leqslant 2\sum_{j\geqslant 1}\mathfrak{d}_j\sum_{1\leqslant k\leqslant j^2}\frac{1}{k} \leqslant 2\sum_{j\geqslant 1}\mathfrak{d}_j\log(j^2) \tag{5.60}$$

which we assumed to be finite.

Concerning the lower bound, denote $m_k := \min_{j=1,\dots,k} r_j$ and apply the inequality $w_{k-j}(0) \geqslant (k-j)^{\tilde\tau}\theta^{\alpha^2(k-j)}m_{k-1}$ in the first sum on the right of (5.32). Dropping the second sum and invoking Lemma 5.7 along with the bound $(1-\frac{j}{k})^{\tilde\tau} \geqslant 1-\tilde\tau j/k$ shows

$$m_k \geqslant \biggl[\mathrm{e}^{-[\tilde\tau+2(b-1)A^2]k^{-1}-c'\mathfrak{s}_k} + \frac{1}{k}\sum_{j=2}^{k} h_j - \frac{\tilde\tau}{k^2}\sum_{j\geqslant 2} jh_j - c'\sum_{j=2}^{k-1}\mathfrak{u}_{k,j}\biggr] m_{k-1} \tag{5.61}$$

for some constant $c' > 0$. We now check that there is $k_0 \geqslant 1$ such that the the term in the square bracket is uniformly positive for $n \geqslant k \geqslant k_0$ and differs from 1 by a quantity that is uniformly summable on $k = 1,\dots,n$. It follows that $m_k \geqslant c'' m_{k_0}$ for all $n \geqslant k \geqslant k_0$. To extend the bound to $k \leqslant k_0$ we call upon Lemma 5.5 which gives $m_k \geqslant c''' k^{-\tilde\tau}$ for $c''' > 0$ independent of $n \geqslant k \geqslant 1$. $\square$

We are now ready for:

Proof of Theorem 1.4, $\beta = \beta_{\mathrm{c}}$. We start by deriving a recursive bound on the difference $r_k - r_{k-1}$. For this we only need to expand a bit on the arguments from the proof of Lemma 5.8. Indeed, using that $\{r_k\}_{k=1}^n$ is bounded, we can trim (5.58) to the form

$$r_k = r_{k-1} - \frac{1}{k}\bigl[\tilde\tau + 2(b-1)A^2\bigr] r_{k-1} + \frac{1}{k}\sum_{j=2}^{k-1} h_j r_{k-j} + O(\mathfrak{v}_k), \tag{5.62}$$

where $\mathfrak{v}_k := k^{-2} + \mathrm{e}^{-\eta k} + \mathfrak{s}_k + \sum_{j=2}^{k-1}\mathfrak{u}_{k,j}$. For our choice of $\tilde\tau$ and η, this gives

$$r_k - r_{k-1} = \frac{1}{k}\sum_{j=2}^{k-1} h_j (r_{k-j} - r_{k-1}) + O(\mathfrak{v}_k). \tag{5.63}$$

With the help of the triangle inequality and a simple interchange of two sums we get

$$|r_k - r_{k-1}| \leqslant \frac{1}{k}\sum_{i=1}^{k-2}\Bigl(\sum_{j\geqslant i+1} h_j\Bigr)|r_{k-i} - r_{k-i-1}| + a\mathfrak{v}_k \tag{5.64}$$

for a constant $a \geqslant 0$ independent of $n \geqslant k \geqslant 1$. Setting

$$k_0 := \left\lceil \sum_{i\geqslant 1}\mathrm{e}^{\eta i}\Bigl(\sum_{j\geqslant i+1} h_j\Bigr)\right\rceil, \tag{5.65}$$

where the sum over i is finite for η as above, we assume that a is so that the bound

$$|r_j - r_{j-1}| \leqslant a\sum_{i=0}^{j-2}\mathrm{e}^{-\eta i}\mathfrak{v}_{j-i} \tag{5.66}$$

holds for all $j = 2,\dots,k_0$.

We now claim that (5.66) is true for all $j = 1, \dots, n$. Indeed, suppose $k \geqslant k_0$ is such that (5.66) holds for $j = 2, \dots, k-1$. Then plugging the bound in (5.64) yields

$$\begin{aligned} |r_k - r_{k-1}| &\leqslant a\mathfrak{v}_k + \frac{a}{k}\sum_{i=1}^{k-2}\Big(\sum_{j\geqslant i+1} h_j\Big)\sum_{\ell=i}^{k-2} \mathrm{e}^{-\eta(\ell-i)}\mathfrak{v}_{k-\ell} \\ &= a\mathfrak{v}_k + \frac{a}{k}\sum_{\ell=1}^{k-2}\Big[\sum_{i=1}^{\ell}\mathrm{e}^{\eta i}\Big(\sum_{j\geqslant i+1} h_j\Big)\Big]\mathrm{e}^{-\eta\ell}\mathfrak{v}_{k-\ell}. \end{aligned} \tag{5.67}$$

Noting that the quantity in the square bracket is bounded by k_0, the fact that $k \geqslant k_0$ then implies (5.66) for $j := k$. This proves (5.66) for all $j = 2, \dots, n$ by induction.

With (5.66) in hand, we proceed similarly as in the subcritical situations. Indeed, abbreviate $n' := \lfloor n/2 \rfloor$, set $C_n := r_{n'}^2$ and note that, by Lemma 5.8, C_n this is positive and finite uniformly in $n \geqslant 1$. The inequality (5.66) implies

$$\big|r_k - \sqrt{C_n}\big| \leqslant \frac{a}{1-\mathrm{e}^{-\eta}}\sum_{j=\min\{k,n-k\}}^{\max\{k,n-k\}}\mathfrak{v}_j + \frac{a}{1-\mathrm{e}^{-\eta}}\sum_{j=1}^{k-1}\mathrm{e}^{-\eta j}\mathfrak{v}_{k-j}. \tag{5.68}$$

The assumption $\sum_{j\geqslant 1}\mathfrak{d}_j \log(j) < \infty$ along with the bounds in the proof of Lemma 5.8 then give $r_k - \sqrt{C_n} \to 0$ as $\min\{k, n-k\} \to 0$. Using this in (5.54) along with $2\tilde{\tau} = \tau(\alpha)$, Lemma 5.4 and standard facts about Fourier series show

$$\big|f_k(z)\big|^2 = \big[C_n + o(1)\big]k^{\tau(\alpha)}\theta^{\alpha^2 k} \tag{5.69}$$

with $o(1) \to 0$ as $\min\{k, n-k\} \to \infty$ uniformly in $z \in \mathbb{R}$. Plugging $k := k(x,y)$ we get (1.18) for $\beta = \beta_c$. □

5.4 Above criticality.

Our last item of business in this section is the asymptotic of the fractional charge for β slightly above β_c. Throughout we assume that the sequence $\{\mathfrak{d}_k\}_{k\geqslant 0}$ in Assumption 1.1 exhibits exponential decay.

We will again rely on the representation from Lemma 5.6 for which we need to identify the asymptotic values of the coefficients $\Gamma_{k,j}(p,0)$. Abbreviate $\Xi_b(q) := \{(\ell_1, \dots, \ell_b) \in \mathbb{Z}^b \colon \ell_1 + \dots + \ell_b = q\}$ and, with $\{\lambda_\star(q)\}_{n\in\mathbb{Z}}$ as in Theorem 3.6, set

$$\gamma_\star(q) := \frac{\sum_{\bar{\ell}\in\Xi_{b-1}(q)}\prod_{i=1}^{b-1}\lambda_\star(\ell_i)}{\sum_{\bar{\ell}'\in\Xi_b(0)}\prod_{i=1}^{b}\lambda_\star(\ell_i')}, \tag{5.70}$$

where $\bar{\ell}$ stands for $(\ell_1, \dots, \ell_{b-1})$ and $\bar{\ell}'$ stands for $(\ell_1', \dots, \ell_b')$. Note that, by (3.30), the sums are finite for $b\theta - 1$ small and $\gamma_\star(p) = \gamma_\star(-p)$ by $\lambda_\star(q) = \lambda_\star(-q)$. Now let

$$\Gamma_1^\star(p) := \gamma_\star(p) \tag{5.71}$$

and, for $j \geqslant 2$, let

$$\Gamma_j^\star(p) := \sum_{\substack{q_1,\dots,q_{j-1}\in\mathbb{Z}\smallsetminus\{0\} \\ q_0 = p,\, q_j = 0}}\Big(\prod_{i=0}^{j-1}\theta^{(q_i+\alpha)^2-\alpha^2}\Big)\prod_{i=1}^{j}\gamma_\star(q_i - q_{i-1}). \tag{5.72}$$

Theorem 3.6 then shows:

Lemma 5.9 *For all $\alpha \in (-1/2, 1/2)$ and $\delta \in (0, 1-2|\alpha|)$ there exists $\epsilon > 0$ and, for all $\beta > \beta_c$ with $1/\beta > 1/\beta_c - \epsilon$, there exist $C > 0$ and $\eta > 0$ such that*

$$\Gamma_j^\star(p) \leqslant C\sqrt{b\theta - 1}\;\; b^{-(1-2|\alpha|-\delta)j} \mathrm{e}^{-\eta|p|(|p|-1)} \tag{5.73}$$

holds for all $j \geqslant 2$ and $p \in \mathbb{Z}$ and

$$\bigl|\Gamma_{k,j}(p,0) - \Gamma_j^\star(p)\bigr| \leqslant Cb^{-(1-2|\alpha|-\delta)j} \mathrm{e}^{-\eta|p|(|p|-1)} \mathrm{e}^{-\eta \min\{k,n-k\}} \tag{5.74}$$

holds for $n \geqslant k > j \geqslant 1$ and $p \in \mathbb{Z}$.

Proof. We start by estimates for the weights γ_k and $\gamma_\star$. For $b\theta - 1$ sufficiently small, the bounds (3.30–3.31) imply that, for some constant $C' > 0$,

$$\max\biggl\{\sum_{\ell\in\mathbb{Z}} \gamma_k(\ell), \sum_{\ell\in\mathbb{Z}} \gamma_\star(\ell)\biggr\} \leqslant 1 + C'\sqrt{b\theta - 1} \tag{5.75}$$

holds uniformly in $1 \leqslant k \leqslant n$. The assumption of exponential decay of $\{\mathfrak{d}_k\}_{k\geqslant 0}$ in turn allows us to summarize the inequalities (3.31–3.32) as

$$\biggl|\frac{a_k(q)}{a_k(0)} - \lambda_\star(q)\biggr| \leqslant C'' \mathrm{e}^{-\eta' \max\{\min\{k,n-k\},|q|\}}. \tag{5.76}$$

Hereby we get

$$\sum_{q\in\mathbb{Z}} \bigl|\gamma_k(q) - \gamma_\star(q)\bigr| \leqslant C''' \mathrm{e}^{-\eta' \min\{k,n-k\}}. \tag{5.77}$$

We will assume that $\eta' \leqslant \frac{1}{2}\delta \log b$.

The reasoning underlying (5.33) gives

$$\Gamma_j^\star(p) \leqslant \theta^{\sigma_{\min}^2|p|(|p|-1)+(1-2|\alpha|)(j-1)} \biggl(\sum_{q\in\mathbb{Z}\smallsetminus\{0\}} \gamma_\star(q)\biggr)\biggl(\sum_{\ell\in\mathbb{Z}} \gamma_\star(\ell)\biggr)^{j-1}, \tag{5.78}$$

where we noted that $q_j - q_{j-1} \neq 0$ in (5.72). The first sum on the right is order $\sqrt{b\theta - 1}$ by the same argument that proved (5.75). The second sum can be made less that b^δ by taking $b\theta - 1$ small, proving (5.73) with $\mathrm{e}^{-\eta} := \theta^{\sigma_{\min}^2}$.

In order to prove (5.74), we telescopically swap the k-dependent terms in the expresion for $\Gamma_{k,j}(0)$ for the corresponding terms in $\Gamma_j^\star$. Using (5.37) this gives

$$\begin{aligned} &\bigl|\Gamma_{k,1}(p,0) - \Gamma_1^\star(p)\bigr| \\ &\quad\leqslant \theta^{-\alpha^2+\sigma_{\min}^2|p|(|p|-1)} \bigl|\gamma_{k-1}(p) - \gamma_\star(p)\bigr| + \bigl|\theta_k^{(p+\alpha)^2}\theta^{-\alpha^2} - \theta^{(p+\alpha)^2}\theta^{-\alpha^2}\bigr|\, \gamma_\star(p). \end{aligned} \tag{5.79}$$

For $j \geqslant 2$, we in turn invoke (5.33) with the result

$$\bigl|\Gamma_{k,j}(p,0) - \Gamma_j^\star(p)\bigr| \leqslant \theta^{-\bar{c}+(1-2|\alpha|)(j-1)+\sigma_{\min}^2|p|(|p|-1)} B_1 + \theta^{-\bar{c}+(1-2|\alpha|)(j-2)} B_2, \tag{5.80}$$

where

$$B_1 := \sum_{m=1}^{j} \biggl(\prod_{i=1}^{m-1} \sum_{\ell\in\mathbb{Z}} \gamma_{k-i}(\ell)\biggr)\biggl(\sum_{\ell\in\mathbb{Z}} \bigl|\gamma_{k-m}(\ell) - \gamma_\star(\ell)\bigr|\biggr)\biggl(\sum_{\ell\in\mathbb{Z}} \gamma_\star(\ell)\biggr)^{j-m} \tag{5.81}$$

and

$$
\begin{aligned}
B_2 := \Bigl(\sum_{\ell\in\mathbb{Z}}\gamma_\star(\ell)\Bigr)^j\Biggl[&\bigl|\theta_{k-1}^{(p+\alpha)^2}\theta^{-\alpha^2}-\theta^{(p+\alpha)^2}\theta^{-\alpha^2}\bigr| \\
&+\theta^{(p+\alpha)^2-\alpha^2}\sum_{i=1}^{j-1}\sum_{q\in\mathbb{Z}}\bigl|\theta_{k-i-1}^{(q_i+\alpha)^2}\theta^{-\alpha^2}-\theta^{(q+\alpha)^2}\theta^{-\alpha^2}\bigr|\Biggr]
\end{aligned}
\tag{5.82}
$$

The bounds (5.75–5.77) give

$$
B_1 \leqslant C'''\mathrm{e}^{\eta' j}\mathrm{e}^{-\eta'\min\{k,n-k\}}j[1+C'\sqrt{b\theta-1}]^{j-1}
\tag{5.83}
$$

If $b\theta-1$ is so small that $1+C'\sqrt{b\theta-1}\leqslant b^{\delta/3}$, then the assumption $\eta'\leqslant\frac{1}{2}\delta\log b$ along with the fact that $\sup_{j\geqslant1} jb^{-j\delta/6}<\infty$ bounds the first term on the right of (5.80) by a quantity proportional to the right-hand side of (5.74).

The first term on the right of (5.79) is bounded via (5.77) so it remains to bound the second terms in (5.79) and (5.80). Here the term involving $\gamma_\star$ controlled with the help of (5.75) so it remains to estimate the quantity in absolute value. The elementary inequality $|\mathrm{e}^{-a}-\mathrm{e}^{-\tilde a}|\leqslant\mathrm{e}^{-\min\{a,\tilde a\}}|a-\tilde a|$ combine into

$$
\bigl|\theta_\ell^{(q+\alpha)^2}\theta^{-\alpha^2}-\theta^{(q+\alpha)^2}\theta^{-\alpha^2}\bigr|\leqslant\log(1/\theta)\,\theta^{\min\{1,\sigma_\ell^2\}(q+\alpha)^2-\alpha^2}(q+\alpha)^2|\sigma_\ell^2-1|.
\tag{5.84}
$$

Assumption 1.1 and the exponential decay of $\{\mathfrak{d}_j\}_{j\geqslant0}$ give

$$
|\sigma_\ell^2-1|\leqslant\widetilde C\mathrm{e}^{-\widetilde\eta\min\{\ell,n-\ell\}}
\tag{5.85}
$$

for some $\widetilde C,\widetilde\eta>0$. In light of $\inf_{n\geqslant\ell\geqslant1}\sigma_\ell^2>0$, this bounds the second term on the right of (5.79) by a quantity proportional to the right the right hand side of (5.74).

The bounds (5.84–5.85) dominate the first term in the square bracket in (5.82) by a constant times $\theta^{\frac12\min\{\sigma_{\min}^2,1\}|p|(|p|-1)}\mathrm{e}^{-\widetilde\eta\min\{k,n-k\}}$, where "half" of the exponential decay in $|p|$ was used to control the term $(p+\alpha)^2$. Similarly, the second term in the square bracket in (5.82) is bounded by the same quantity as the first times $j\mathrm{e}^{\widetilde\eta j}$. Summarizing,

$$
B_2\leqslant\widetilde C'\theta^{\frac12\min\{\sigma_{\min}^2,1\}|p|(|p|-1)}\mathrm{e}^{-\widetilde\eta\min\{k,n-k\}}[1+C'\sqrt{b\theta-1}]^j\bigl[1+j\mathrm{e}^{\widetilde\eta j}\bigr],
\tag{5.86}
$$

where (5.75) was used for the $\gamma_\star$-dependent prefactor. Assuming that $\widetilde\eta<\delta/2$ and $b\theta-1$ so small that $1+C'\sqrt{b\theta-1}\leqslant b^{\delta/6}$, the last two terms on the right are at most a constant times $b^{\delta j}$. Inserting this on the right of (5.80), we get the claim. □

Given $\alpha\in(-1/2,1/2)$ and $\beta>\beta_\mathrm{c}$ as in Lemma 5.9, define $t_\star=t_\star(\alpha,\beta)$ by

$$
t_\star:=\inf\Bigl\{t>0\colon\ \sum_{j\geqslant1}\Gamma_j^\star(0)t^{-j}\leqslant1\Bigr\}.
\tag{5.87}
$$

Clearly, $t_\star\in(0,\infty)$ and, by Fatou's lemma, $\sum_{j\geqslant1}\Gamma_j^\star(0)t_\star^{-j}\leqslant1$. Key for our use of this quantity is the fact that equality holds. Indeed, we have:

Lemma 5.10 *For each $\alpha\in(-1/2,1/2)$ there exists $\tilde\epsilon>0$ such that*

$$
\sum_{j\geqslant1}\Gamma_j^\star(0)t_\star^{-j}=1
\tag{5.88}
$$

holds true for all $\beta>\beta_\mathrm{c}$ with $1/\beta>1/\beta_\mathrm{c}-\tilde\epsilon$.

Proof. The above Fatou argument gives $\Gamma_1^\star(0)t_\star^{-1} \leqslant 1$ and so $t_\star \geqslant \gamma_\star(0)^{-1}$. By (3.30), we have $\gamma_\star(0) \to 1$ as $b\theta$ decreases to 1 and so, by the uniform exponential decay (5.73), $t_\star$ lies in the region of continuity of $t \mapsto \sum_{j\geqslant 1}\Gamma_j^\star(0)t^{-j}$ for β close to β_c. The equality (5.88) must therefore hold at the infimum. □

We now follow the same blueprint as in the critical case. Define $\{\tilde r_k\}_{k=1}^n$ by

$$\tilde r_k := (t_\star\theta^{\alpha^2})^{-k} w_k(0). \tag{5.89}$$

This quantity depends also on n but we keep that dependence implicit. Our aim is to show that $\tilde r_k$ is close to an n-dependent constant once $\min\{k, n-k\}$ is sufficiently large. As in this critical situations, for this we first prove:

Lemma 5.11 *For each $\alpha \in (-1/2, 1/2)$ there exists $\epsilon > 0$ such that for all $\beta > \beta_c$ satisfying $1/\beta > 1/\beta_c - \tilde\epsilon$ we have*

$$0 < \inf_{n\geqslant k\geqslant 1} \tilde r_k \leqslant \sup_{n\geqslant k\geqslant 1} \tilde r_k < \infty. \tag{5.90}$$

Proof. Applying (5.89) in (5.32) yields

$$\tilde r_k = \sum_{j=1}^{k-1} \Gamma_{k,j}(0,0)t_\star^{-j}\tilde r_{k-j} + \sum_{q\in\mathbb{Z}\smallsetminus\{0\}} \Gamma_{k,k-1}(0,q)t_\star^{-k}\theta^{-\alpha^2}w_1(q). \tag{5.91}$$

An inspection of (5.11) shows that w_1 is bounded. Since $t_\star$ is close to one for $b\theta - 1$ small, the uniform exponential decay (5.33) implies that the last term is $O(\mathrm{e}^{-\eta' k})$ for some $\eta' > 0$. Invoking Lemma 5.9 to swap $\Gamma_{k,j}(0,0)$ for $\Gamma_j^\star(0)$ gives

$$\tilde r_k \leqslant O(\mathrm{e}^{-\eta' k}) + \sum_{j=1}^{k-1} \Gamma_j^\star(0)t_\star^{-j}\tilde r_{k-j} + \mathrm{e}^{-\eta\min\{k,n-k\}} \sum_{j=1}^{k} b^{-(1-2|\alpha|-\delta)}t_\star^{-j}\tilde r_{k-j}. \tag{5.92}$$

Setting $M_k := \max_{1\leqslant j\leqslant k}\tilde r_j$, the same reasoning applied to the second sum along with the bound $\sum_{j=1}^{k-1}\Gamma_j^\star(0)t_\star^{-j} \leqslant 1$ yields $M_k \leqslant O(\mathrm{e}^{-\eta' k}) + (1 + O(\mathrm{e}^{-\eta\min\{k,n-k\}}))M_{k-1}$ with the implicit constants uniform in n. This now gives the upper bound in (5.90).

For the complementary direction we first prove that $\{\tilde r_k\}_{k\geqslant 0}$ cannot decay exponentially fast. Indeed, for this we pick $\delta > 0$ and note that Lemma 5.9 along with (5.88) show that, for some $\ell \geqslant 1$ and $k_0 > \ell$,

$$\sum_{j=1}^{\ell} \Gamma_{k,j}(0,0)t_\star^{-j} \geqslant \mathrm{e}^{-\delta} \tag{5.93}$$

holds once $\min\{k, n-k\} \geqslant k_0$. Now observe that plugging (5.89) for $w_j(0)$ in (5.32) and retaining only the terms with $j \leqslant \ell$ from the first sum (and dropping the second sum) yields

$$\tilde r_k \geqslant \sum_{j=1}^{\ell} \Gamma_{k,j}(0,0)t_\star^{-j}\tilde r_{k-j}. \tag{5.94}$$

Setting $m_k := \min_{1\leqslant j\leqslant k}\tilde r_j$ we get $m_k \geqslant \mathrm{e}^{-\delta}m_{k-1}$ once $\min\{k, n-k\} \geqslant k_0$. For k violating this inequality, we in turn use that $\tilde r_k \geqslant \Gamma_{k,1}(0)m_{k-1}$ and observe that the product of $\min\{\Gamma_{k,1}(0), 1\}$ for $k = 0, \dots, k_0$ and $k = n-k_0, \dots, n$ is positive uniformly in $n \geqslant 1$.

Writing c for this product we conclude that $m_k \geqslant c\mathrm{e}^{-\delta(k-k_0)}\tilde{r}_1$. As $\delta > 0$ is arbitrary, we have ruled out exponential decay.

We now redo the argument leading to (5.94) while invoking Lemma 5.9 and the boundedness of $\{\tilde{r}_k\}_{k=1}^n$ proved earlier to get

$$\tilde{r}_k \geqslant \sum_{j=1}^{k-1} \Gamma_j^\star(0) t_\star^{-j} \tilde{r}_{k-j} - O(\mathrm{e}^{-\eta \min\{k,n-k\}}) \tag{5.95}$$

The boundedness of $\{\tilde{r}_k\}_{k\geqslant 1}$ along with the exponential decay (5.73) allows us to extend the sum all the way to infinity at the cost of an $O(\mathrm{e}^{-\eta' k})$ error. From (5.88) we then get $m_k \geqslant m_{k-1} - a\mathrm{e}^{-\eta \min\{k,n-k\}}$ for some constant $a > 0$. But the fact that $\{m_k\}_{k\geqslant 1}$ does not decay exponentially means that we can wrap this as $m_k \geqslant (1 - a\mathrm{e}^{-\eta k/2})m_{k-1}$ once k is sufficiently large. Any positive sequence satisfying this recursive bound is necessarily bounded away from zero. □

Next we prove an iterative bound on the increments of $\{\tilde{r}_k\}_{k=1}^n$:

Lemma 5.12 *For each $\alpha \in (-1/2, 1/2)$ there exists $\epsilon > 0$ and $\widetilde{C}, \widetilde{\eta} > 0$ such that for all $\beta > \beta_{\mathrm{c}}$ satisfying $1/\beta > 1/\beta_{\mathrm{c}} - \tilde{\epsilon}$,*

$$|\tilde{r}_k - \tilde{r}_{k-1}| \leqslant \widetilde{C}\mathrm{e}^{-\widetilde{\eta} \min\{k,n-k\}} \tag{5.96}$$

holds for all $n \geqslant k \geqslant 2$.

Proof. We start by using (5.74) in (5.91) along with (5.33), the boundedness of $\{\tilde{r}_k\}_{k=1}^{n+1}$ and the fact that $t_\star$ is close to 1 when $b\theta - 1$ is small to get

$$\tilde{r}_k = O(\mathrm{e}^{-\eta \min\{k,n-k\}}) + \sum_{j=1}^{k-1} \Gamma_j^\star(0) t_\star^{-j} \tilde{r}_{k-j}. \tag{5.97}$$

Next we invoke (5.88) to rewrite this as

$$\tilde{r}_k - \tilde{r}_{k-1} = O(\mathrm{e}^{-\eta \min\{k,n-k\}}) + \sum_{j=2}^{k-1} \Gamma_j^\star(0) t_\star^{-j} (\tilde{r}_{k-j} - r_{k-1}), \tag{5.98}$$

where we also noted that the $j = 1$ term cancels on the right-hand side. Using the same argument as in (5.64), this yields

$$|\tilde{r}_k - \tilde{r}_{k-1}| \leqslant \sum_{i=1}^{k-2} \Biggl(\sum_{j=i+1}^{k-1} \Gamma_j^\star(0) t_\star^{-j} \Biggr) |r_{k-i} - r_{k-i-1}| + a\mathrm{e}^{-\eta \min\{k,n-k\}} \tag{5.99}$$

for some constant $a > 0$ which we will for convenience assume exceeds $\mathrm{e}^{2\eta}|\tilde{r}_2 - \tilde{r}_1|$.

Next observe that, since $t_\star$ is close to one for β close to β_{c}, for all $j \geqslant 2$ we have $\Gamma_j^\star(0) t_\star^{-j} \leqslant C'\sqrt{b\theta - 1}\,\mathrm{e}^{-\eta' j}$ with C' a constant and η' close to $(1 - 2|\alpha|)\log b$. Assuming $C'\sqrt{b\theta - 1} \leqslant 1/2$ we thus get $\sum_{j=i}^k \Gamma_j^\star(0) t_\star^{-j} \leqslant 2C'\sqrt{b\theta - 1}\,\mathrm{e}^{-\eta' i}$ for all $i \geqslant 2$. Abbreviating $\eta'' := \min\{\eta, \eta'\}$, the bound

$$|\tilde{r}_\ell - \tilde{r}_{\ell-1}| \leqslant 2a\mathrm{e}^{-\frac{1}{2}\eta'' \min\{\ell,n-\ell\}}, \quad \ell = 2, \dots, k-1, \tag{5.100}$$

then iterates via (5.99) to

$$
\begin{aligned}
|\tilde r_k-\tilde r_{k-1}|&\leqslant 2aC'(b\theta-1)\sum_{i=2}^k \mathrm{e}^{-\eta' i}\mathrm{e}^{-\frac12\eta''\min\{k-i,n-k+i\}}+a\mathrm{e}^{-\eta''\min\{k,n-k\}}\\
&\leqslant\biggl(\frac{4C'\sqrt{b\theta-1}}{1-\mathrm{e}^{-\eta'/2}}+1\biggr)a\mathrm{e}^{-\frac12\eta''\min\{k,n-k\}},
\end{aligned}
\tag{5.101}
$$

where we also used $\mathrm{e}^{-\frac12\eta''\min\{k-i,n-k+i\}}\leqslant \mathrm{e}^{-\frac12\eta''(k-i)}+\mathrm{e}^{-\frac12\eta''(n-k)}$ and applied $\eta'\geqslant\eta''$. Noting that C' and η' do not depend on β, for $b\theta-1$ so small that $4C'\sqrt{b\theta-1}\leqslant 1-\mathrm{e}^{-\eta'/2}$ we proved (5.100) for $\ell:=k$ from (5.100) for $\ell<k$. Since (5.100) holds for $\ell:=2$ by our assumption on a, it holds for all $\ell\geqslant1$ by induction. □

We are now ready for:

Proof of Theorem 1.4, $\beta>\beta_{\mathrm{c}}$. For each $n\geqslant1$ abbreviate $n':=\lfloor n/2\rfloor$. We start by noting that the exponential decay (5.96) implies

$$
|\tilde r_k-\tilde r_{n'}|\leqslant\frac{\widetilde C\mathrm{e}^{\tilde\eta}}{1-\mathrm{e}^{-\tilde\eta}}\,\mathrm{e}^{-\tilde\eta\min\{k,n-k\}},\quad 1\leqslant r\leqslant n.
\tag{5.102}
$$

Next we note that the identity (5.32) rewrites via (5.89) as

$$
(t_\star\theta^{\alpha^2})^{-k}w_k(p)=\sum_{j=1}^{k-2}\Gamma_{k,j}(p,0)t_\star^{-j}\tilde r_{k-j}+\sum_{q\in\mathbb{Z}\smallsetminus\{0\}}\Gamma_{k,k-1}(p,q)t_\star^{-k}\theta^{-\alpha^2}w_1(q).
\tag{5.103}
$$

Denoting

$$
\tilde r_{n'}(p):=r_{n'}\sum_{j\geqslant1}\Gamma_j^\star(p)t_\star^{-j}
\tag{5.104}
$$

we thus get

$$
\begin{aligned}
\bigl|(t_\star\theta^{\alpha^2})^{-k}w_k(p)-\tilde r_{n'}(p)\bigr|&\leqslant\sum_{j=1}^{k-2}\bigl|\Gamma_j^\star(p,0)-\Gamma_j^\star(p)\bigr|t_\star^{-j}\tilde r_{n-j}+r_{n'}\sum_{j\geqslant k-1}\Gamma_j^\star(p)t_\star^{-j}\\
&\quad+\sum_{j=1}^{k-2}\Gamma_j^\star(p)t_\star^{-j}|\tilde r_{k-j}-\tilde r_{n'}|+\sum_{q\in\mathbb{Z}\smallsetminus\{0\}}\Gamma_{k,k-1}(p,q)t_\star^{-k}\theta^{-\alpha^2}w_1(q).
\end{aligned}
\tag{5.105}
$$

Using that $\{\tilde r_k\}_{k=0}^n$ is bounded uniformly in n, invoking the bounds (5.73–5.74) in the first two terms, the decay (5.102) in the third term and the bound (5.33) in the last term along with the fact that $t_\star$ is close to one shows

$$
\bigl|(t_\star\theta^{\alpha^2})^{-k}w_k(p)-\tilde r_{n'}(p)\bigr|\leqslant C'\mathrm{e}^{-\eta'|p|(|p|-1)}\mathrm{e}^{-\eta'\min\{k,n-k\}}
\tag{5.106}
$$

for some $C',\eta'>0$ independent of n provided $b\theta-1$ is sufficiently small.

We now define

$$
f_\star(z):=\mathrm{e}^{\tilde v_\star(z)}\sum_{p\in\mathbb{Z}}\sum_{j\geqslant1}\Gamma_j^\star(p)t_\star^{-j}\mathrm{e}^{2\pi\mathrm{i}pz},
\tag{5.107}
$$

where

$$
\mathrm{e}^{-\tilde v_\star(z)}:=\sum_{q\in\mathbb{Z}}\lambda_\star(q)\mathrm{e}^{2\pi\mathrm{i}qz}
\tag{5.108}
$$

and where the sum in (5.107) converges absolutely thanks to (5.73) and the fact that $t_\star$ is close to one. The definition of f_k in (5.2) then shows

$$\begin{aligned}\bigl|(t_\star\theta^{\alpha^2})^{-k}f_k(z)-r_{n'}f_\star(z)\bigr| \leqslant & r_{n'}\bigl|\mathrm{e}^{-v_\star(z)}-a_{k-1}(0)^{-1}\mathrm{e}^{-v_{k-1}(z)}\bigl|a_{k-1}(0)\mathrm{e}^{v_{k-1}(z)}\bigr|f_n(z)\bigr| \\ &+a_{k-1}(0)\mathrm{e}^{v_{k-1}(z)}\sum_{p\in\mathbb{Z}}\bigl|(t_\star\theta^{\alpha^2})^{-k}w_k(p)-\tilde r_{n'}(p)\bigr|.\end{aligned}\tag{5.109}$$

Now observe that

$$\bigl|\mathrm{e}^{-v_\star(z)}-a_{k-1}(0)^{-1}\mathrm{e}^{-v_{k-1}(z)}\bigr|\leqslant\sum_{q\in\mathbb{Z}}\Bigl|\frac{a_{k-1}(q)}{a_{k-1}(0)}-\lambda_\star(q)\Bigr|\tag{5.110}$$

which by (5.76) decays $C'\mathrm{e}^{-\eta'\min\{k,n-k\}}$. Noting that $r_{n'}$ is bounded, we get

$$\Bigl|(t_\star\theta^{\alpha^2})^{-k}E\bigl(f_k(\phi_k)^2)\bigr)-r_{n'}^2E_\star\bigl(f_\star(\phi)^2\bigr)\Bigr|\leqslant C''\mathrm{e}^{-\eta\min\{k,n-k\}}\tag{5.111}$$

for some constant $C''>0$.

Set $C_n:=r_{n'}^2E_\star\bigl(f_\star(\phi)^2\bigr)$. Plugging (5.111) in Corollary 5.3, for all distinct $x,y\in\Lambda_n$ we thus obtain

$$\langle\mathrm{e}^{2\pi\mathrm{i}\alpha(\phi_x-\phi_y)}\rangle_{n,\beta}=\bigl[C_n+O(\mathrm{e}^{-\eta\min\{k,n-k\}})\bigr](t_\star\theta^{\alpha^2})^{-2k}\tag{5.112}$$

where $k:=k(x,y)$. Denoting

$$\kappa(\alpha,\beta):=\frac{4\pi^2}{\beta}\alpha^2-2\log t_\star(\alpha,\beta)\tag{5.113}$$

we obtain (1.18). The sequence $\{C_n\}_{n\geqslant1}$ is uniformly positive and finite for $b\theta-1$ small thanks to Lemma 5.11 and the fact that $f_\star$ is dominated by the $(p,j):=(0,1)$ term with the rest being at least order $\sqrt{b\theta-1}$.

It remains to show that $\kappa(\alpha,\beta)$ obeys the inequality in (1.17). For this we will have to extract the leading-order asymptotic of $t_\star$ in powers of $\epsilon:=\sqrt{b\theta-1}$, for ϵ small positive. We start by noting that

$$\gamma_\star(0)=1-2(b-1)\lambda_\star(1)^2+O(\epsilon^4)\tag{5.114}$$

while

$$\gamma_\star(1)=(b-1)\lambda_\star(1)+O(\epsilon^3)\tag{5.115}$$

and $\sum_{n\geqslant2}\gamma_\star(n)=O(\epsilon^2)$. This now gives

$$\Gamma_1^\star(0)=1-2(b-1)\lambda_\star(1)^2+O(\epsilon^4)\tag{5.116}$$

and, for $j\geqslant2$,

$$\Gamma_j^\star(0)=(b-1)^2\bigl(\theta^{(1+2\alpha)(j-1)}+\theta^{(1-2\alpha)(j-1)}\bigr)\lambda_\star(1)^2+O(\epsilon^4)b^{-(1-2|\alpha|)j}\tag{5.117}$$

where the implicit constant in the $O(\epsilon^3)$ does not depend on j. Using this in (5.88) while noting that $\lambda_\star(1)=O(b\theta-1)$, a calculation shows

$$t_\star=1+(b-1)\biggl[\frac{b-1}{b^{1+2\alpha}-1}+\frac{b-1}{b^{1-2\alpha}-1}-2\biggr]\lambda_\star(1)^2+O\bigl((b\theta-1)^2\bigr).\tag{5.118}$$

Differentiating with respect to α we check that the square brackets is strictly positive once $\alpha\neq0$. It follows that $t_\star>0$ and thus $\kappa(\alpha,\beta)<\frac{4\pi^2}{\beta}\alpha^2$ for $b\theta-1$ small positive. $\square$

Remark 5.13 In the derivation asymptotic (6.24–6.25) we show that

$$\lambda_\star(1) = \sqrt{\frac{2(b^3-1)}{(b-1)^2(b+1)^3}}\,\sqrt{b\theta-1} + O\bigl(|b\theta-1|^{3/2}\bigr) \tag{5.119}$$

Inserting this in (5.118) then shows

$$t_\star(\alpha,\beta) = 1 + \tau(\alpha)(b\theta-1) + O\bigl((b\theta-1)^2\bigr). \tag{5.120}$$

Invoking (5.113) along with $b\theta - 1 = \frac{2\pi^2}{\beta_c^2}(\beta-\beta_c) + O((\beta-\beta_c)^2)$ we get (1.20). The numerical closeness of the critical and near-critical coefficients again stems from the structure of $\Gamma_j^\star(0)$ and the similarity of leading order term of $\lambda_k(1)$ in the critical and slightly supercritical regimes.

Remark 5.14 The above proofs were tailored to the asymptotic of the fractional dipole-charge correlator but the structure applies to, and the conclusion is in fact much easier for its monopole counterpart. Indeed, we get the identity

$$\langle \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_x}\rangle_{n,\beta} = f_{n+1}(0) \tag{5.121}$$

for f_{n+1} obtained by taking $k := n+1$ in (5.2). (Here we used that the underlying Markov chain effectively takes value zero at the initial time so no expectation is needed.). Setting

$$C_n' := \begin{cases} (t_\star\theta^{\alpha^2})^{-n}\langle \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_x}\rangle_{n,\beta}, & \text{if } \beta \neq \beta_c,\\ k^{-\tilde\tau}\theta^{-\alpha^2 n}\langle \mathrm{e}^{2\pi\mathrm{i}\alpha\phi_x}\rangle_{n,\beta}, & \text{if } \beta = \beta_c, \end{cases} \tag{5.122}$$

where $t_\star := 1$ when $\beta < \beta_c$, the arguments used above show that C_n' is bounded away from zero and infinity. Writing this using $\kappa(\alpha,\beta)$ we get (1.21) as desired.

6. Supercritical iterations

The principal objective of this section is the proof of Theorem 3.6. This requires studying the flow of the iterations (3.18) under the conditions when these admit a "non-trivial" fixed point. The analysis carries a significant technical overhead that goes quite beyond what was sufficient for the subcritical and critical cases.

6.1 Renormalization-group flow.

We start by casting the iterations in a more convenient and also somewhat more general form. Recall the definition of θ_k from (3.19) and θ from (3.29) and note that $\beta > \beta_c$ is equivalent to $b\theta > 1$. Let Σ be the set of (doubly-infinite) positive sequences $\boldsymbol{\lambda} = \{\lambda(q)\}_{q\in\mathbb{Z}}$ satisfying the symmetry condition $\lambda(-q) = \lambda(q)$ for all $q \in \mathbb{Z}$ and such that $\lambda(0) = 1$ and $\sup_{q\geqslant 0} \lambda(q+1)/\lambda(q) < \infty$ hold true. For each $\boldsymbol{\lambda} \in \Sigma$ and $q \in \mathbb{Z}$ set

$$G_q(\boldsymbol{\lambda}) := \sum_{\substack{\ell_1,\dots,\ell_b\in\mathbb{Z}\\ \ell_1+\dots+\ell_b=q}} \prod_{i=1}^b \lambda(\ell_i) \tag{6.1}$$

and, for each $k \geqslant 0$, let

$$F_q^{(k)}(\boldsymbol{\lambda}) := \frac{G_q(\boldsymbol{\lambda})}{G_0(\boldsymbol{\lambda})}\theta_k^{q^2}. \tag{6.2}$$

We will write $F^{(k)}$ for the map assigning $\boldsymbol{\lambda}$ the sequence $\{F_q^{(k)}(\boldsymbol{\lambda})\}_{q\in\mathbb{Z}}$ and use the notation F for the corresponding map in which the θ_k's have been replaced by θ.

In order to make the connection to the problem at hand, note that extending our earlier notation (3.59) to

$$\lambda_k(q) := \frac{a_k(q)}{a_k(0)}, \quad q \in \mathbb{Z}, \tag{6.3}$$

and denoting $\boldsymbol{\lambda}_k := \{\lambda_k(q)\}_{q\in\mathbb{Z}}$, the iterations (3.18) become

$$\boldsymbol{\lambda}_k = F^{(k)}(\boldsymbol{\lambda}_{k-1}), \quad k \geqslant 1, \tag{6.4}$$

where the parametrization reflects that $a_0(\cdot)/a_0(0)$ corresponds to $\boldsymbol{\lambda}_0$. We are thus interested in the convergence/limit properties of the flow of compositions of functions $\{F^{(k)}\colon k \geqslant 0\}$ evaluated on elements from Σ.

Our control of the iterations turns out to be slightly stronger when b is even. Indeed, in this case we can work with any starting $\boldsymbol{\lambda}_0 \in \Sigma$ while for b odd we have to assume that the initial $\boldsymbol{\lambda}_0$ arises from the setting of the present work. We thus set $\Sigma' := \Sigma$ when b is even and let Σ' be the set of $\boldsymbol{\lambda} \in \Sigma$ that are the Fourier coefficients of a positive measure on $[0, 1)$ when b is odd. Then we restate the key part of Theorem 3.6 as:

Theorem 6.1 *Let $b \geqslant 2$. There exists $\epsilon > 0$ and, for each $\beta > 0$ satisfying $1 < b\theta < 1 + \epsilon$, there exists a unique $\boldsymbol{\lambda}_\star \in \Sigma'$ such that*

$$F(\boldsymbol{\lambda}_\star) = \boldsymbol{\lambda}_\star. \tag{6.5}$$

Moreover, under Assumption 1.1, for each β as above there exist $\eta > 0$ and $C > 0$ and, for each $\boldsymbol{\lambda}_0 \in \Sigma'$, there exists $k_0 \geqslant 0$ such that, for all $n > 2k_0$, the sequence $\{\boldsymbol{\lambda}_k\}_{k=0}^n$ defined from $\boldsymbol{\lambda}_0$ via (6.4) *obeys*

$$\sum_{q\geqslant 1} \bigl[16 b^{3/2}(b\theta - 1)\bigr]^{\frac{q-1}{2}} \bigl|\lambda_k(q) - \lambda_\star(q)\bigr| \leqslant C\biggl[\mathrm{e}^{-\eta k} + \sum_{j=0}^{k} \mathrm{e}^{-\eta(k-j)}\, \mathfrak{d}_{\min\{j,n-j\}}\biggr] \tag{6.6}$$

whenever $\min\{k, n-k\} \geqslant k_0$. *Here $\{\mathfrak{d}_j\}_{j\geqslant 0}$ is the sequence from Assumption 1.1*

While the above may appear to be a run-off-the-mill conclusion of the Banach Fixed Point Theorem, the proof is considerably more complicated. A key problem is that F is not contractive on Σ when $\beta > \beta_c$ due to the "subcritical" fixed point (corresponding to $\lambda(q) = \delta_{q,0}$) lingering on the "boundary" of Σ. This fixed point is unstable for $\beta > \beta_c$ which mucks up uniform control of the iterations.

Our way to overcome this is by following the iterations until they reach a suitable subset $\Sigma_0 \subseteq \Sigma$ where contractivity can be proved. We assume $b\theta - 1$ small as, under this condition, the evolution of $\boldsymbol{\lambda}_k$ is completely controlled by $\lambda_k(1)$ and $\lambda_k(2)$, just as we saw happen for the critical case in Lemma 3.12 and the proof of Theorem 3.5. Indeed, these two coordinates evolve autonomously (modulo error terms) according to (3.64) while the remaining ones are just "swept along."

Working near critical β unfortunately means that the convergence $\boldsymbol{\lambda}_k \to \boldsymbol{\lambda}_\star$ is very slow; in fact, it is the slower the closer is $b\theta$ to 1. Indeed, our proof gives (6.6) with η proportional to $b\theta - 1$ which, as is easy to check, reflects also the true decay rate when $\sigma_k^2 = 1$ for all $k \geqslant 0$. The inhomogeneity of $\{\sigma_k^2\}_{k=0}^n$ causes further errors that are governed by the tails of the convergent series $\sum_{j\geqslant 0} \mathfrak{d}_j$.

6.2 Preliminary observations.

We start by some preliminary technical estimates. As noted above, our control of the iterations is better when b is even. This is due to the availability of:

Lemma 6.2 *Suppose that $b \geqslant 2$ is even and let G_q be as in* (6.1)*. Then for all $\boldsymbol{\lambda} \in \Sigma$,*

$$G_q(\boldsymbol{\lambda}) \leqslant G_0(\boldsymbol{\lambda}), \quad q \in \mathbb{Z}. \tag{6.7}$$

In particular, we have $F_q(\boldsymbol{\lambda}) \leqslant \theta^{q^2} \leqslant 1$ for all $q \in \mathbb{Z}$.

Proof. Assume b even and recall that $\Xi_b(n) := \{(\ell_1, \dots, \ell_b) \in \mathbb{Z}^b \colon \ell_1 + \dots + \ell_b = n\}$. Our goal is to show that, for all $\boldsymbol{\lambda} \in \Sigma$ and $n \in \mathbb{Z}$,

$$\sum_{\bar\ell \in \Xi_b(n)} \prod_{i=1}^b \lambda(\ell_i) \leqslant \sum_{\bar\ell \in \Xi_b(0)} \prod_{i=1}^b \lambda(\ell_i), \tag{6.8}$$

where $\bar\ell = (\ell_1, \dots, \ell_b)$. For this we note that, since b is even, $b' := b/2$ is a natural and the distributive law yields

$$\sum_{\bar\ell \in \Xi_b(q)} \prod_{i=1}^b \lambda(\ell_i) = \sum_{j \in \mathbb{Z}} \biggl(\sum_{\bar\ell' \in \Xi_{b'}(j)} \prod_{i=1}^{b'} \lambda(\ell'_i) \biggr) \biggl(\sum_{\bar\ell'' \in \Xi_{b'}(q-j)} \prod_{i=1}^{b'} \lambda(\ell''_i) \biggr). \tag{6.9}$$

On the other hand, the same argument and the symmetry condition $\lambda(-\ell) = \lambda(\ell)$ shows

$$\sum_{j \in \mathbb{Z}} \biggl(\sum_{\bar\ell' \in \Xi_{b'}(j)} \prod_{i=1}^{b'} \lambda(\ell'_i) \biggr)^2 = \sum_{\bar\ell \in \Xi_b(0)} \prod_{i=1}^b \lambda(\ell_i). \tag{6.10}$$

To get the desired claim (6.8), it suffices to invoke the Cauchy-Schwarz inequality in (6.9) and apply (6.10). □

The distinction between b even and b odd now enters solely through the following enhancement of Lemmas 3.7–3.8:

Lemma 6.3 *For any $\boldsymbol{\lambda}_0 \in \Sigma'$ and all $n \geqslant k \geqslant 1$,*

$$\sup_{q \geqslant 0} \frac{\lambda_k(q+1)}{\lambda_k(q)} \leqslant \biggl(\prod_{i=0}^k \max\{b\theta_i^3, 1\} \biggr) \max\biggl\{ 1, \sup_{q \geqslant 0} \frac{\lambda_0(q+1)}{\lambda_0(q)} \biggr\}. \tag{6.11}$$

Proof. Let $\boldsymbol{\lambda}_0 \in \Sigma'$ and, setting $a_0(0) := 1$ if b is even, let $\{a_0(q)\}_{q \in \mathbb{Z}}$ be such that $a_0(q)/a_0(0) = \lambda_0(q)$ for each $q \in \mathbb{Z}$. Lemma 3.7 along with the fact that $(q+1)^2 - q^2 \geqslant 3$ once $q \geqslant 1$ imply

$$\sup_{q \geqslant 1} \frac{a_k(q+1)}{a_k(q)} \leqslant b\theta_k^3 \sup_{q \geqslant 0} \frac{a_{k-1}(q+1)}{a_{k-1}(q)}. \tag{6.12}$$

Now note that, for b even, Lemma 6.2 shows that $a_k(n) \leqslant a_k(0)$ for all $n \in \mathbb{Z}$ and $k \geqslant 0$ while for b odd this holds by the fact that the Fourier coefficients of a positive measure are bounded by the total mass of the measure. Denoting, as before, the supremum on the left of (6.11) as c_k, we are thus led to the inequality

$$c_k \leqslant \max\{1, b\theta_k^3 c_{k-1}\}. \tag{6.13}$$

To iterate this, set $\tilde{c}_k := \max\{1, c_k\}$ are note that the above gives $\tilde{c}_k \leqslant \max\{1, b\theta_k^3\}\tilde{c}_{k-1}$. This now readily implies (6.11). □

As a consequence we get:

Corollary 6.4 *Suppose $b\theta^3 < 1$. Then, under Assumption 1.1, for all $\boldsymbol{\lambda}_0 \in \Sigma'$,*

$$A_1 := \Bigl(\sup_{n\geqslant 1}\prod_{i=0}^{n}\max\{b\theta_i^3, 1\}\Bigr)\max\Bigl\{1, \sup_{q\geqslant 0}\frac{\lambda_0(q+1)}{\lambda_0(q)}\Bigr\} < \infty \tag{6.14}$$

and, for each $n \geqslant 1$, the iterations $\{\boldsymbol{\lambda}_k\}_{k=0}^n$ generated from $\boldsymbol{\lambda}_0$ via (6.4) *obey*

$$\sup_{n\geqslant k\geqslant 0}\,\sup_{q\geqslant 0}\frac{\lambda_k(q+1)}{\lambda_k(q)} \leqslant A_1. \tag{6.15}$$

Proof. The condition $b\theta^3 < 1$ along with Assumption 1.1 imply that the product is uniformly bounded. Invoking (6.11), we get the claim. □

Remark 6.5 To phrase the above in the vernacular of the renormalization group theory, under the condition $b\theta^3 < 1$, the estimate (6.12) says that $\lambda(q)$ with $|q| \geqslant 2$ are irrelevant (i.e., contracting) "directions" of the renormalization-group flow. Due to the normalization $\lambda(0) = 1$, the only possibly relevant (i.e., expanding) "direction" in this regime is thus $\lambda(1) = \lambda(-1)$. The punchline of Corollary 6.4 is that the expansion in this coordinate is still clamped down by a uniform bound.

Before we move to the consequences of above observations, let us record the following general bound that shows up repeatedly in the sequel:

Lemma 6.6 *Suppose $\boldsymbol{\lambda} \in \Sigma$ and $A \in (0,1)$ are such that $\sup_{q\geqslant 0}\lambda(q+1)/\lambda(q) \leqslant A$. For all $r \geqslant 1$ and $q_1, \dots, q_r \geqslant 0$ we then have*

$$\sum_{\substack{\ell_1,\dots,\ell_r\in\mathbb{Z}\\ \forall i\leqslant r\colon |\ell_i|\geqslant q_i}}\prod_{i=1}^{r}\lambda(\ell_i) \leqslant \Bigl(\frac{1+A}{1-A}\Bigr)^r\prod_{\substack{i=1,\dots,r\\ q_i\geqslant 1}}\bigl[\lambda(1)A^{q_i-1}\bigr], \tag{6.16}$$

with the product on the right no larger than $A^{q_1+\dots+q_r}$.

Proof. It suffices to deal with $r := 1$ where the sum equals $\delta_{q_1,0} + 2\sum_{\ell>q_1}\lambda(\ell)$. The assumptions imply that $\lambda(\ell) \leqslant \lambda(1)A^{\ell-1}$ for $\ell \geqslant 1$. Using this bounds the sum by $\lambda(1)A^{q_1-1}\frac{2A}{1-A}$ when $q_1 \geqslant 1$ and by $\frac{1+A}{1-A}$ when $q_1 = 0$. The second part of the claim follows from $\lambda(1) \leqslant A$. □

6.3 Near-critical bounds.

We will now improve the above crude estimate on $\sup_{q\geqslant 0}\lambda_k(q+1)/\lambda_k(q)$ to a bound that is small in the "near-critical" regime, i.e., for $b\theta - 1$ small positive.

Lemma 6.7 *Suppose $\beta > 0$ is such that $(b - 1/5)\theta < 1 < b\theta$ and $b\theta^3 < 1$. Under Assumption 1.1, for each $\boldsymbol{\lambda}_0 \in \Sigma'$ there exists $k_1 \geqslant 0$ such that*

$$\sup_{q\geqslant 0} \frac{\lambda_k(q+1)}{\lambda_k(q)} \leqslant A_3\sqrt{b\theta - 1} \tag{6.17}$$

holds with

$$A_3 := 2^{5/6}\sqrt{\frac{1}{1-(b-1)\theta}} \tag{6.18}$$

provided that $\min\{k, n-k\} \geqslant k_1$.

Proof. For each $n \geqslant 1$ fix $\{\sigma_k\}_{k=0}^n$ obeying (1.6). We first make some observations for a fixed $n \geqslant 1$. For $k = 0, \dots, n$, denote the supremum in (6.17) as c_k, abbreviate $\delta := \sum_{j\geqslant 0} \mathfrak{d}_j$ and set $C := \inf_{0\leqslant j\leqslant n} c_j$. Then recall the inequality (3.51) from the proof of Lemma 3.10, where α_k is defined in (3.47). Note that $\alpha_k \geqslant (1 + C^2)^{-1}$ by the fact that $\alpha_k \leqslant 1$, abbreviate

$$\bar\alpha := \frac{1}{1+C^2}\theta^{4\delta} \tag{6.19}$$

and set

$$h(u) := \frac{\theta u}{1+\bar\alpha u^2} + (b-1)\theta u. \tag{6.20}$$

From the argument following (3.51) we then get that the quantity $\tilde c_k$ defined from c_k via (3.52) obeys the iterative bound $\tilde c_{j+1} \leqslant h_k(\tilde c_j)$ for all $j = 0, \dots, n-1$.

Under $(b-1)\theta < 1 < b\theta$, the function h is increasing and concave on positive reals with two fixed points: one at zero and the other at

$$u_\star := \sqrt{\frac{1}{\bar\alpha}\frac{1}{1-(b-1)\theta}}\,\sqrt{b\theta-1}. \tag{6.21}$$

Moreover, iterations started at $u \leqslant u_\star$ never rise above $u_\star$ while those started at $u > u_\star$ decrease, due to $\mathrm{e}^{-\eta} := \sup_{u\geqslant u_\star} h'(u) < 1$, geometrically fast towards $u_\star$. As $\tilde c_k \leqslant c_k\theta^{-\delta}$, it follows that, once $j \geqslant 0$ is such that $C\theta^{-\delta}\mathrm{e}^{-\eta j} \leqslant 2^{1/3}u_\star$ we have $\tilde c_{k+j} \leqslant h_k^j(\tilde c_k) \leqslant 2^{1/3}u_k$. An inspection of (6.19) and (6.21) shows that j can be chosen independently of C.

We will now apply the same argument repeatedly to subsequences of the form $\{\boldsymbol{\lambda}_i\}_{i=k}^{n-k}$. Set $C_k := \sup_{n\geqslant k}\max_{k\leqslant i\leqslant n-k} c_i$ and let $\delta_k := \sum_{j\geqslant k}\mathfrak{d}_j$. The above then shows that there exists $j \geqslant 0$ such that for all $k \geqslant 0$,

$$C_{k+j} \leqslant 2^{1/6}\theta^{-3\delta_k}\sqrt{1+C_k^2}\sqrt{\frac{b\theta-1}{1-(b-1)\theta}} \tag{6.22}$$

where one factor of $\theta^{-\delta_k}$ arises from returning to variables c_ℓ on the left-hand side and another factor $\theta^{-2\delta_k}$ arises from the term in (6.19). We now claim that the sequence $\{C_k\}_{k\geqslant 0}$ will eventually drop below one. Indeed, if $C_k \geqslant 1$ for some k with $\theta^{-3\delta_k} \leqslant 2^{1/6}$, then $2^{1/6}\theta^{-3\delta_k}\sqrt{1+C_k^2} \leqslant 2C_k$ which using $x := 1 - (b - 1/5)\theta$ yields

$$C_{k+j} \leqslant 2\sqrt{\frac{b\theta-1}{1-(b-1)\theta}}\,C_k = \sqrt{\frac{1-5bx}{1-5(b-1)x}}\,C_k. \tag{6.23}$$

For $x > 0$, which is equivalent to $(b - 1/5)\theta < 1$, the square root on the right is less than one showing that $\ell \mapsto C_{k+\ell j}$ decreases exponentially. In particular, there exists $\ell \geqslant 1$ such that $C_{k+\ell j} \leqslant 1$.

To finish the proof we now note that $C_{k'} \leqslant 1$ along with $\theta^{-3\delta_{k'}} \leqslant 2^{1/6}$ via (6.22) implies $C_{k'+j} \leqslant A_3\sqrt{b\theta - 1}$. Hence (6.17) holds once $\min\{k, n-k\} \geqslant k_1 := k' + j$. □

The power of 2 in (6.18) was chosen to ensure that $A_3 \leqslant 2\sqrt{b}$ for $b\theta - 1$ small. This will aid some numerical computations later. We now use the above to show that the supremum is, for $b\theta - 1$, dominated by the first two components and, in fact, nail their asymptotic values in this regime.

Lemma 6.8 *For each $\delta, \delta' \in (0,1)$ there exists $\epsilon > 0$ such that for all $\boldsymbol{\lambda}_0 \in \Sigma'$ and all $\beta > 0$ with $1 < b\theta < 1 + \epsilon$ and there exists $k_3 \geqslant 0$ for which*

$$\lambda_k(1) \geqslant \left(\frac{(b-1)^2}{2}\frac{(b+1)^3}{b^3-1} + \delta\right)^{-1/2}\sqrt{b\theta - 1} \tag{6.24}$$

$$\lambda_k(1) \leqslant \left(\frac{(b-1)^2}{2}\frac{(b+1)^3}{b^3-1} - \delta\right)^{-1/2}\sqrt{b\theta - 1} \tag{6.25}$$

and

$$\left|\frac{\lambda_k(2)}{b\theta - 1} - \frac{1}{(b-1)(b+1)^3}\right| \leqslant \delta' \tag{6.26}$$

hold when $\min\{k, n-k\} \geqslant k_3$.

Proof. As before, we will use the shorthands $\lambda_k := \lambda_k(1)$ and $\gamma_k := \lambda_k(2)$ and, committing major abuse of notation, abbreviate $\epsilon := \sqrt{b\theta - 1}$. Our first goal is to show that λ_k will eventually be at least order ϵ. For this we invoke the inequality (3.60) from the proof of Lemma 3.11 which reads

$$\lambda_k \geqslant \frac{b\theta_k}{1 + \alpha' c_k^3}\frac{\lambda_{k-1}}{1 + b(b-1)\lambda_{k-1}^2} \tag{6.27}$$

for some constant $\alpha' > 0$, whenever $c_{k-1} \leqslant 1/2$. The latter is enabled by assuming $\min\{k, n-k\} > k_1$ and $A_3\epsilon \leqslant 1/2$, for k_1 and A_3 as in Lemma 6.7. Plugging in the bound $c_{k-1} \leqslant A_3\epsilon$ while noting that $\theta_k = \theta^{\sigma_k^2}$ we then get

$$\frac{b\theta_k}{1 + \alpha' c_k} \geqslant \frac{1 + \epsilon^2}{1 + \alpha' A_3^3\epsilon^3}\theta^{|1-\sigma_k^2|} \tag{6.28}$$

We now take $k_1' \geqslant k_1$ so large that $\mathfrak{d}_j \leqslant \epsilon^3$ once $j \geqslant k_1'$ and ϵ so small that right-hand side of (6.28) is at least $1 + \epsilon^2/2$. It follows that

$$\lambda_k \geqslant \left(1 + \frac{\epsilon^2}{2}\right)\frac{\lambda_{k-1}}{1 + b(b-1)\lambda_{k-1}^2} \tag{6.29}$$

once $\min\{k, n-k\} \geqslant k_1'$. Interpreting the right-hand side as $h(\lambda_{k-1})$ for $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ increasing and convex shows that λ_k converges under iterations to the unique non-zero fix point of h which, as a computation shows, occurs at $[2b(b-1)]^{-1/2}\epsilon$. Hence there exists $k_1'' \geqslant k_i'$ such that

$$\lambda_k \geqslant \frac{1}{2\sqrt{b(b-1)}}\epsilon \tag{6.30}$$

once $\min\{k, n-k\} \geqslant k_1''$.

Next we observe that, with c_k bounded by a constant times λ_k, whenever k obeys $\min\{k, n-k\} \geqslant k_1''$ the identities (3.64) from Lemma 3.12 are still in force. The calculation leading up to (3.74) still applies. As δ_k there is order ϵ which is order λ_k, instead of (3.76) we then get

$$\frac{\gamma_k}{\lambda_k^2} = \frac{1}{2}\frac{b-1}{b^3-1} + t_k''\lambda_k \tag{6.31}$$

for some bounded sequence $\{t_k''\}_{k=0}^n$. Plugging this in the first line of (3.64) yields

$$\lambda_{k+1} = b\theta_{k+1}\frac{\lambda_k + \frac{b-1}{2}(b-2+\frac{b-1}{b^3-1})\lambda_k^3 + r_k'\lambda_k^4}{1+b(b-1)\lambda_k^2 + s_k\lambda_k^3} \tag{6.32}$$

where $r_k' := r_k + (b-1)t_k''$.

In order to analyze this further, note that for $\min\{k, n-k\}$ so large that $\mathfrak{d}_k \leqslant \epsilon$ the above leads to the inequalities

$$b\theta\frac{\lambda_k}{1+(A+B\epsilon)\lambda_k^2} \leqslant \lambda_{k+1} \leqslant b\theta\frac{\lambda_k}{1+(A-B\epsilon)\lambda_k^2}, \tag{6.33}$$

where

$$\begin{aligned} A := b(b-1) - \frac{b-1}{2}\Bigl(b-2+\frac{b-1}{b^3-1}\Bigr) &= \frac{b-1}{2}\Bigl[b+2-\frac{1}{b^2+b+1}\Bigr] \\ &= \frac{b-1}{2}\frac{(b+1)^3}{b^2+b+1} = \frac{(b-1)^2}{2}\frac{(b+1)^3}{b^3-1}. \end{aligned} \tag{6.34}$$

and B is a positive constant derived from the bounds on the sequences r_k' and s_k. Noting iterations of $h(u) = b\theta\frac{u}{1+Au^2}$ are attracted to $u_\star = A^{-1/2}\sqrt{b\theta-1}$, following the iterations (6.33) we then get that, after a finite number of steps, we have

$$(A+2B\epsilon)^{-1/2}\epsilon \leqslant \lambda_k(1) \leqslant (A-2B\epsilon)^{-1/2}\epsilon \tag{6.35}$$

For ϵ such that $B\epsilon < \delta$, this gives (6.24–6.25). With the help (6.31) and $\lambda_k \leqslant A_3\epsilon$ we then get (6.26) as well. $\square$

6.4 Contractive region.

We now proceed to define a subdomain of Σ on which we later prove uniform contractivity of the map F. This subdomain will depend on $\beta > 0$, which we assume is such that $b\theta > 1$, and numbers $\delta, \delta' \in (0,1)$ and $A > 0$ as

$$\Sigma_0 := \Biggl\{ \boldsymbol{\lambda} \in \Sigma' \colon \lambda(1) \geqslant \Bigl(\frac{(b-1)^2}{2}\frac{(b+1)^3}{b^3-1} + \delta\Bigr)^{-1/2}\sqrt{b\theta-1}$$
$$\wedge\ \lambda(1) \leqslant \Bigl(\frac{(b-1)^2}{2}\frac{(b+1)^3}{b^3-1} - \delta\Bigr)^{-1/2}\sqrt{b\theta-1} \tag{6.36}$$
$$\wedge\ \sup_{q\geqslant 0}\frac{\lambda(q+1)}{\lambda(q)} \leqslant A\sqrt{b\theta-1}\ \wedge\ \Bigl|\frac{\lambda(2)}{b\theta-1} - \frac{1}{(b-1)(b+1)^3}\Bigr| \leqslant \delta' \Biggr\}.$$

Assuming that Assumption 1.1 holds, we now summarize the previous observations in:

Lemma 6.9 *For all $A \geqslant 2\sqrt{b}$ and $\delta, \delta' \in (0,1)$, there exists $\epsilon > 0$ such that, for Σ_0 defined by δ, δ' and A as above, the following is true for all β satisfying $1 < b\theta < 1+\epsilon$: For all $\boldsymbol{\lambda}_0 \in \Sigma'$ there exists $k_4 \geqslant 0$ such that $\boldsymbol{\lambda}_k \in \Sigma_0$ holds whenever $\min\{k, n-k\} \geqslant k_4$ for the iterations $\{\boldsymbol{\lambda}_k\}_{k=0}^n$ defined from $\boldsymbol{\lambda}_0$ via (6.4).*

Proof. This follows from Lemmas 6.7 and 6.8 along with the fact that A_3 in (6.18) tends to $2^{5/6}\sqrt{b}$ as $b\theta$ decreases to 1. □

We will henceforth focus on the evolution driven by F, i.e., for $\theta_k = \theta$ for all k. Here we need to check that F maps Σ_0 into itself.

Lemma 6.10 *For each $A > \sqrt{\frac{2}{b-1}}$ and $\delta' > 0$ there exists $\delta > 0$ and $\epsilon > 0$ such that*

$$F(\Sigma_0) \subseteq \Sigma_0 \tag{6.37}$$

holds for all $\beta > 0$ with $1 < b\theta < 1+\epsilon$.

Proof. Fix A and δ' as above and, abusing notation again, abbreviate $\epsilon := \sqrt{b\theta-1}$. Pick $\boldsymbol{\lambda} \in \Sigma_0$ and note that, since $F(\boldsymbol{\lambda}) \in \Sigma'$, we only need to verify that $F(\boldsymbol{\lambda})$ obeys the conditions in (6.36).

For the first two conditions in (6.36), we repeat the calculations underlying the proof of Lemma 6.8 to get

$$F_1(\boldsymbol{\lambda}) = b\theta\frac{\lambda(1) + [\binom{b-1}{2} + \frac{1}{2}\frac{(b-1)^2}{b^3-1} + \eta\epsilon^2]\lambda(1)^3}{1 + [b(b-1) + \eta'\epsilon^2]\lambda(1)^2} \tag{6.38}$$

for some non-negative η and η' depending only on δ' and A. Once ϵ is sufficiently small (depending only on δ), this implies

$$b\theta\frac{\lambda(1)}{1+(A+\delta)\lambda(1)^2} \leqslant F_1(\boldsymbol{\lambda}) \leqslant b\theta\frac{\lambda(1)}{1+(A-\delta)\lambda(1)^2} \tag{6.39}$$

for some constant B derived from η and η'. Now check that (for $b\theta = 1+\epsilon^2$) the expression on the left preserves the inequality $\lambda(1) \geqslant (A+\delta)^{-1}\epsilon$ while that on the right preserves the inequality $\lambda(1) \leqslant (A-\delta)^{-1}\epsilon$. We conclude that $F_1(\boldsymbol{\lambda})$ obeys the first two lines in (6.36).

For the third condition in (6.36) we first note that Lemma 3.9 along with the third condition for $\boldsymbol{\lambda}$ give

$$F_1(\boldsymbol{\lambda}) \leqslant \theta \frac{\lambda(1)}{1+\binom{b}{2}\lambda(1)^2} + (b-1)\theta \sup_{n\geqslant 0} \frac{\lambda(n+1)}{\lambda(n)} \leqslant \left(\frac{\theta}{1+\binom{b}{2}A^2\epsilon^2} + (b-1)\theta\right)A\epsilon, \tag{6.40}$$

where we assumed ϵ small enough so that the first term is non-decreasing in $\lambda(1) \leqslant A\epsilon$. As a calculation shows, under $(b-1)\theta < 1$, the prefactor of $A\epsilon$ is less than one for ϵ sufficiently small if $A > \sqrt{\frac{2}{b-1}}$. Since Lemma 3.7 gives

$$\frac{F_{q+1}(\boldsymbol{\lambda})}{F_q(\boldsymbol{\lambda})} \leqslant b\theta^{1+2q} \sup_{\ell\geqslant 0} \frac{\lambda(\ell+1)}{\lambda(\ell)} \leqslant b\theta^3 A\epsilon, \quad q \geqslant 1, \tag{6.41}$$

under $b\theta^3 \leqslant 1$, the third condition in (6.36) thus applies to $F(\boldsymbol{\lambda})$.

Finally, for the last condition in (6.36) abbreviate $C := [(b-1)(b+1)^3]^{-1}$. We now proceed as in the derivation of (3.67) to get

$$F_2(\boldsymbol{\lambda}) - b\theta^4\Big[\lambda(2) - \frac{b-1}{2}\lambda(1)^2\Big] = O(\epsilon^4). \tag{6.42}$$

Noting that $b\theta^4 = b^{-3} + O(\epsilon^2)$, this implies

$$\big|F_2(\boldsymbol{\lambda}) - C\epsilon^2\big| \leqslant b^{-3}\big|\lambda(2) - C\epsilon^2\big| + b^{-3}\left|\frac{b-1}{2}\lambda(1)^2 - (b^3-1)C\epsilon^2\right| + A''\epsilon^4, \tag{6.43}$$

where A'' is a constant that depends only on A. Invoking the conditions from Σ_0, a calculation shows

$$\big|\epsilon^{-2}F_2(\boldsymbol{\lambda}) - C\big| \leqslant b^{-3}\delta' + \left[\frac{(b^3-1)(b-1)}{(b+1)^2}\right]^2\delta + A''\epsilon^2. \tag{6.44}$$

We now choose δ so that the corresponding term is less than, say, $\delta'/2$. For ϵ small, $F(\boldsymbol{\lambda})$ then obeys also the last condition in the definition of Σ_0 and so $F(\boldsymbol{\lambda}) \in \Sigma_0$. □

6.5 Contractivity.

Having identified Σ_0 and shown that F maps it into itself, we now prove that F is actually contractive on it. Note that, for $\sup_{q\in\mathbb{Z}} \lambda(q) < 1$, each component of $F(\boldsymbol{\lambda})$ is the ratio of two positive convergent sums and so F is continuously differentiable. A natural way to prove contractivity is thus to estimate the derivatives of F in a suitable norm. However, this will only be useful if we first show:

Lemma 6.11 *Σ_0 is a convex set.*

Proof. The first and third condition in the definition of Σ_0 are clearly preserved by convex combinations. For the second condition we note that

$$t \mapsto \frac{tA + (1-t)A'}{tB + (1-t)B'} \tag{6.45}$$

is, for any $A, A', B, B' > 0$, monotone on $[0,1]$. If the ratio is less than a constant at $t = 0$ and $t = 1$, it is less than that constant for all $t \in [0,1]$. □

We will for brevity write $\partial_\ell F_q$ to denote the partial derivative of the q-th component of F with respect to $\lambda(\ell)$. We start with estimates on these:

Lemma 6.12 *Let $\beta > \beta_c$. For each $A > 0$ there exist $\delta_0 > 0$ and $\epsilon_0 > 0$ such that, for Σ_0 defined by A and any $\delta, \delta' \in (0, \delta_0]$,*

$$\big|\partial_\ell F_q(\boldsymbol{\lambda})\big| \leqslant \begin{cases} b(b-1)\theta^{q^2}\lambda(1) + O(\epsilon^2), & \text{if } |q-\ell| = 1,\\ b\theta^{q^2} + O(\epsilon^2), & \text{if } q = \ell \geqslant 2,\\ (b\theta^{q^2} + O(\epsilon^2))(bA\epsilon)^{|q-\ell|}, & \text{if } |q-\ell| \geqslant 1,\\ 1 - b(b-1)\lambda(1)^2, & \text{if } q = \ell = 1, \end{cases} \tag{6.46}$$

holds for all $\boldsymbol{\lambda} \in \Sigma_0$ and all $q, \ell \geqslant 1$ provided that $\epsilon := \sqrt{b\theta - 1} \in (0, \epsilon_0)$. Here the implicit constants in $O(\epsilon^2)$ terms do not depend on q and ℓ.

Proof. Fix $q \geqslant 1$ and $\ell \geqslant 1$. We start with some general considerations. Denote by $\widetilde{G}_q$ the quantity G_q with b replaced by $b-1$. The symmetry $\lambda(-\ell) = \lambda(\ell)$ then gives

$$\partial_\ell G_q(\boldsymbol{\lambda}) = b\widetilde{G}_{q-\ell}(\boldsymbol{\lambda}) + b\widetilde{G}_{q+\ell}(\boldsymbol{\lambda}) \tag{6.47}$$

and so, by the quotient rule,

$$\partial_\ell F_q(\boldsymbol{\lambda}) = \Big[\big(\widetilde{G}_{q-\ell}(\boldsymbol{\lambda}) + \widetilde{G}_{q+\ell}(\boldsymbol{\lambda})\big)G_0(\boldsymbol{\lambda})^{-1} - 2\widetilde{G}_\ell(\boldsymbol{\lambda})G_q(\boldsymbol{\lambda})G_0(\boldsymbol{\lambda})^{-2}\Big]b\theta^{q^2}. \tag{6.48}$$

Note that one term in the square bracket is positive and the other is negative. It thus suffices to estimate each of them separately.

Moving to actual estimates, let us begin with the the third line in (6.46). Here we first observe that the argument in Lemma 3.7 and the second condition in the definition of Σ_0 give $G_{q+1}(\boldsymbol{\lambda})/G_q(\boldsymbol{\lambda}) \leqslant b\sup_{\ell\geqslant 1}\lambda(\ell+1)/\lambda(\ell) \leqslant bA\epsilon$ whenever $q \geqslant 1$. Using also that $G_1(\boldsymbol{\lambda}) \leqslant bA\epsilon + O(\epsilon^3)$ and $G_0(\boldsymbol{\lambda}) \geqslant 1$, the first term in the square bracket in (6.48) is at most $(bA\epsilon)^{|q-\ell|}(1 + O(\epsilon^2))$ while the second is order $\epsilon^{q+\ell} = \epsilon^{|q-\ell|}O(\epsilon^2)$. This proves the bound on the third line in (6.46).

The first and second lines in (6.46) require explicit treatment of the leading-order term contributing to $\widetilde{G}_{n-\ell}(\boldsymbol{\lambda})$. This is easy for $n = \ell$ where we only need $\widetilde{G}_0(\boldsymbol{\lambda}) \leqslant 1 + O(\epsilon^2)$, which is proved from (3.65). This, along with the aforementioned estimates, bounds the square bracket in (6.48) by the maximum of $1 + O(\epsilon^2)$ and $4(bA\epsilon)^2$. For the first line (i.e., $|q-\ell| = 1$) we in turn need

$$\widetilde{G}_1(\boldsymbol{\lambda}) \leqslant \big(b - 1 + O(\epsilon^2)\big)\lambda(1), \tag{6.49}$$

which is proved from (3.66). This dominates the square bracket in (6.48) by $(b-1)\lambda(1) + O(\epsilon^2)$, thus showing that $|\partial_\ell F_q(\boldsymbol{\lambda})| \leqslant b(b-1)\theta^{q^2}\lambda(1) + O(\epsilon^2)$.

Unlike the previous cases, both terms in the square bracket in (6.48) will contribute to the alternative $\ell = q = 1$ in (6.46). This results in potential cancellations that force us to extract terms of order up to ϵ^2 explicitly. Since the first term in the square bracket in (6.48) is of order unity while the second term is of order ϵ^2, it suffices to derive an upper bound on $\partial_\ell F_q(\boldsymbol{\lambda})$. Here bounding the first term requires the upper bounds

$$\widetilde{G}_0(\boldsymbol{\lambda}) \leqslant 1 + (b-1)(b-2)\lambda(1)^2 + O(\epsilon^4) \tag{6.50}$$

and

$$\begin{aligned}\widetilde{G}_2(\boldsymbol{\lambda}) &\leqslant (b-1)\lambda(2) + \binom{b-1}{2}\lambda(1)^2 + O(\epsilon^4)\\ &\leqslant \left(\frac{b-1}{2}\Bigl[b-2+\frac{b-1}{b^3-1}\Bigr] + \delta''\right)\lambda(1)^2 + O(\epsilon^4),\end{aligned}\tag{6.51}$$

where δ'' is a quantity of order of $\delta' + \delta$. Since we are tracking terms up to order ϵ^2, bounding $G_0(\boldsymbol{\lambda}) \geqslant 1$ is not sufficient; instead we need

$$G_0(\boldsymbol{\lambda}) \geqslant 1 + b(b-1)\lambda(1)^2.\tag{6.52}$$

In the second term we in turn need the lower bounds

$$\begin{aligned}G_1(\boldsymbol{\lambda}) &\geqslant b\lambda(1)\\ \widetilde{G}_1(\boldsymbol{\lambda}) &\geqslant (b-1)\lambda(1)\end{aligned}\tag{6.53}$$

along with the upper bound

$$G_0(\boldsymbol{\lambda}) \leqslant 1 + O(\epsilon^2).\tag{6.54}$$

Putting these together we get

$$\begin{aligned}\partial_1 F_1(\boldsymbol{\lambda}) &\leqslant \frac{1 + [(b-1)(b-2) + \frac{b-1}{2}[b-2+\frac{b-1}{b^3-1}] + \delta'']\lambda(1)^2 + O(\epsilon^4)}{[1+b(b-1)\lambda(1)^2]^2} - \frac{b(b-1)\lambda(1)^2}{1+O(\epsilon^2)}\\ &= 1 + \left((b-1)(b-2) + \frac{b-1}{2}\Bigl[b-2+\frac{b-1}{b^3-1}\Bigr] + \delta'' - 3b(b-1)\right)\lambda(1)^2 + O(\epsilon^4)\\ &= 1 - \left(\frac{b-1}{2}\Bigl(3b+6-\frac{b-1}{b^3-1}\Bigr) - \delta''\right)\lambda(1)^2 + O(\epsilon^4).\end{aligned}\tag{6.55}$$

Using $\frac{b-1}{b^3-1} \leqslant 1$ and $O(\epsilon^4) = \lambda(1)^2 O(\epsilon^2)$ by $\boldsymbol{\lambda} \in \Sigma_0$, this is at most $1 - b(b-1)\lambda(1)^2$ once δ, δ' and ϵ are sufficiently small. □

We now use these to prove:

Lemma 6.13 *Let Σ_0 be defined using $A > \sqrt{\frac{2}{b-1}}$ and $\delta \in (0, \delta_0)$, for δ_0 as in Lemma 6.12. Given $t > 0$ and $\beta > 0$ with $0 < b\theta - 1 < (tbA^2)^{-1}$, we have*

$$\varrho(\boldsymbol{\lambda}, \boldsymbol{\lambda}') := \sum_{q\geqslant 1} (tbA)^{q-1}(b\theta - 1)^{\frac{q-1}{2}}\bigl|\lambda(q) - \lambda'(q)\bigr| < \infty\tag{6.56}$$

for all $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Sigma_0$. Moreover, if t obeys

$$\sqrt{2}\,\frac{\sqrt{1-b^{-3}}}{(1-b^{-2})\sqrt{1+b^{-1}}}\, b^2 > tA\tag{6.57}$$

and

$$\sup_{\ell\geqslant 2}\left(b^{1-\ell^2} + \sum_{q=1}^{\ell-1} t^{q-\ell} b^{1-q^2}\right) < 1,\tag{6.58}$$

then there exist $\delta_1 > 0$, $\epsilon_1 > 0$ and $\eta > 0$ such that

$$\varrho\bigl(F(\boldsymbol{\lambda}), F(\boldsymbol{\lambda}')\bigr) \leqslant \bigl[1 - \eta(b\theta - 1)\bigr]\varrho(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\tag{6.59}$$

holds for all $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Sigma_0$ provided $\delta < \delta_1$ and $b\theta - 1 \leqslant \epsilon_1$.

Proof. Let us again abbreviate $\epsilon := \sqrt{b\theta - 1}$ and let $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Sigma_0$. The conditions defining Σ_0 then imply $|\lambda(q) - \lambda'(q)| \leqslant 2(A\epsilon)^{|q|}$ and so the series in (6.56) converges whenever $tbA^2\epsilon^2 < 1$. Next recall that, by Lemma 6.11, the convex combination $\boldsymbol{\lambda}_u := (1-u)\boldsymbol{\lambda} + u\boldsymbol{\lambda}'$ lies in Σ_0 for all $u \in [0,1]$. Moreover, elementary calculus shows

$$\begin{aligned} \big|F_q(\boldsymbol{\lambda}') - F_q(\boldsymbol{\lambda})\big| &= \left|\int_0^1 \sum_{\ell \geqslant 1} \partial_\ell F_q(\boldsymbol{\lambda}_u)\big(\lambda'(\ell) - \lambda(\ell)\big)\mathrm{d}u\right| \\ &\leqslant \int_0^1 \bigg(\sum_{\ell \geqslant 1} \big|\partial_\ell F_q(\boldsymbol{\lambda}_u)\big|\big|\lambda'(\ell) - \lambda(\ell)\big|\bigg)\mathrm{d}u \end{aligned} \tag{6.60}$$

where the second line follows by the triangle inequality.

Multiplying (6.60) by $(tA\epsilon)^{q-1}$ and summing over $q \geqslant 1$ we find out that, in order to prove (6.59), it suffices to show that for all $\ell \geqslant 1$ and all $\boldsymbol{\lambda} \in \Sigma_0$,

$$\sum_{q \geqslant 1} (tbA\epsilon)^{q-1}\big|\partial_\ell F_q(\boldsymbol{\lambda})\big| \leqslant (1 - \eta\epsilon^2)(tbA\epsilon)^{\ell-1}. \tag{6.61}$$

Starting first with the cases $\ell \geqslant 2$, here we plug in the second and third line in (6.46) with the result

$$\begin{aligned} &\sum_{q \geqslant 1} (tbA\epsilon)^{q-1}\big|\partial_\ell F_q(\boldsymbol{\lambda})\big| \\ &\quad \leqslant \bigg(b\theta^{\ell^2} + \sum_{q=1}^{\ell-1} bt^{q-\ell}\theta^{q^2} + \sum_{q > \ell+1} b\big(\sqrt{t}\, bA\epsilon\big)^{2(q-\ell)}\theta^{q^2} + O(\epsilon^2)\bigg)(tbA\epsilon)^{\ell-1}, \end{aligned} \tag{6.62}$$

where the $O(\epsilon^2)$ term collects the contribution of $O(\epsilon^2)$-terms in (6.46). Now observe that, in the limit as $\epsilon \downarrow 0$, the term in the large parenthesis is bounded by the supremum in (6.58), proving (6.61) for $\ell \geqslant 2$ once ϵ is small enough.

For $\ell = 1$ we in turn invoke the first and last line in (6.46) to the leading order terms and bound the rest using the third line with the result

$$\begin{aligned} \sum_{q \geqslant 1} (tbA\epsilon)^{n-1}\big|\partial_1 F_q(\boldsymbol{\lambda})\big| &\leqslant 1 - b(b-1)\lambda(1)^2 \\ &\quad + \big(b(b-1) + O(\epsilon^2)\big)\theta^4\lambda(1)(tbA\epsilon) + \sum_{q \geqslant 3}\big(\sqrt{t}\, bA\epsilon\big)^{2(q-\ell)}\theta^{q^2}. \end{aligned} \tag{6.63}$$

Invoking the upper and lower bounds on $\lambda(1)$, the right-hand side is bounded by

$$1 - \left(\frac{b(b-1)}{\frac{1}{2}\frac{(b-1)^2(b+1)^3}{b^3-1} + \delta} - tbA\theta^4 \frac{b(b-1)}{\sqrt{\frac{1}{2}\frac{(b-1)^2(b+1)^3}{b^3-1} - \delta}}\right)\epsilon^2 + O(\epsilon^3). \tag{6.64}$$

We now check that the term in the large parentheses will be positive for θ close to $1/b$ and δ sufficiently small if (6.57) holds. For small-enough ϵ, the term then dominates the expansion in powers of ϵ which validates (6.61) for $\ell = 1$. $\square$

6.6 Convergence proofs.

We are now finally in a position to address the proofs of Theorems 6.1 and 3.4. One last technical hurdle to get out of the way is the choice of parameters t and A:

Lemma 6.14 *For all $b \geqslant 2$, the inequalities (6.57–6.58) are true when*

$$tA \leqslant 4\sqrt{b} \quad \text{and} \quad t \geqslant \frac{3}{2}. \tag{6.65}$$

Proof. We start by proving that

$$\inf_{\substack{b\in\mathbb{N}\\ b\geqslant 2}} \sqrt{2}\,\frac{\sqrt{1-b^{-3}}}{(1-b^{-2})\sqrt{1+b^{-1}}}\,b^{3/2} > 4. \tag{6.66}$$

Indeed, for $b \geqslant 3$ we invoke $\sqrt{1-b^{-3}} > \sqrt{1-b^{-2}}$ and $\sqrt{1+b^{-1}} \leqslant \sqrt{2}$ to dominate the expression by $b^{3/2}$ from below. Since $b^{3/2} \geqslant 3^{3/2} \geqslant 5$ for $b \geqslant 3$, we are down to $b = 2$. Here we calculate the expression explicitly to be $\frac{8\sqrt{7}}{3\sqrt{3}}$ which is above 4, albeit just barely. It follows that (6.57) holds if $tA \leqslant 4\sqrt{b}$.

As for the second condition, for $\ell = 2$ we need that $2^{-8} + t^{-1} < 1$ which is true whenever $t \geqslant 8/7$. For $\ell \geqslant 3$ we bound the expression by $b^{-8} + t^{-1} + b^{-3}\sum_{\ell\geqslant 3} t^{1-\ell}$ and so we need

$$t^{-1} + \frac{1}{8}\frac{t^{-2}}{1-t^{-1}} < 1 - 2^{-8} \tag{6.67}$$

The left-hand side is decreasing in t and, at $t := 3/2$, equals $5/6$ which is indeed less than the right-hand side. Hence (6.58) holds for all $t \geqslant 3/2$. □

We are now ready for:

Proof of Theorem 6.1. We assume throughout that $b\theta - 1$ is positive and small enough so that the statements of above lemmas apply. As to the choice of t and A, relying on Lemma 6.14, we set $t := 3/2$ and put $A := \frac{8}{3}\sqrt{b}$. Notice that this enables Lemma 6.9 as well as other claims where a bound on A appeared. Also note that $tA = 4\sqrt{b}$ so ϱ from (6.56) coincides with the expression in (6.6).

Next observe that ϱ is a metric on Σ_0 and, relying on product topology and completeness of the space of probability measures on $[0,1)$, that (Σ_0, ϱ) is complete. By Lemma 6.13, F is a strict contraction on Σ_0. Using Lemma 6.9 along with the Banach contraction principle, iterations of F on any $\boldsymbol{\lambda} \in \Sigma'$ thus converge to some $\boldsymbol{\lambda}_\star \in \Sigma_0$ which is then also a unique fixed point of F in Σ'.

Let us now consider a sequence $\{\boldsymbol{\lambda}_k\}_{k=0}^n$ obtained by $\boldsymbol{\lambda}_k := F^{(k)}(\boldsymbol{\lambda}_{k-1})$ starting from some $\boldsymbol{\lambda}_0 \in \Sigma'$. In order to control the approach of this sequence to $\boldsymbol{\lambda}_\star$, we need to compare the action of F and $F^{(k)}$. For this we first note that, for all $\boldsymbol{\lambda} \in \Sigma_0$ and all $q \in \mathbb{Z}$,

$$\begin{aligned} \bigl|F_q^{(k)}(\boldsymbol{\lambda}) - F_q(\boldsymbol{\lambda})\bigr| &= \frac{G_q(\boldsymbol{\lambda})}{G_0(\boldsymbol{\lambda})}\bigl|\theta^{q^2\sigma_k^2} - \theta^{q^2}\bigr| \\ &\leqslant (bA\epsilon)^q \theta^{q^2\min\{1,\sigma_k^2\}} \log(1/\theta)|\sigma_k^2 - 1|, \end{aligned} \tag{6.68}$$

where $\epsilon := \sqrt{b\theta - 1}$. Assuming $\sigma_k^2 \geqslant 1/2$ and noting that $A \leqslant 4b^{3/2}$, we thus get

$$\begin{aligned}\varrho\bigl(F^{(k)}(\boldsymbol{\lambda}), F(\boldsymbol{\lambda})\bigr) &= \sum_{q\geqslant 1}(4b^{3/2}\epsilon)^{q-1}\bigl|F_n^{(k)}(\boldsymbol{\lambda}) - F_n(\boldsymbol{\lambda})\bigr| \\ &\leqslant \biggl(\sum_{q\geqslant 1}(4b^{3/2}\epsilon)^{2q-1}\theta^{q^2/2}\log(1/\theta)\biggr)|\sigma_k^2 - 1|.\end{aligned} \tag{6.69}$$

Write $\widehat{C}$ for the expression in the parenthesis and recall the sequence $\{\mathfrak{d}_k\}_{k\geqslant 0}$ from Assumption 1.1. Abbreviate $\mathfrak{d}'_k := \mathfrak{d}_{\min\{k,n-k\}}$. For all $\boldsymbol{\lambda} \in \Sigma_0$ the triangle inequality along with (6.59) and (6.69) show

$$\begin{aligned}\varrho\bigl(F^{(k)}(\boldsymbol{\lambda}), \boldsymbol{\lambda}_\star\bigr) &\leqslant \varrho\bigl(F^{(k)}(\boldsymbol{\lambda}), F(\boldsymbol{\lambda})\bigr) + \varrho\bigl(F(\boldsymbol{\lambda}), F(\boldsymbol{\lambda}_\star)\bigr) \\ &\leqslant \widehat{C}\,\mathfrak{d}'_k + (1 - \eta\epsilon^2)\varrho(\boldsymbol{\lambda}, \boldsymbol{\lambda}_\star).\end{aligned} \tag{6.70}$$

Using this for $\boldsymbol{\lambda} := \boldsymbol{\lambda}_{k-1}$ yields

$$\varrho(\boldsymbol{\lambda}_k, \boldsymbol{\lambda}_\star) \leqslant \widehat{C}\,\mathfrak{d}'_k + (1 - \eta\epsilon^2)\varrho(\boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_\star). \tag{6.71}$$

whenever k is such that $\boldsymbol{\lambda}_k \in \Sigma_0$ and $\sigma_k^2 \geqslant 1/2$.

To finish the proof, consider the family of $\{\sigma_k^2\}_{k=0}^n$ conforming to Assumption 1.1 with sequence $\{\mathfrak{d}_k\}_{k\geqslant 1}$. Denote $\operatorname{diam}(\Sigma_0) := \sup\{\varrho(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\colon \boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Sigma_0\}$ and, for $n \geqslant 1$, let

$$k_0 := 1 + \sup_{n\geqslant 1}\max\bigl\{k \leqslant n/2\colon \{\boldsymbol{\lambda}_k, \boldsymbol{\lambda}_{n-k}\} \nsubseteq \Sigma_0 \vee \mathfrak{d}'_k > 1/2\bigr\}, \tag{6.72}$$

where the maximum is set to be $k_0 := 0$ if the set is empty. The above lemmas show that $k_0 < \infty$ for each $\boldsymbol{\lambda}_0 \in \Sigma'$. Since $\boldsymbol{\lambda}_{k_0} \in \Sigma_0$, iterations of (6.71) then show

$$\varrho(\boldsymbol{\lambda}_k, \boldsymbol{\lambda}_\star) \leqslant \widehat{C}\sum_{j=0}^{k-k_0-1}(1 - \eta\epsilon^2)^j\mathfrak{d}'_{k-j} + (1 - \eta\epsilon^2)^{k-k_0}\operatorname{diam}(\Sigma_0) \tag{6.73}$$

whenever $\min\{k, n-k\} \geqslant k_0$. Now set $C := \max\{\widehat{C}, \operatorname{diam}(\Sigma_0)\}$, write $\mathrm{e}^{-\eta}$ instead of $1 - \eta\epsilon^2$ and extend the range of the sum to all $j \leqslant k$. □

With this we now quickly finish also:

Proof of Theorem 3.6. Let $\epsilon > 0$ be such that Theorem 6.1 applies. This yields the existence of $\boldsymbol{\lambda}_\star$ which obeys (3.30) by extension of the bounds from Lemma 6.7. The bound (3.32) in turn follows from (6.6) by retaining only the term corresponding to n in the sum on the left and redefining C correspondingly.

Let $\tilde{v}_\star$ be as defined in (5.108). For the convergence of v_k and its derivative, we need the uniform bound

$$\left|\frac{\mathrm{e}^{-v_k(z)}}{a_k(0)} - \mathrm{e}^{-\tilde{v}_\star(z)}\right| \leqslant \sum_{q\in\mathbb{Z}}\bigl|\lambda_k(q) - \lambda_\star(q)\bigr|. \tag{6.74}$$

The bounds (3.30) and (3.32) then show that the sum on the right tends to zero as $\min\{k, n-k\} \to \infty$. Under the additional assumption that $\{\mathfrak{d}_k\}_{k\geqslant 0}$ decays exponentially we can unite the estimates (3.31–3.32) as

$$\bigl|\lambda_k(q) - \lambda_\star(q)\bigr| \leqslant C'\mathrm{e}^{-\eta'\max\{k,|q|\}}. \tag{6.75}$$

This now readily shows that the sum on the right of (6.74) decays exponentially with k, proving

$$\sup_{z\in\mathbb{R}}\bigl|v_k(z)+\log a_k(0)-\tilde v_\star(z)\bigr|\leqslant C'\mathrm{e}^{-\eta'\min\{k,n-k\}}. \tag{6.76}$$

To derive (3.34) from this, note that a simple telescoping argument gives

$$\begin{aligned}\Biggl|\frac{a_k(0)}{a_{k-1}(0)^b}-\sum_{\substack{q_1,\dots,q_b\in\mathbb{Z}\\ q_1+\dots+q_b=0}}\prod_{i=1}^b\lambda_\star(q_i)\Biggr|&\\ \leqslant\sum_{i=1}^b\Bigl(\sum_{q\in\mathbb{Z}}\lambda_{k-1}(q)\Bigr)^{i-1}&\Bigl(\sum_{q\in\mathbb{Z}}\bigl|\lambda_k(q)-\lambda_\star(q)\bigr|\Bigr)\Bigl(\sum_{q\in\mathbb{Z}}\lambda_\star(q)\Bigr)^{b-i}.\end{aligned} \tag{6.77}$$

The right hand side is now bounded by $C''\mathrm{e}^{-\eta'\min\{k,n-k\}}$. Since $v_\star(z)-bv_\star(z')$ differs from $\tilde v_\star(z)-b\tilde v_\star(z')$ by the logarithm of the giant sum on the left, (3.34) follows by combining the previous two estimates.

To extend the convergence to the derivatives, we note that $v_\star-\tilde v_\star$ differ by a constant and so $v_\star'=\tilde v_\star'$. Here we get

$$v_k'(z)=\mathrm{e}^{v_k(z)}a_k(0)\sum_{q\in\mathbb{Z}}\lambda_k(q)(2\pi\mathrm{i}q)\mathrm{e}^{2\pi\mathrm{i}qz} \tag{6.78}$$

and

$$v_\star'(z)=\mathrm{e}^{\tilde v_\star(z)}\sum_{q\in\mathbb{Z}}\lambda_\star(q)(2\pi\mathrm{i}q)\mathrm{e}^{2\pi\mathrm{i}qz} \tag{6.79}$$

This implies

$$\begin{aligned}\bigl|v_k'(z)-v_\star'(z)\bigr|\leqslant \mathrm{e}^{v_k(z)}a_k(0)\mathrm{e}^{\tilde v_\star(z)}&\biggl|\frac{\mathrm{e}^{-v_k(z)}}{a_k(0)}-\mathrm{e}^{-\tilde v_\star(z)}\biggr|\sum_{q\in\mathbb{Z}}\lambda_k(q)2\pi|q|\\ &+\mathrm{e}^{\tilde v_\star(z)}\sum_{q\in\mathbb{Z}}\bigl|\lambda_k(q)-\lambda_\star(q)\bigr|(2\pi|q|)\end{aligned} \tag{6.80}$$

Since (3.31) implies that $\mathrm{e}^{-\tilde v_k(z)}/a_k(0)\geqslant 1/2$ while (3.30) gives $\mathrm{e}^{-\tilde v_\star(z)}\geqslant 1/2$ once $b\theta-1$ is sufficiently small while $\sum_{q\in\mathbb{Z}}\lambda_k(q)|q|$ is bounded uniformly in k, both terms on the right decay to zero as $\min\{k,n-k\}\to\infty$. The decay is exponentially fast if $\{\mathfrak{d}_j\}_{j\geqslant 0}$ decays exponentially. This yields (3.35) as desired. □

Remark 6.15 The above proofs are tailored for the near-critical regime, meaning with b fixed and $b\theta-1$ positive but small. Another interesting asymptotic regime which can be analyzed is that of large b. Focussing for simplicity on b even and $\theta_k=\theta$, here Lemma 6.2 shows that $\lambda_k(q)\leqslant\theta^{q^2}$ for all $k\geqslant 0$ and all $q\in\mathbb{Z}$. As θ is close to $1/b$, this suggests introduction of the scaled variables

$$\lambda_k'(q):=b^{q^2}\lambda_k(q). \tag{6.81}$$

In terms of these, the iterations (6.4) take the form

$$\lambda_k'(1)=b\theta\frac{\sum_{\ell\geqslant 0}\frac{1}{\ell!(\ell+1)!}\lambda_{k-1}'(1)^{2\ell+1}+O(1/b)}{1+2\sum_{\ell\geqslant 0}\frac{1}{\ell!\ell!}\lambda_{k-1}'(1)^{2\ell}+O(1/b)} \tag{6.82}$$

and

$$\lambda_k'(q) = (b\theta)^{q^2} \frac{\sum_{\ell\geqslant 0} \frac{1}{\ell!(\ell+|q|)!}\lambda_{k-1}'(1)^{2\ell+|q|} + O(1/b)}{1 + 2\sum_{\ell\geqslant 0} \frac{1}{\ell!\ell!}\lambda_{k-1}'(1)^{2\ell} + O(1/b)} \tag{6.83}$$

for $|q| \geqslant 2$. Using these one can show that $\lambda_k'(1)$ converges to a positive quantity characterized, modulo errors that vanish as $b \to \infty$, as a fix point of the ratio of two modified Bessel functions. The other reduced variables are then simply computed from the limit version of (6.83).

We have in fact carried our initial proof in this framework except that, in order to overcome the non-linearity of the right-hand side (6.82), we ultimately also had to assume that $b\theta$ is close to 1. However, we expect that with increasing b large, one should be able to control larger and larger intervals of $b\theta$.

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