RESEARCH STATEMENT OF MAREK BISKUP

Here is the list of of what I presently regard as my five best papers (labeled [A-E] below). Further work is described in my Prague summer school notes [9] or can be found in my publication list.

[A] N. Berger and M. Biskup, *Quenched invariance principle for simple random walk on percolation clusters*, Prob. Theory Rel. Fields **137** (2007), no. 1-2, 83–120.

The main result is the proof that the simple random walk on a.e. supercritical percolation cluster scales to Brownian motion under the usual diffusive scaling of space and time.

Sidoravicius and Sznitman [47] previously proved this in dimensions $d \geq 4$ by comparing the quenched and annealed path distributions; their estimates used the fact that two random walk paths in high-enough dimensions are unlikely to meet at a large number of points. Much work has been done in recent years on estimates of the heat kernel (Mathieu-Remy [35], Barlow [4]).

The approach of [A] is different from that of Sidoravicius and Sznitman and is based on the notion of *harmonic deformation* of the infinite cluster. This is an embedding of the percolation graph on which the random walk is a martingale. The construction of the deformation invokes the so called *corrector*, which has been a standard tool in homogenization theory, and it follows the line of e.g. Kipnis and Varadhan's paper [31]. The hard part is the proof that the deformation grows sublinearly with the distance which is needed in order to show that the paths of the walk on the natural embedding and the harmonic embedding stay sufficiently close to each other. This is achieved by combining facts about percolation (comparison of the graph distance and the Euclidean distance, a'la Antal-Pisztora [3]) and ergodic theory. The proof in $d = 2$ does not need anything beyond that; in $d \geq 3$ we also need rather sophisticated heat-kernel estimates proved recently by Barlow [4].

An independent proof of this result was simultaneously given by Mathieu-Piatnitski [34] who employ more traditional tools of homogenization theory.

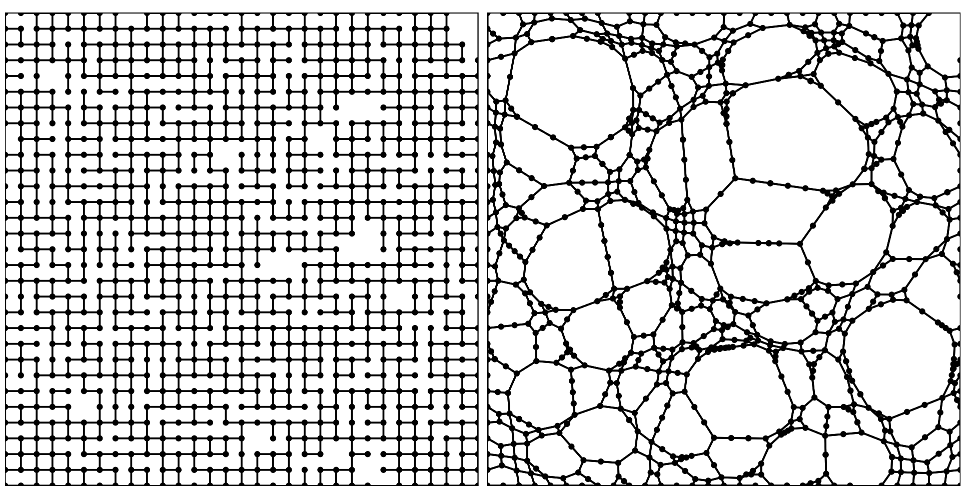

FIGURE 1. The percolation cluster for $p = 0.7$ in a 30×30 box, and its harmonic deformation. The random walk on the deformed graph is a martingale.

The novel contribution of this work is the use of geometric methods as opposed to homogenization which has been dominating this area in the past. This manifests itself both in the interpretation of the corrector as a harmonic deformation as well as in the estimates of the corrector growth, which are done in purely geometric terms. The harmonic embedding is a subject of its own interest and is currently studied by a number of people. Further interesting phenomena occur for the random conductance model (Berger-Biskup-Hoffman-Kozma [8]).

[B] M. Biskup, L. Chayes and R. Kotecký, *Critical region for droplet formation in the two-dimensional Ising model*, Commun. Math. Phys. **242** (2003), no. 1-2, 137-183.

The main result is that, for 2D Ising model in a box of side L, temperature below critical and magnetization smaller than equilibrium value by a constant times $L^{4/3}$, a droplet of the opposite phase appears "discontinuously" once the constant exceeds a certain critical value.

This work can be regarded as the final dot after the long program of "2D Wulff construction." This program started with the work of Alexander-Chayes-Chayes [2] and Dobrushin-Kotecký-Shlosman [22], and continued via Dobrushin-Shlosman [23], Pfister [41], Pfister-Velenik [42], Ioffe [28] to Ioffe-Schonmann [29]. These works covered deviations of the magnetization from the equilibrium value up to $L^{4/3+\epsilon}$ for all subcritical temperatures.

It turns out that the case of deviations proportional to $L^{4/3}$ is the most interesting. The phenomenon that occurs for such cases is that only part of the magnetization goes into a droplet and the rest dissolves in background fluctuations. If the excess is smaller than some critical value, all of it dissolves. The droplet thus "appears" discontinuously.

The hardest part is the proof that if there there is at most one droplet larger than a suitable power of $\log L$. This is complicated because if there is a droplet, it will be mesoscopic at best, and because we do not currently have control of the large deviations of droplet shape beyond the leading order provided by the skeleton calculus. The help comes from the Gaussian estimates on the background fluctuations which can be pushed to a local-CLT level (i.e., order $O(1)$ multiplicative terms).

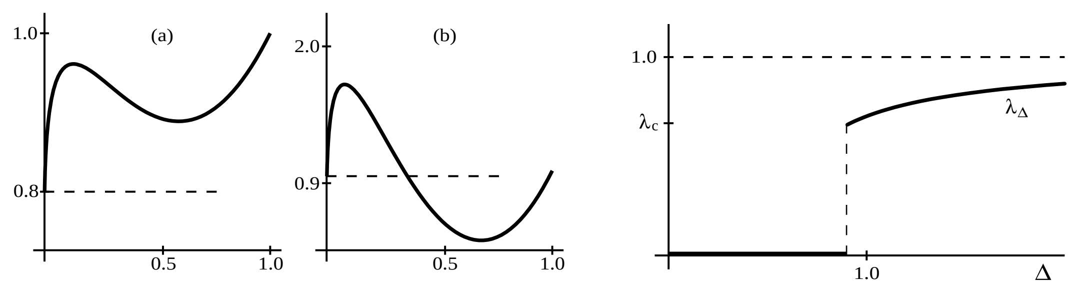

FIGURE 2. Left: The large-deviation rate function $\Phi_\Delta(\lambda)$ for the configurations with a droplet containing λ-fraction of the excess magnetization, where Δ is a dimensionless parameter encompassing all details of the system. Right: The plot of the global minimizer λ_Δ of the rate function $\lambda \mapsto \Phi_\Delta(\lambda)$ as Δ varies through $(0,\infty)$. The jump to $\lambda_c = {}^2\!/_3$ occurs at $\Delta_c = \frac{1}{2}({}^3\!/_2)^{3/2}$.

The novel contribution of this work is primarily the discovery of the discontinuous droplet-formation phenomenon, supported by its fully rigorous proof in the case of the 2D Ising model.

Physicists and simulation people seem to be particularly excited about the "universal formulation" of the result (Biskup-Chayes-Kotecký [13])—e.g., the fraction of excess going to the droplet should be independent of the model—and have tested various predictions numerically with great precision (Nussbaumer et al [39]) in relatively small systems (L of order 10^3). Their fascinating conclusion is that the findings in $L^{4/3}$-regime dominate the finite-size scaling picture.

[C] M. Biskup, *On the scaling of the chemical distance in long range percolation models*, Ann. Probab. **32** (2004), no. 4, 2938-2977.

The main result is that, for long range percolation on $\mathbb{Z}^d$ where x and y get connected by an edge with probability proportional to $|x-y|^{-s}$ with $d < s < 2d$, the graph distance for sites at Euclidean distance N grows like $(\log N)^{\Delta+o(1)}$ as $N \to \infty$, where $\Delta^{-1} = \log_2(\frac{2d}{s})$.

The question about graph distance growth arose from a paper of Benjamini-Berger [5] who came to it in the discussion of the role of geometry for small-world phenomena. Long-range percolation is a natural model in this respect; and a lot is known about the existence of the infinite connected component (Schulman [46], Newman-Schulman [38], Aizenman-Newman [1], Berger [7]). Concerning the graph-distance asymptotic, the regime $s \le d$ was addressed in previous work of Benjamini-Kesten-Peres-Schramm [6] ($s < d$) and Coppersmith-Gamarnik-Sviridenko [21] ($s = d$). For $d < s < 2d$, Benjamini-Berger could prove polylogarithmic bounds on the growth but the precise asymptotic was open.

It turns out the edges in path of close to optimal length can be organized into a hierarchical structure: There is one edge of length N, two edges of length $N^{\frac{s}{2d}}$, four edges of length $N^{(\frac{s}{2d})^2}$, etc. This provides a fairly easy upper bound; the hard part is the proof that this strategy is more or less optimal. This is achieved by controlling all hierarchical path structures of this form that can be found in the vicinity of the desired endpoints.

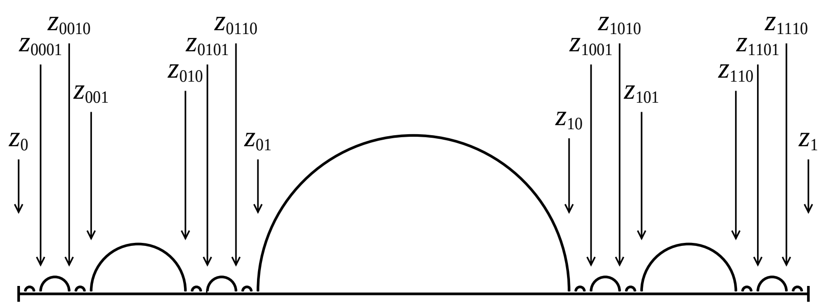

FIGURE 3. A schematic picture of the shortest path connecting points at distance N. Most of the distance is covered by a single edge $(z_{01}, z_{10}$ of length about N. The parts to the endpoints of this edge are covered by edges (z_{001}, z_{010}) and (z_{101}, z_{110}) of length about N^γ, where $\gamma = \frac{s}{2d}$; these are in turn connected by four edges of length N^{γ^2}, etc. The hierarchy needs to be developed for about $\log\log N$ generations to arrive at the desired conclusions.

The principal contribution of this work is the development of a multiscale approach to this problem which works under the sole condition that there exists a percolation cluster. The ideas obtained thereby may prove useful in studying random walk on such clusters, and for the percolation theory in its own right.

[D] M. Biskup and L. Chayes, *Rigorous analysis of discontinuous phase transitions via mean-field bounds*, Commun. Math. Phys. **238** (2003), no. 1-2, 53-93.

The main result is that any ferromagnetic spin model with the interaction Hamiltonian of the form $H = -\sum_{\langle x,y\rangle} S_x \cdot S_y$ undergoes a first-order phase transition in sufficiently high dimension, provided a corresponding transition occurs in the mean field theory.

Mean-field theory is one of the standard tools used by physicists; however, its rigorous justification has been for the most part missing. Mathematical results have been available for the convergence of the free energy and the magnetization in the $d \to \infty$ limit (Pearce-Thompson [40], Bricmont-Kesten-Lebowitz-Schonmann [18], Kesten-Schonmann [30]), but no conclusion could be drawn for phase transitions in finite dimensions. Exceptions are the Kac models with interactions smeared-out over large regions of the lattice (e.g., Cassandro-Pressutti [20], Bovier-Zahradník [16, 17], Cassandro-Ferrari-Merola-Presutti [19]) but, due to the large technical overhead, progress there has for the most part been limited to symmetric, Ising-like situations.

The principal tool of the above work is the infrared bound—going back to the work of Fröhlich-Simon-Spencer [26] and Dyson-Lieb-Simon [24]—combined with a number of convexity inequalities. The main technical result is a bound on the large-deviation rate function Φ_β for the magnetization on the complete graph—the ultimate playground of the mean-field theory—evaluated at the magnetization $m_\star$ *of an actual system* on $\mathbb{Z}^d$:

$$\Phi_\beta(m_\star) \le \inf_m \Phi_\beta(m) + C\beta\mathcal{I}_d$$

where C is a model-dependent constant and

$$\mathcal{I}_d = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{\hat{J}(k)^2}{1-\hat{J}(k)}$$

with $\hat{J}(k) = \frac{1}{d}\sum_{i=1}^d \cos(k_j)$. As $\mathcal{I}_d \to 0$ for $d \to \infty$, in $d \gg 1$ the physical magnetization is a *near minimizer* of the large-deviation rate function for the *mean-field* magnetization. This permits proofs of phase transitions in actual systems on $\mathbb{Z}^d$ by studying the associated mean-field theory—which is generally a manageable problem in multivariable calculus.

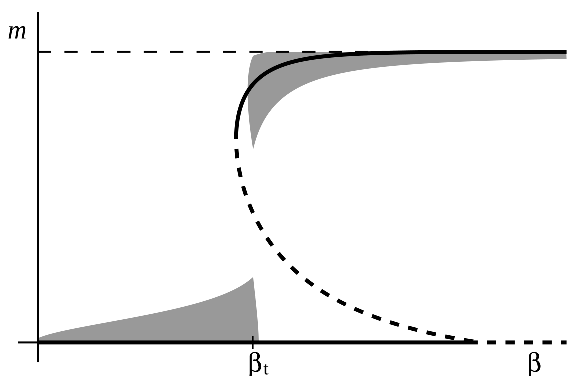

FIGURE 4. The plot of the extrema of the mean-field large-deviation rate function (solid lines) and the values allowed for the physical magnetization (shaded areas) by the above bounds in the Potts ferromagnet with $q = 15$ and $d \gg 1$. Forced to vary between two disconnected regions, the physical magnetization undergoes a discontinuity at some β_t.

The principal contribution of this work is the development of a rigorous connection between phase transitions in mean-field theory and those in actual lattice systems. This connection can be extended to all interactions for which one can prove the infrared bound (Biskup-Chayes-Crawford [12]). An ultimate product is a proof of first-order phase transition in systems in which other techniques failed so far; e.g., the 3-state Potts ferromagnet in $d \geq 3$. The proofs are fairly modest (a few pages compared to 40+ page proof by Gobron-Merola [27] of the corresponding theorem for the Kac-version of the Potts model).

[E] M. Biskup, C. Borgs, J.T. Chayes, L.J. Kleinwaks, R. Kotecký, *Partition function zeros at first-order phase transitions: A general analysis*, Commun. Math. Phys. **251** (2004) 79–131.

The loci of the zeros of partition functions with periodic boundary conditions are described in a class of models amenable to analysis by Pirogov-Sinai theory. The zeros are determined up to errors that are exponentially small in the linear size of the system.

One of the most celebrated results of lattice statistical mechanics is the Lee-Yang Circle Theorem [32] from 1951. The theorem states that the zeros of the partition function of the Ising model with complex external field h and periodic boundary conditions lie on the unit circle in the complex $z = \mathrm{e}^h$ plane. The appearance of phase transitions corresponds to zeros pinching the physical part of the z-plane in the thermodynamic limit. However, despite numerous Lee-Yang type results (e.g., Fisher [25], Ruelle [45], Newman [37], Lieb-Sokal [33], Nashimori-Griffiths [36]), the program of understanding phase transitions by means of the partition function zeros was never fulfilled for the lack of tools to control the loci of the zeros.

As was shown in [E], the situation improves drastically for models amenable to analysis by the Pirogov-Sinai theory (Pirogov-Sinai [43, 44], Zahradník [48]). The latter is an analytic version of the Peierls' argument and, when applicable, it offers full analytic control of most of the relevant quantities even under complexified parameters (Borgs-Imbrie [14], Borgs-Kotecký [15]).

The hard part of the analysis was to develop a corresponding machinery for computing the zeros while maintaining the local analyticity (which ensures that no extraneous roots appear, etc). This is difficult because the zeros occur exactly at the points where the limiting quantities (free energy, magnetization) fail to be analytic. A convenient tool here is Rouché's Theorem.

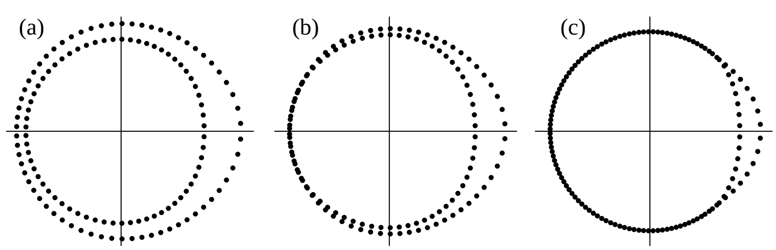

FIGURE 5. The partition function zeros for the Blume-Capel model on spins taking values ± 1 and 0, in an external field h distinguishing $+1$ from -1 and under a varying parameter λ that alters the balance between the 0's and ± 1's. The zeros generally lie on two non-circular closed curves in the $z = \mathrm{e}^h$ plane, but as λ varies to give more favor to ± 1's, part of them collapses onto the unit circle. Despite a perturbative approach, the non-degeneracy of the zeros forces them exactly on the unit circle once they fall sufficiently close to it.

The principal contribution of this work—and its follow-up [10]—is the development of finite-size scaling for the partition function, and its derivatives, deep in the complex plane. The work settled a controversy about the existence of zeros inside the unit circle for the Potts model in complex external field [11]. A "perturbative Lee-Yang" theorem was proved stating that, in systems with $h \to -h$ symmetry, the zeros that get too close to the unit circle necessary fall on it.

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