Lipschitz functions, proper colorings and cutsets in high dimensions

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Random surfaces

- **Surface:** $f:\Lambda \to \mathrm{R}$ or $f:\Lambda \to \mathrm{Z}$. Λ a box in Z^{d}.
- **Hamiltonian:**

$$\mathrm{H(f)} = \sum_{x\sim y} \mathrm{V}\big(\mathrm{f(x)} - \mathrm{f(y)}\big) \text{ for a potential } \mathrm{V}:\mathrm{R}\to\mathrm{R}.$$

$$\mathrm{DGFF}: \mathrm{V(x)} = \mathrm{x}^2.$$

- **Random surface:** sampled with probability (density) proportional to $e^{-\beta H(f)}$ with parameter $\beta > 0$ representing inverse temperature.
- **Boundary conditions:** zero on boundary, zero at one point, sloped boundary conditions, etc.

Properties of random surfaces

- Properties predicted to be universal. Independent of potential V under minor assumptions.
- When Λ is a box of side length n, have:
  $\sqrt{n}$ fluctuations in 1 dimensions.
  $\sqrt{\log n}$ fluctuations in 2 dimensions.
  Bounded fluctuations in ≥ 3 dimensions.
- When V is strictly convex, many universal properties established by a long list of authors.
- Some recent progress also for continuous, non-convex potentials (Adams, Biskup, Cotar, Deuschel, Kotecký, Müller, Spohn).

Integer-valued random surfaces

- When the surface takes integer values $f : \Lambda \to Z$, a new transition is expected: a two-dimensional roughening transition.
- Transition from previous logarithmic fluctuations to bounded fluctuations as the temperature decreases.
- Established only in 2 models! The integer-valued DGFF: $V(x) = x^2$ and the Solid-On-Solid model: $V(x) = |x|$ (Fröhlich and Spencer 81)

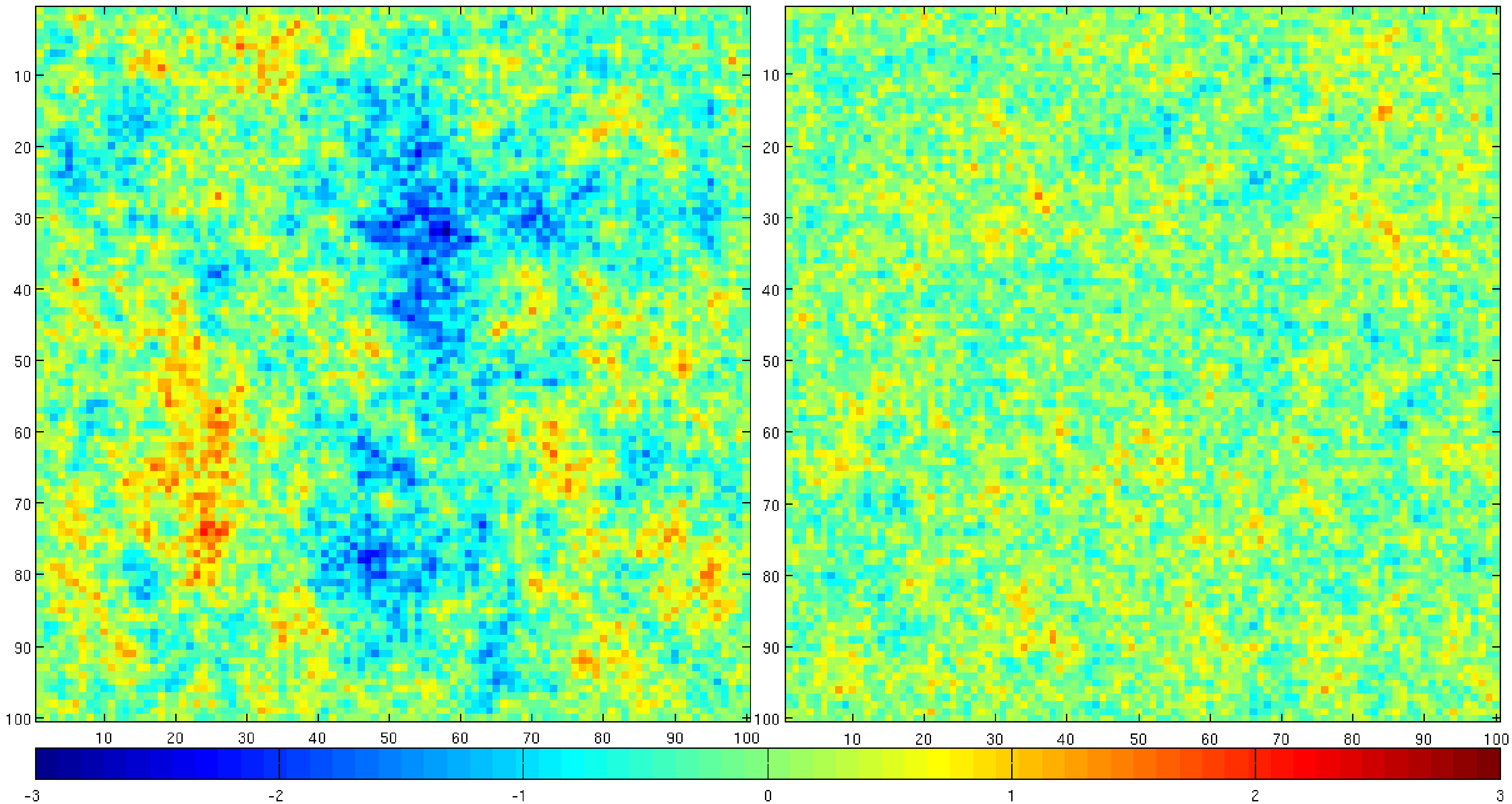
Random Lipschitz functions

- A random Lipschitz function is the case

$$V(x) = \begin{cases} 0 & -1 \le x \le 1 \\ \infty & \text{otherwise} \end{cases}$$

- The so-called hammock potential.
- Here, the parameter β is irrelevant and the function is a just a uniformly sampled Lipschitz function on Λ.
  Natural also from an analytic point of view.
- Analysis of this case is wide open for all $d \geq 2$!

A uniform Lipschitz function in 2 and 3 dimensions



Integer-valued Lipschitz functions

- A random M-Lipschitz function is an $f : \Lambda \to Z$ sampled with the potential

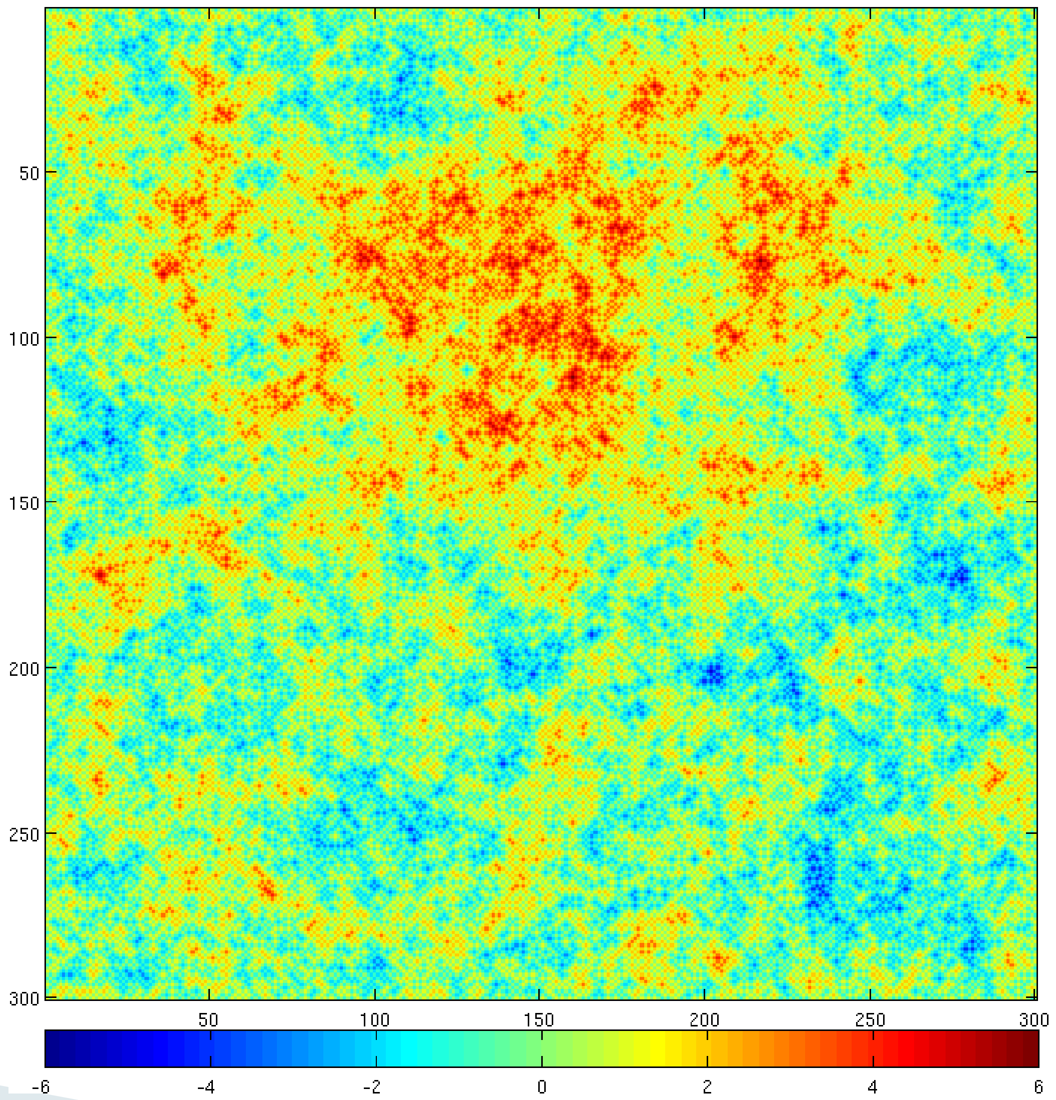
$$V(x) = \begin{cases} 0 & -M \le x \le M \text{ is an integer} \\ \infty & \text{otherwise} \end{cases}$$

- Almost no analysis available for these functions.
- A random (graph) homomorphism function is the case

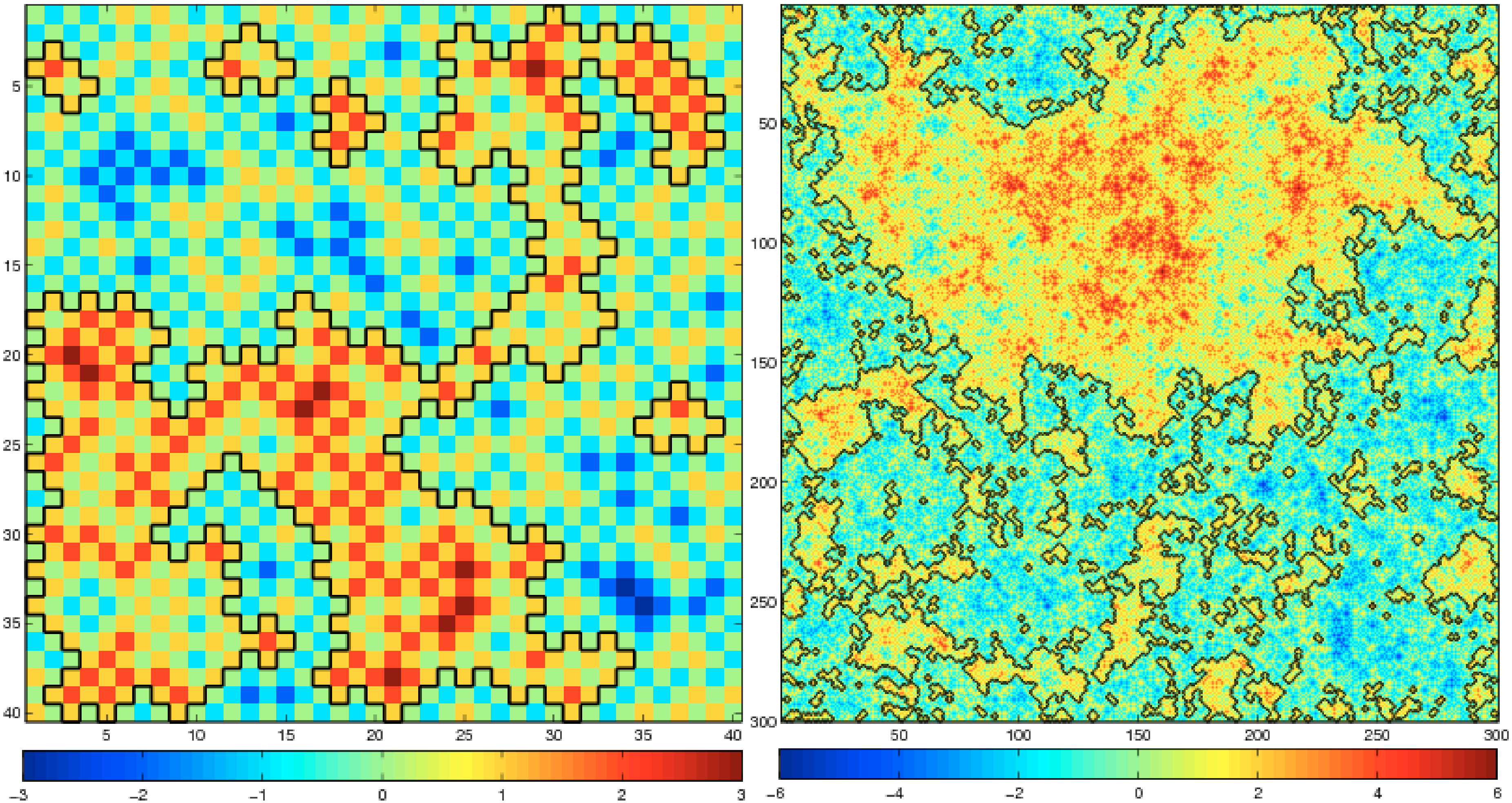
$$V(x) = \begin{cases} 0 & x \in \{-1, 1\} \\ \infty & \text{otherwise} \end{cases}$$

- Investigated on general graphs as a generalization of SRW.
  In Z^2, it is the height function for the square ice model.
- Benjamini,Yadin, Yehudayoff 07:lower bound $(\log n)^{1/d}$ on maximum.
- Galvin, Kahn 03-04: On hypercube $\{0,1\}^d$, takes ≤ 5 values with high probability as $d \to \infty$.

A 2-dimensional homomorphism

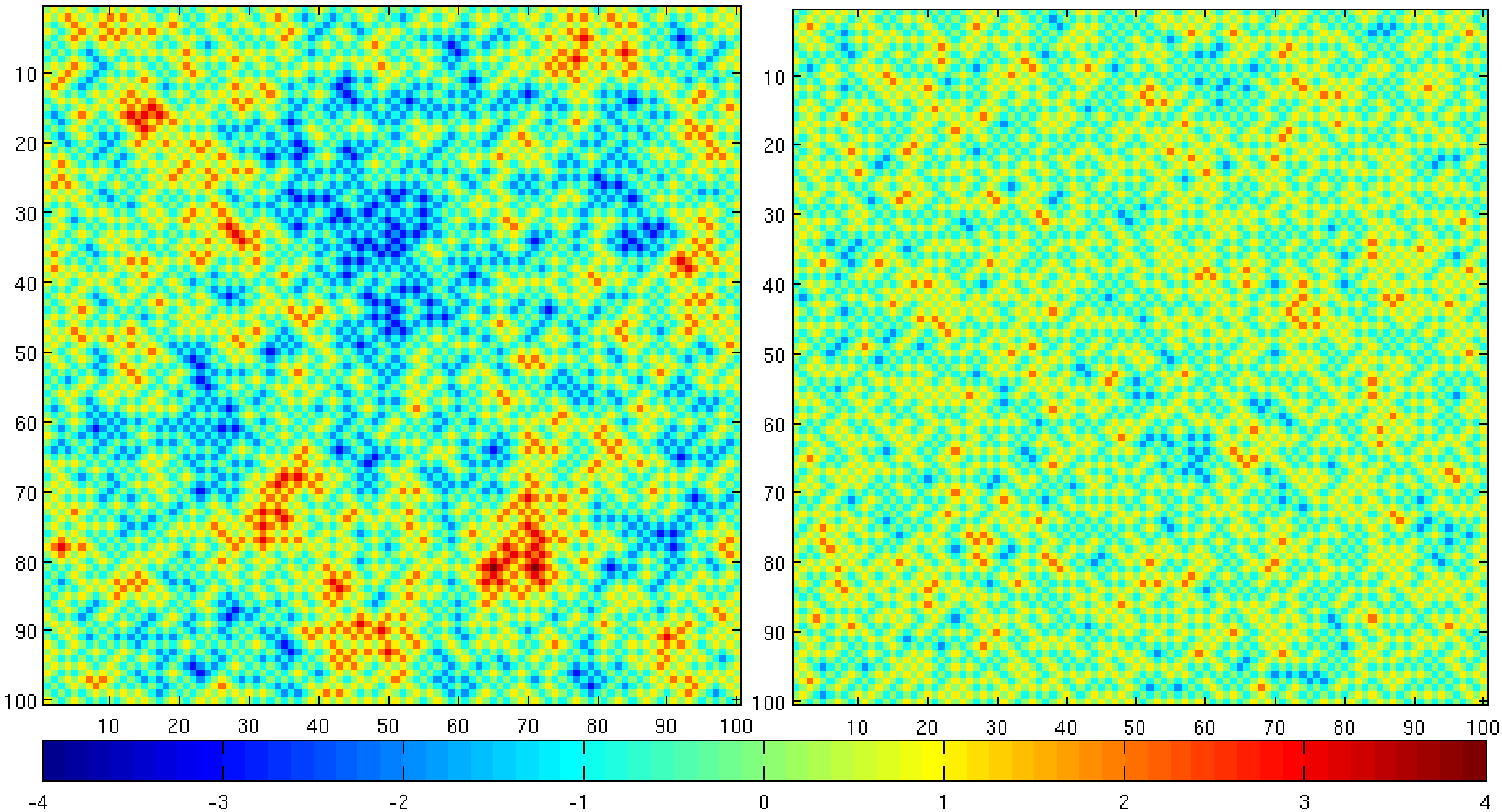


Level sets of 2D homomorphism



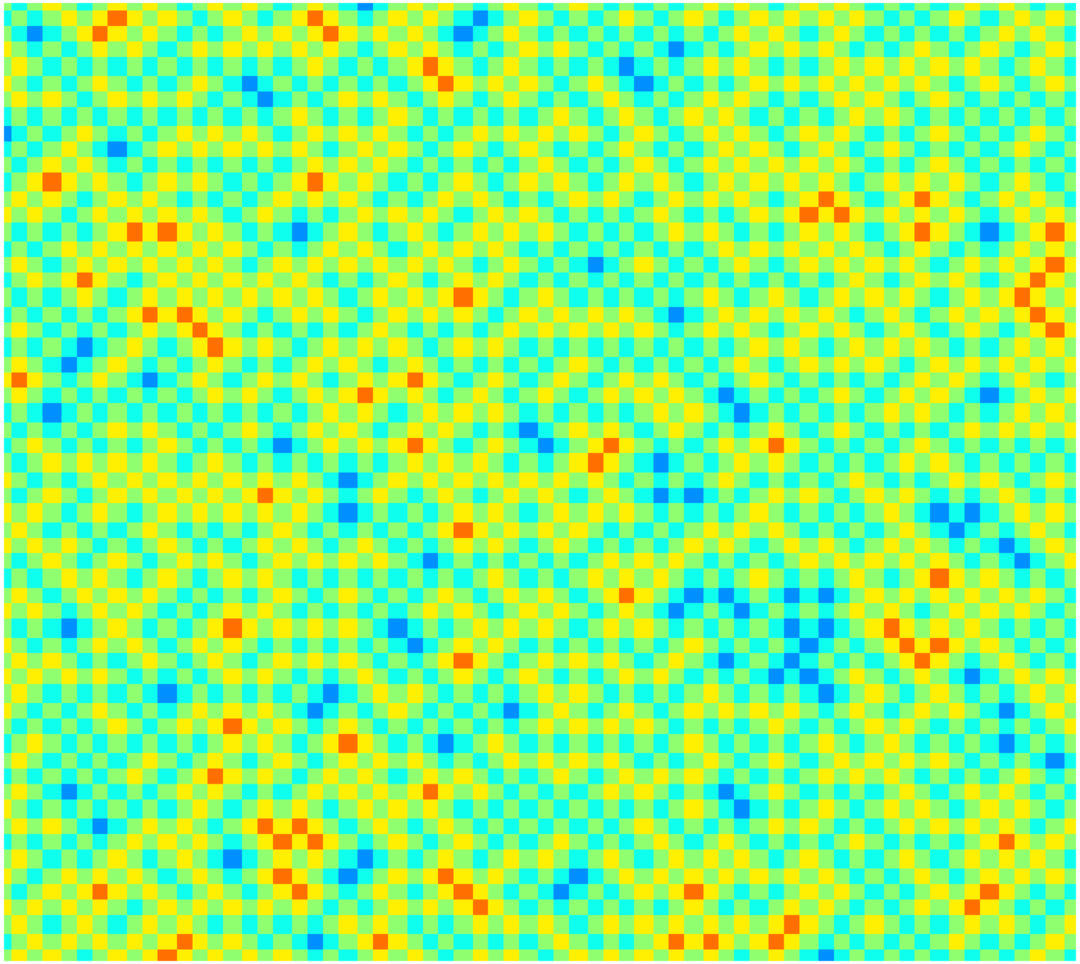
Outermost 0→1 level sets of homomorphism functions
Trivial level sets (having exactly 4 edges) not drawn

2 and 3-dimensional homomorphism functions



Rigidity of 3-dimensional homomorphism function

- Equals 0 (Green in the picture) on nearly all of the even sub-lattice (when the boundary values are 0 on even boundary vertices).
- Obtain 2 values for 1-Lipschitz functions (M=1 case).



Proper colorings of Z^d

- Given a box $\Lambda \subseteq Z^d$, uniformly sample a proper q-coloring of Λ.
- How does this coloring look? Does it have long-range order? Are there multiple Gibbs measures as $\Lambda \to Z^d$?
- Predicted interplay between d and q as follows:
- No structure when q is large compared to d – a unique Gibbs measure. Proven when $q \geq 11d/3$ (Vigoda 00).
- Rigidity when d is large compared to q – most colorings use only about half of the colors on the even sub-lattice and about half of the colors on the odd sub-lattice. Little is proven about this regime.
- Proper q-colorings are the same as the anti-ferromagnetic q-state Potts model at zero temperature.
  Transition from above behavior to unique Gibbs measure as temperature increases.

Homomorphisms $\leftrightarrow$ 3-colorings

- Normalized homomorphism height functions:
  $f : Z^d \to Z, \ |f(u) - f(v)| = 1 \text{ for } u \sim v \text{ and } f(0) = 0$
  are in bijection with proper 3-colorings taking the color zero at the origin.
- The bijection is given simply by $f \mapsto f \bmod 3$.
- Works also for $f : \Lambda \to Z$ when Λ is a box in Z^d.
- Does not work when $\Lambda = Z_n^d$ is a torus due to existence of non-trivial cycles (there exist 3-colorings on Z_n^d which are not modulo 3 of homomorphisms).

Main results (I) – fluctuations

- Theorem(P.): $\exists d_0$ such that if $d \geq d_0$ and f is a uniformly sampled homomorphism function on Z_n^d then $P(|f(x)| \geq t) \leq \exp(-c_d t^d)$ for all x,

  $\max_x |f(x)| \leq C_d \log(n)^{1/d}$ with high probability as $n \to \infty$.
- The boundary values for the above theorem are fixing f to be 0 on an arbitrary subset of Z_n^d.
- Matching lower bound on maximum follows from results of BYY in the case of a one-point boundary condition.
- The results extend to 1-Lipschitz functions via a bijection of Yadin.

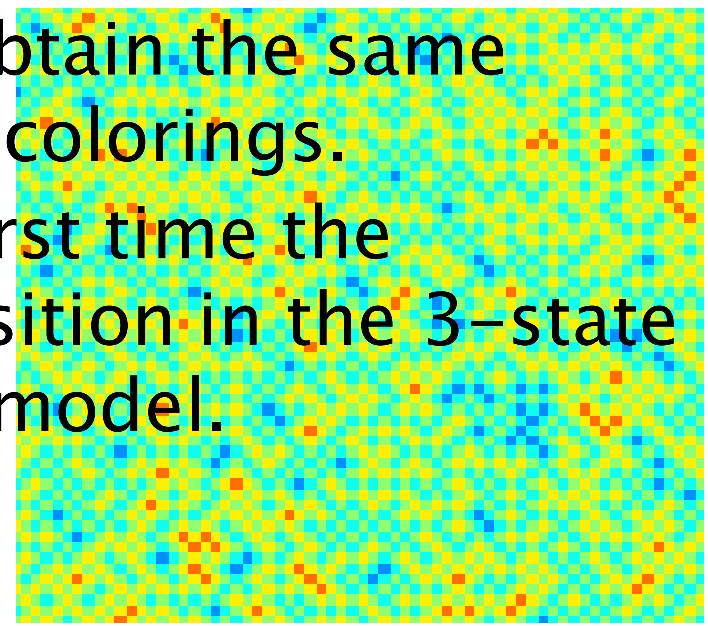
Main results (II) – level sets

- A level set of f is a (minimal) set of edges separating the boundary set from a point and having 0's on one side and 1's on the other.
- Main Theorem(P., P. & Feldheim): $\exists d_0$ such that if $d \geq d_0$, $x \in Z_n^d$ and f is a uniformly sampled homomorphism function on Z_n^d then
  $$P(\exists \text{ level set of length } L \text{ around } x) \leq \exp(-c_d L)$$
- We prove $c_d = c/d\log^4 d$.

0	1	0	−1	0	1
−1	0	1	0	1	0
0	1	2	1	2	1
1	2	3	2	1	0
0	1	2	1	0	−1
−1	0	1	0	1	0

Structure of homomorphism functions and proper 3-colorings

- It follows that if f is sampled with zeros on the even boundary of a large box, then it will be zero on nearly all even vertices inside the box, with the largest breakup of the pattern having logarithmic boundary length.
- By taking modulo 3, we obtain the same rigidity also for proper 3-colorings.
- This establishes for the first time the existence of a phase transition in the 3-state anti-ferromagnetic Potts model.



Roughening transition

- The results do not apply to the homomorphism model in 2 dimensions (the square ice model).
- However, they may be applied to tori with non-equal side lengths: $n_1 \le n_2 \le \cdots \le n_d$. Need only that d is large and that the torus is non-linear:

$$n_d \le \exp\left(\frac{1}{d\log^3(d)}\prod_{i=1}^{d-1} n_i\right)$$

- Thus the model on the n x n x 2 x 2 x … x 2 torus (with a fixed number of 2’s) has bounded fluctuations.
  I.e., the roughening transition occurs when adding a critical number of such 2’s!
- We prove the full roughening transition occurs between the model on an n x log(n) torus and the model on an n x log(n) x 2 x … x 2 torus.
- Refutes a conjecture of Benjamini, Yadin, Yehudayoff and answers a question of Benjamini, Häggström and Mossel.

Limits of homomorphism functions (joint with Feldheim)

- Infinite volume limit: We prove that as $\Lambda \to Z^d$ with zero boundary conditions, the model converges to a limiting Gibbs measure – gives a meaning to a uniformly sampled homomorphism or 1-Lipschitz function on the whole Z^d.
- For 3-colorings prove existence of 6 different Gibbs measures (in each, one of the 3 colors is dominant on one of the two sub-lattices).
- Scaling limit: Embedding Λ in $[0,1]^d$ and taking finer and finer mesh, we find that the scaling limit of the model is white noise. Integrals over disjoint regions converge to independent Gaussian limits.

Proof ideas for level set theorem

- Work on Z_n^d . Place zeros on the even boundary and uniformly sample a homomorphism function.
- Fix a point x and consider outermost level set around x. Denote it by LS(f).
- We want to show that $P\left(\left|LS(f)\right| \geq L\right) \leq \exp(-c_d L)$.

0	1	0	−1	0	1
−1	0	1	0	1	0
0	1	2	1	2	1
1	2	3	2	1	0
0	1	2	1	0	−1
−1	0	1	0	1	0

Shift transformation I

0	1	0	−1	0	1
−1	0	1	0	1	0
0	1	2	1	2	1
1	2	3	2	1	0
0	1	2	1	0	−1
−1	0	1	0	1	0

Shift Transformation →

0	1	0	−1	0	1
−1	0	−1	0	−1	0
0	1	0	1	0	−1
1	2	1	0	−1	0
0	1	0	−1	0	−1
−1	0	−1	0	1	0

- The value at each vertex inside the level set is replaced by the value to its right minus 1. Remain with a homomorphism height function!

Shift transformation II

0	1	0	−1	0	1
−1	0	1	0	1	0
0	1	2	1	2	1
1	2	3	2	1	0
0	1	2	1	0	−1
−1	0	1	0	1	0

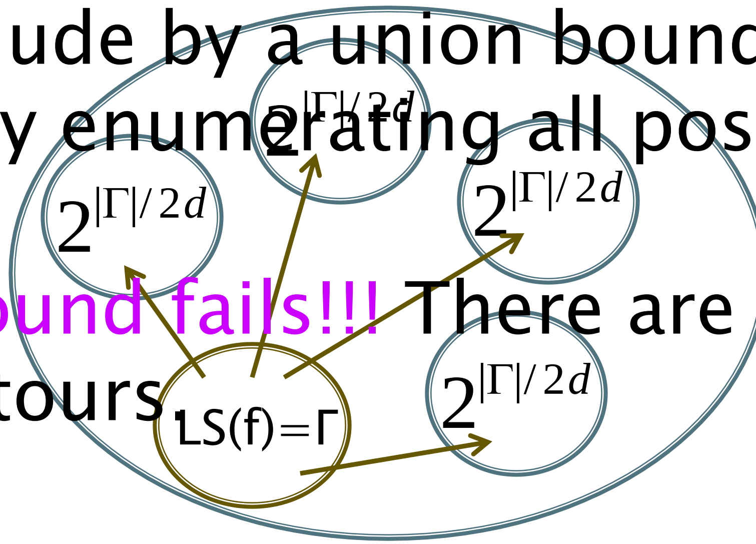
Shift Transformation →

0	1	0	−1	0	1
−1	0	−1	0	−1	0
0	1	0	1	0	−1
1	2	1	0	−1	0
0	1	0	−1	0	−1
−1	0	−1	0	1	0

- Vertices with level set on right are surrounded by zeros after the shift! Can change their values arbitrarily to ± 1.
  There are exactly $|LS(f)|/2d$ such vertices.
  Transformation still invertible given the level set.

Estimate for given contour

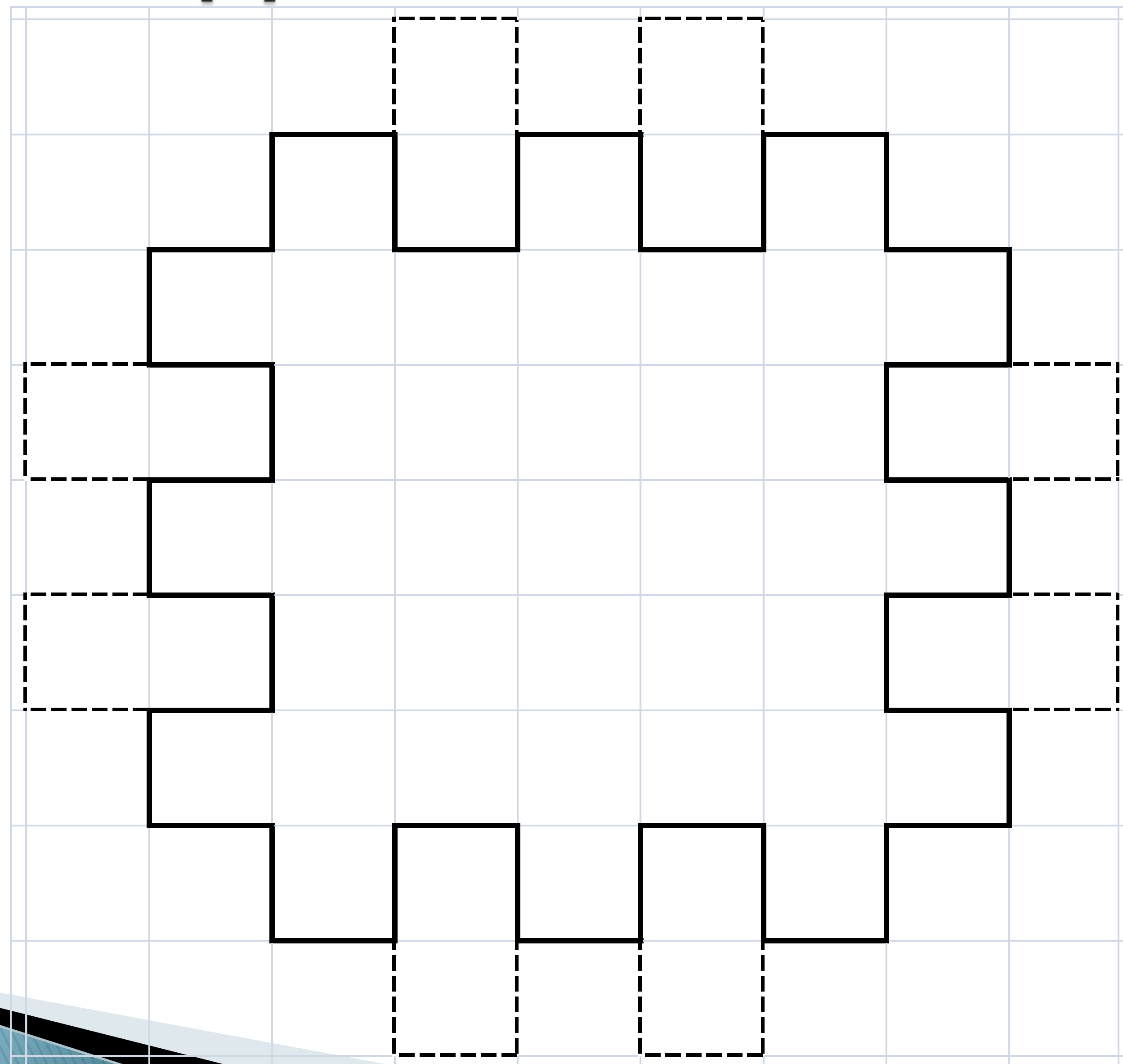
- Thus, given a contour, or cutset, Γ, we associate to each f with LS(f) = Γ a set of $2^{|\Gamma|/2d}$ other homomorphisms in an invertible way.
- It follows that $P\left(LS(f) = \Gamma\right) \leq 2^{-|\Gamma|/2d}$.
- Can we conclude by a union bound (Peierls argument), by enumerating all possible contours?
- The union bound fails!!! There are too many possible contours.



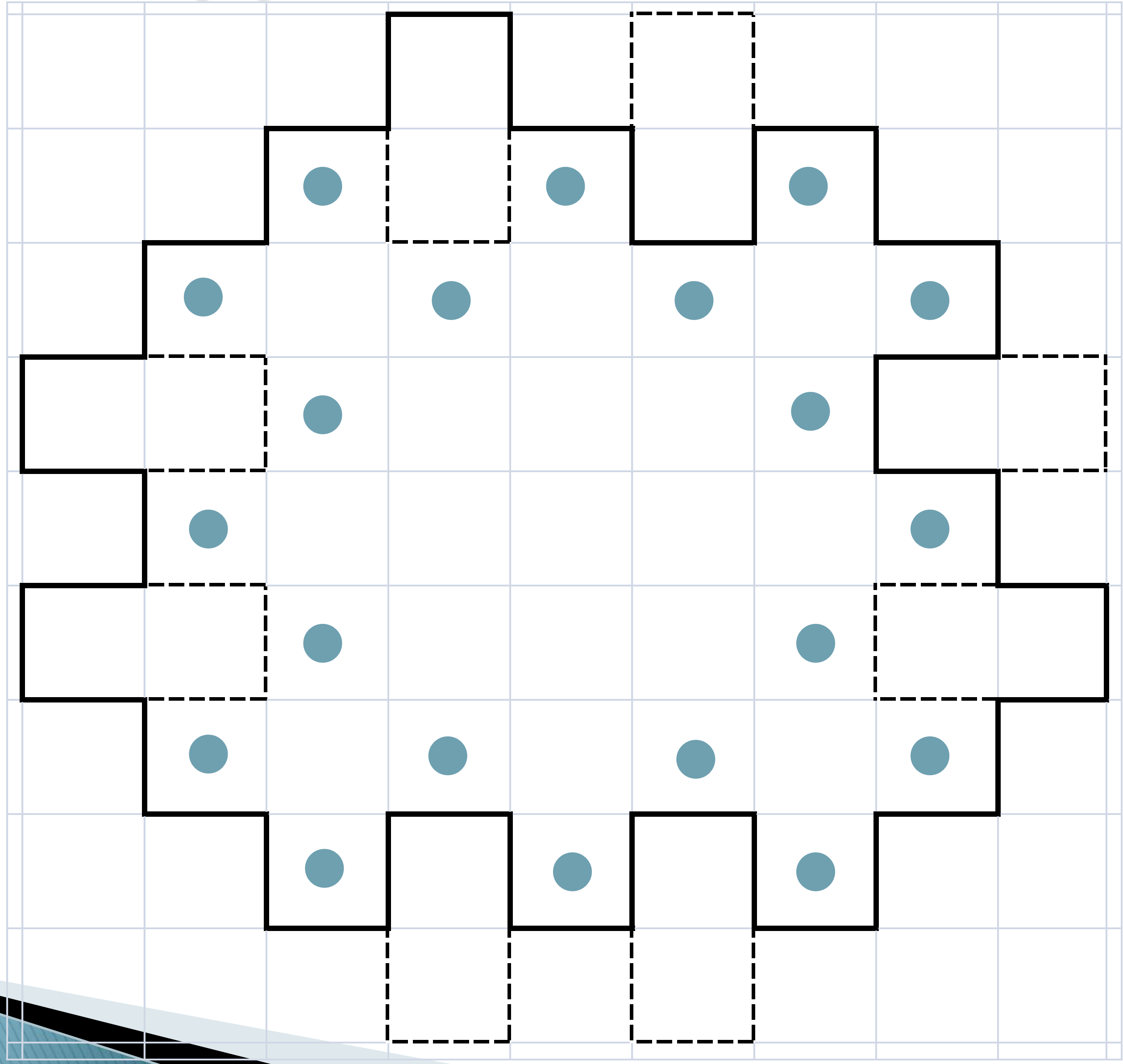
Coarse graining

- Instead of using a union bound, we use a coarse graining technique, grouping the cutsets into sets according to common features and bounding the probability of each set.
- We note that our cutsets have a distinguishing feature – they are odd cutsets. Cutsets whose interior boundary is on the odd sub-lattice.
- Denote by $\mathrm{OCut}_{\mathrm{L}}$ the set of all odd cutsets with exactly L edges.
- It turns out that while $\left|\mathrm{OCut}_{\mathrm{L}}\right| >> 2^{L/2d}$, there are much fewer "global shapes" for cutsets and most cutsets are minor perturbations of some shape.

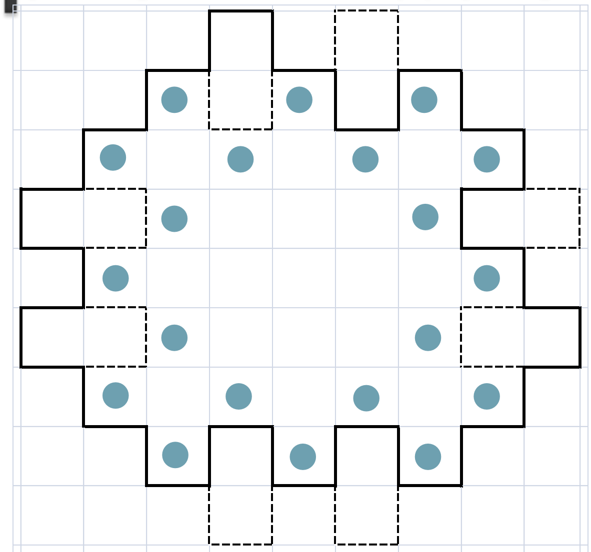
Interior approximation I



Interior approximation I



Interior approximation II



- Say that $A \subseteq Z_n^d$ is an interior approximation to $\Gamma \in \mathrm{OCut_L}$ if

$$\partial_{\mathrm{in}}\Gamma - \mathrm{exposed}(\Gamma) \subseteq \mathrm{A} \subseteq \mathrm{interior}(\Gamma)$$

where exposed(Γ) is the set of vertices adjacent to many edges of Γ($\geq 2\mathrm{d} - \sqrt{\mathrm{d}}$ edges).

Interior approximation III

- Theorem: For any L, there exists a set Ω containing an interior approximation to every cutset in $\Gamma \in \mathrm{OCut}_L$ with

$$|\Omega| \le \exp\left(\frac{C\log^2 d}{d^{3/2}} L\right).$$

- Much smaller if d is large than $|\mathrm{OCut}_L| >> 2^{L/2d}$.
- Thus, to conclude the proof it is sufficient to give a good bound on the probability that LS(f) is any of the cutsets with a given interior approximation.
- This is what we end up doing, but our bound is sufficiently good only for cutsets having a certain regularity. That they do not have almost all their edges on exposed vertices.

Counting irregular cutsets

- We conclude by proving that there are relatively few irregular cutsets.
  Fewer than exp(L/100d) of them in $OCut_L$.
- Thus, we may separately apply a union bound using our previous estimate
  $$P\left(LS(f) = \Gamma\right) \le 2^{-|\Gamma|/2d}$$
  to show that none of these cutsets occur as a level set of f.

Open questions

1. Analyze other Lipschitz function models. Is the behavior similar? As mentioned, analysis of the 1-Lipschitz model follows from our results by a bijection of Yadin. Analyze sloped boundary values.
2. Analyze proper q-colorings for $q>3$. Show rigidity of a typical coloring when d is sufficiently high. There does not seem to be a similar connection to height functions any more.
3. Analyze regular (non odd) cutsets and prove similar structure theorems. Can be useful in many models (Ising, percolation, colorings, etc.) as the cutsets form the phase boundary between two pure phases.
4. Improve structure theorems for odd cutsets – will reduce the minimal dimension for our theorems and will help in analyzing other models.

   With W. Samotij we use these theorems to improve the bounds on the phase transition point in the hard-core model.