Optimal error bounds in stochastic homogenization

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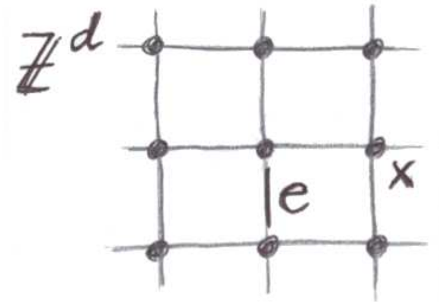
A model problem

Discrete random media

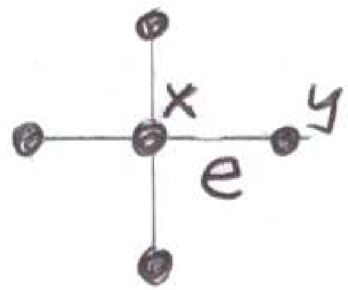


d-dimensional **lattice** $\mathbb{Z}^d$, sites x, edges e

Conductivities on edges

$a(e) > 0$

Operator $-\nabla \cdot a \nabla$ and **form** $\sum \nabla v \cdot a \nabla u$



$$-\nabla \cdot (a \nabla u)(x) = \sum_{y:\, e=[x,y]} a(e)(u(x) - u(y))$$

$$\sum \nabla v \cdot a \nabla u = \sum_e (v(x) - v(y)) a(e) (u(x) - u(y))$$

Homogenization of random discrete elliptic media

Uniform ellipticity:

$\lambda \le a(e) \le 1$ for all edges e for some $0 < \lambda \le 1$.

Statistics:

$\{a(e)\}_e$ independently and identically distributed (i. i. d.)

On large scales: $(T^{-1} - \nabla\cdot a\nabla)^{-1} \approx (T^{-1} - \nabla\cdot a_{hom}\nabla)^{-1}$

with a_{hom} homogeneous and deterministic

[Kozlov (79), Papanicolaou & Varadhan ($\approx$ 81),

Künnemann (83)]

Prediction of a_{hom} from statistics of $\{a(e)\}$?

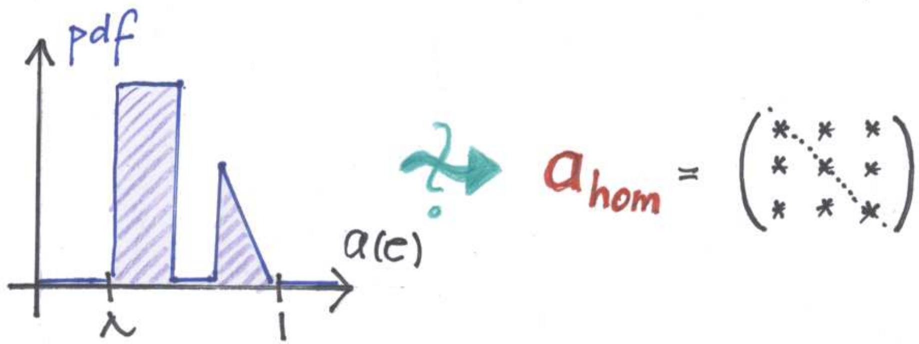
Conductivity^{-1} = Resistance

For d=1: $\langle a(e)^{-1}\rangle^{-1} \; = \; a_{hom}$

For any d: $\langle a(e)^{-1}\rangle^{-1} \; \leq \; a_{hom} \; \leq \; \langle a(e)\rangle$

in series   in parallel

No simple general formula for

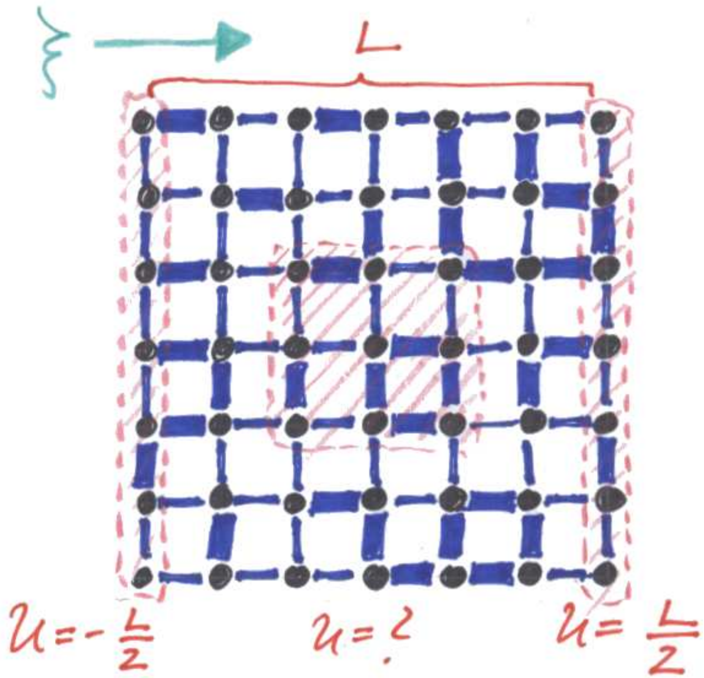


Concept of Representative Volume Element

Linear map $\mathbb{R}^d \ni \xi \;\mapsto\; a_{hom}\xi \in \mathbb{R}^d$

describes relation *average potential gradient* $\mapsto$ *average current*

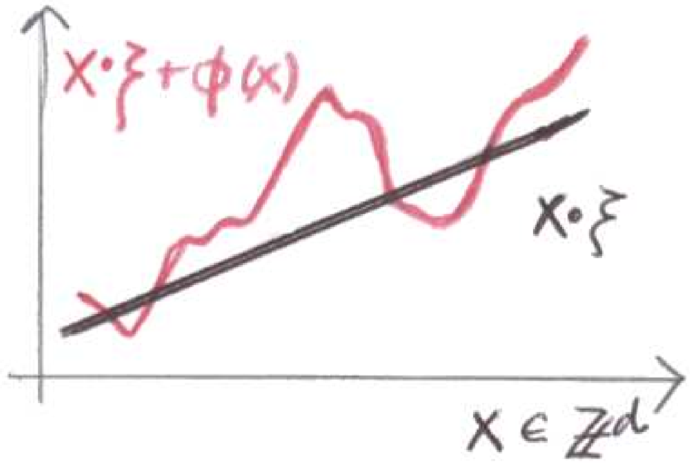
Use large subsystem
(size $L \gg 1$)
to get approximation
of this relation



Mathematical idealization: *corrector field*

Notion of corrector field ϕ

Corrector ϕ for direction $\xi \in \mathbb{R}^d$:



- $x \mapsto x \cdot \xi + \phi(x)$ is a-**harmonic**, i. e.

  $-\nabla \cdot a(\xi + \nabla \phi) = 0$

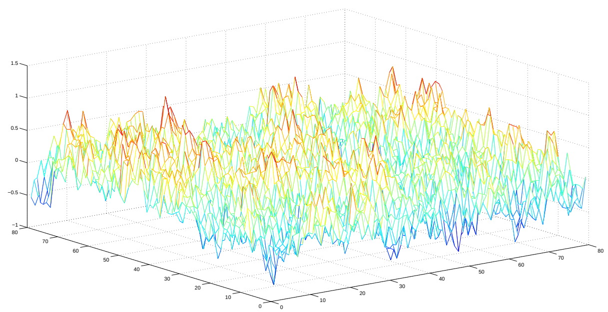
- $\nabla \phi$ is **stationary**, i. e.

  $x \mapsto \nabla \phi(x)$ and $x \mapsto \nabla \phi(x + z)$

  have same statistics

  for all shifts $z \in \mathbb{Z}^d$

  Stationarity of ϕ?



Corrector $\rightsquigarrow$ exact but unpractical formula for a_{hom}

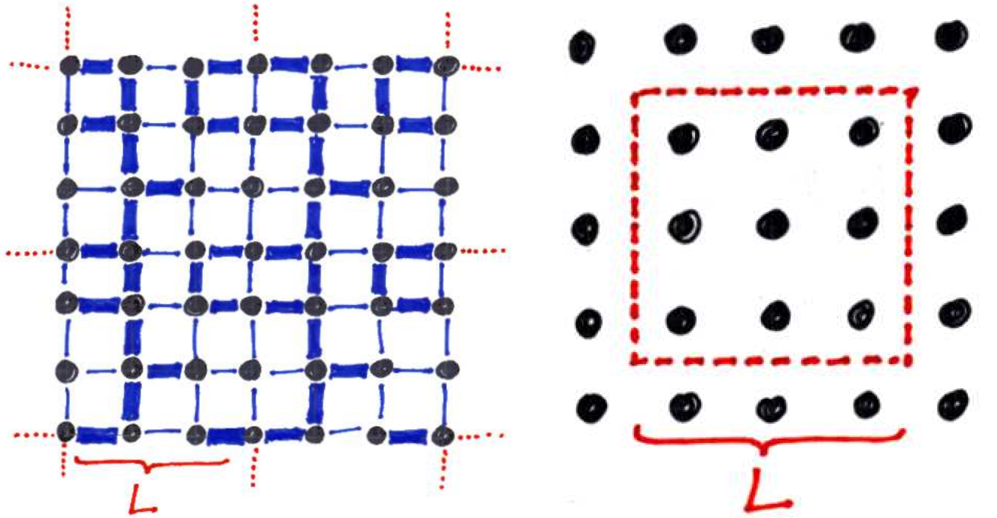
Formula: $\xi \cdot a_{hom}\xi = \langle(\xi+\nabla\phi)\cdot a(\xi+\nabla\phi)\rangle$

... of no practical use:

$-\nabla \cdot a(\xi+\nabla\phi) = 0$ has to be solved

- for *every realization* of coefficients $\{a(e)\}_e$
- on *whole lattice* $\mathbb{Z}^d$

Back to Representative Volume Element



Period L

$a^L(e+Le_i)=a^L(e)$

$\phi^L(x+Le_i)=\phi^L(x)$

Spatial average $[0,L)^d$

Systematic error (wrong statistics)

$$\langle(\xi+\nabla\phi)\cdot a(\xi+\nabla\phi)\rangle \overset{L\gtrsim 1}{\approx} \langle(\xi+\nabla\phi^L)\cdot a^L(\xi+\nabla\phi^L)\rangle$$

Random error (lack of ergodicity)

$$\langle(\xi+\nabla\phi^L)\cdot a^L(\xi+\nabla\phi^L)\rangle \overset{L\gtrsim 1}{\approx} L^{-d}\sum_{x\in[0,L)^d}\Big((\xi+\nabla\phi^L)\cdot a^L(\xi+\nabla\phi^L)\Big)(\omega,x$$

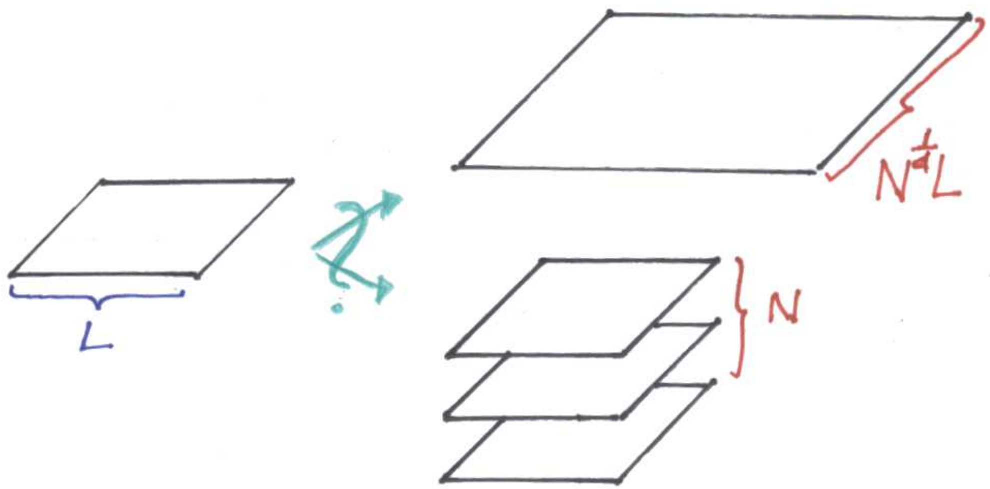
Strategy: size of system vs. number of samples?

N samples (N independent realizations):

$$\langle (\xi + \nabla\phi^L)\cdot a^L(\xi+\nabla\phi^L)\rangle \overset{N\gg 1}{\approx} \mathrm{N}^{-1}\sum_{\mathrm{n}=1}^{\mathrm{N}} L^{-d}\sum_{x\in[0,L)^d}\Big((\xi+\nabla\phi^L)\cdot a^L(\xi+\nabla\phi^L)\Big)(\omega_n,x)$$

L ↑ reduces systematic error and random error

N ↑ reduces only random error

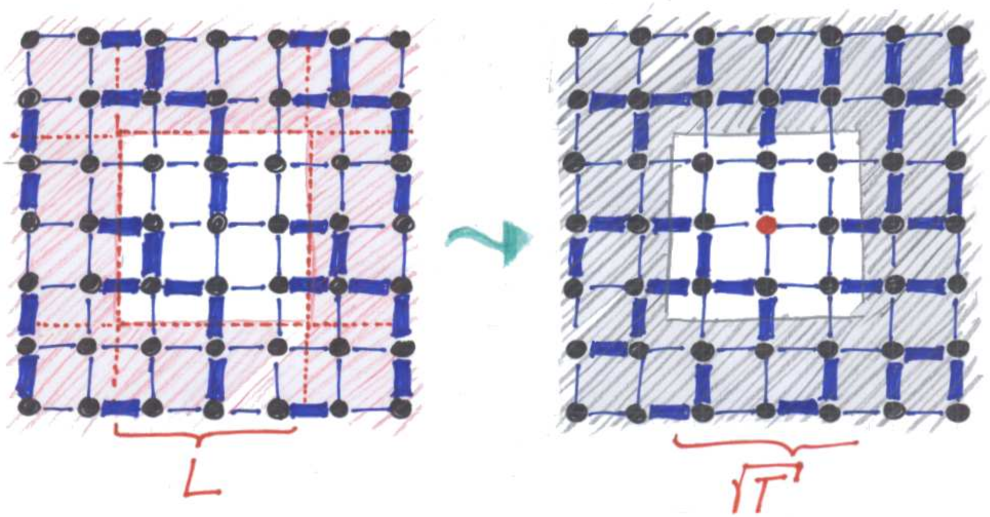


Simplified concept of systematic error

ϕ_T as a proxy for ϕ^L:

$T^{-1}\phi_T - \nabla\cdot a(\xi+\nabla\phi_T) = 0$

in all of $\mathbb{Z}^d$ with $T = L^2$



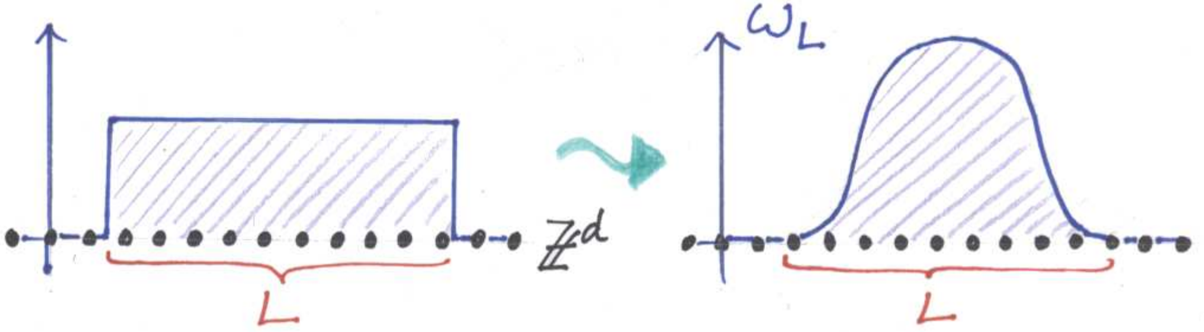
Systematic error (from desorption term)

$$:= \langle(\xi+\nabla\phi_T)\cdot a(\xi+\nabla\phi_T)\rangle - \langle(\xi+\nabla\phi)\cdot a(\xi+\nabla\phi)\rangle$$

$$= \langle\nabla(\phi_T-\phi)\cdot a\nabla(\phi_T-\phi)\rangle \sim \langle|\nabla(\phi_T-\phi)|^2\rangle$$

Simplified concept of random error

Smooth spatial average



$\mathrm{supp}\, w_L \subset [0,L)^d$,

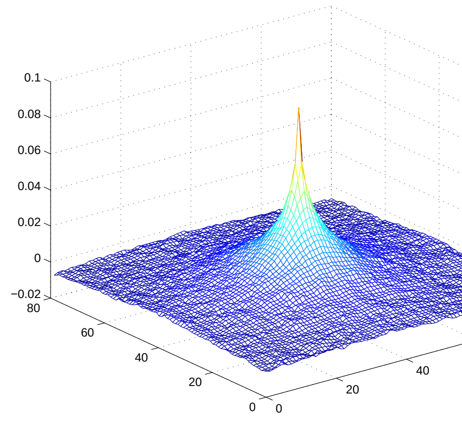
$\sum_x w_L = 1$,

$|\nabla w_L| \lesssim L^{-d-1}$

Random error (from long range correlation in energy density)

$$:= \left\langle \left| \sum_x w_L (\xi + \nabla\phi_T) \cdot a(\xi + \nabla\phi_T) - \langle (\xi + \nabla\phi_T) \cdot a(\xi + \nabla\phi_T) \rangle \right|^2 \right\rangle^{1/2}$$

$$= \mathrm{var}^{1/2} \left[\sum_x w_L (\xi + \nabla\phi_T) \cdot a(\xi + \nabla\phi_T) \right]$$



What estimates to expect?

Vanishing conductivity contrast $1 - \lambda \ll 1$:
(asymptotics and explicit solution)

$$\textbf{systematic error}^2 \sim \left\{ \begin{array}{ll} \mathbf{T}^{-\mathbf{d}} & \text{for } d < 4 \\ \mathbf{T}^{-4} \ln \mathbf{T} & \text{for } d = 4 \\ \mathbf{T}^{-4} & \text{for } d > 4 \end{array} \right\}$$

$$\textbf{random error}^2 \sim \mathbf{L}^{-\mathbf{d}}$$

Random error dominates systematic error

for $L = \sqrt{T}$ and $d < 8$:

$\Longrightarrow$ **many samples** $N \gg 1$ **advantageous**

Our nearly optimal result

For all conductivity ratios $\lambda > 0$ and dimensions $d > 2$:

$$\text{system}^2 \;:=\; \langle \nabla(\phi_T - \phi)\cdot a \nabla(\phi_T - \phi)\rangle^2 \;\leq\; C(\lambda, d) \begin{cases} T^{-d} & d < 4 \\ T^{-4} \ln T & d = 4 \\ T^{-4} & d > 4 \end{cases}$$

$$\text{random}^2 \;:=\; \mathbf{var}\left[\, \sum w_L (\xi + \nabla \phi_T)\cdot a(\xi + \nabla\phi_T)\,\right] \;\leq\; C(\lambda, d)\, L^{-d}$$

For all conductivity ratios $\lambda > 0$ and dimension $d = 2$:

$$\text{system}^2 \;\leq\; C(\lambda)\, T^{-2} (\ln T)^{q(\lambda)}$$

$$\text{random}^2 \;\leq\; C(\lambda)\, L^{-2} (\ln T)^{q(\lambda)}$$

[Gloria&O., Ann.Prob. (to appear), Ann.Appl.Prob. (accepted)]

Mathematical context

Yurinskii ('86) result

Suboptimal estimate

$$\textbf{system}^2 \sim \langle\, |\nabla\phi_T - \nabla\phi|^2\, \rangle^2 \lesssim T^{-\frac{2d-4}{4+d}+} \qquad (\, T^{-\min\{d,4\}+}\,)$$

... relies on suboptimal **variance estimate** for ϕ_T

$$\mathrm{var}\left[\sum \omega_L \phi_T\right] \lesssim T\left(TL^{-d}\right)^{\frac{1}{2}-} \qquad (\, L^{2-d}\,)$$

... relies on **random walk decomposition**

$$\phi_T = \sum_{S\subset\{\mathrm{edges}\}} \phi_{T,S}$$

with

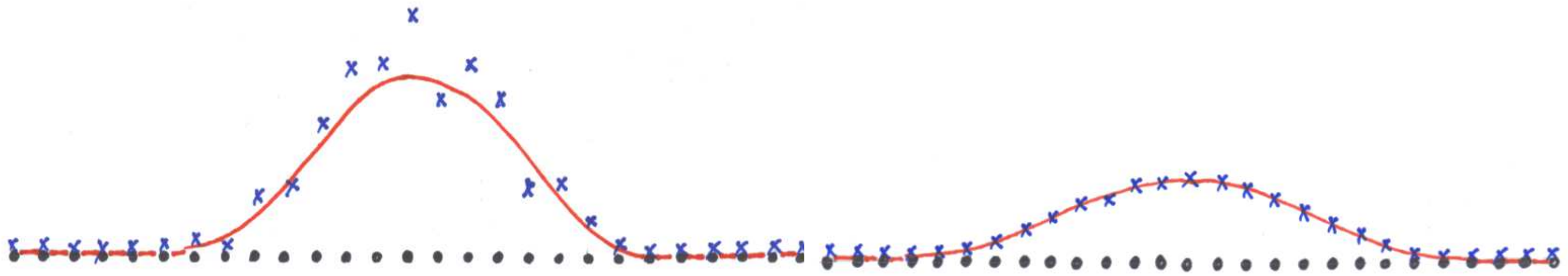
$$\phi_{T,S} = \phi_{T,S}(\{a(e)\}_{e\in S})$$

Random walk in random environment I

Large scale behavior of parabolic Green's function $G(\omega,t,x)$

$$\partial_t G - \nabla\cdot a\nabla G \;=\; 0 \quad G(t=0,x) \;=\; \left\{\begin{array}{ll} 1 & \text{for } x=0 \\ 0 & \text{for } x\neq 0 \end{array}\right\}$$

is Gaussian with variance a_{hom} : $G_{hom}(t,x) = \frac{c_d}{t^{d/2}}\exp(-\frac{x\cdot a_{hom} x}{4t})$



Almost sure ("quenched") *qualitative* result:

Corrector ϕ provides harmonic coordinates

[Sidoravicius & Sznitman 04]

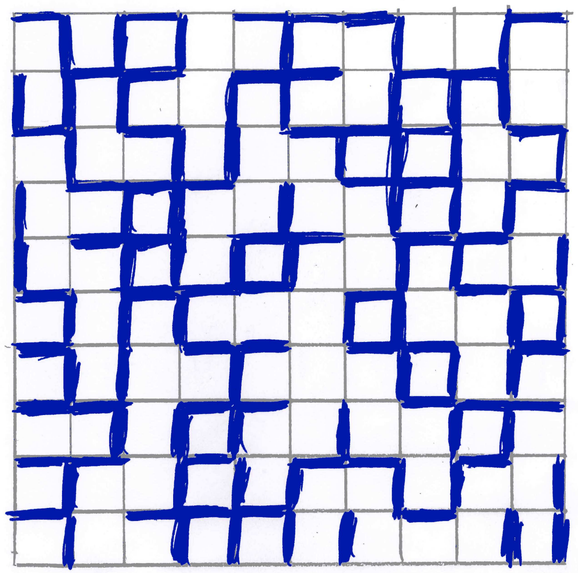
Random walk in random environment II

Random walk on

percolation cluster

Bernoulli, supercritical

i. e. $\lambda = 0$



Almost surely behaves like Brownian motion on large scales:

Sidoravicius & Sznitman '04 ($d \geq 4$), Mathieu & Piatnitski '07 ($d \geq 2$), Berger & Biskup '07 ($d \geq 2$)	} sublinear growth of corrector ϕ

Naddaf & Spencer (unpublished) result

Spectral gap estimate: $\mathrm{var}[X] \;\le\; C(a)\left\langle \sum_{\text{edges } e} \left(\frac{\partial X}{\partial a(e)}\right)^2 \right\rangle$

Energy density: $X := \sum \nabla u \cdot a \nabla u$ where $-\nabla \cdot a \nabla u = \nabla \cdot f$

have $\frac{\partial X}{\partial a(e)} = -|\nabla u(e)|^2$.

Meyer's estimate: $\exists\, p(d,\lambda) > 2$ s. t. $\sum |\nabla u|^p \le C(d,\lambda) \sum |f|^p$

For small ellipticity ratio $1-\lambda \ll 1$: $\sum |\nabla u|^4 \le C(d) \sum |f|^4$

Optimal conclusion for $1-\lambda \le c(\mathrm{d})$:

$\mathrm{var}[\sum \nabla u \cdot a \nabla u] \;\le\; C(d,a) \sum |f|^4$

Gradient Gibbs measures I (Funaki & Spohn '97)

From *quenched* $\quad \min_{\phi} \sum_{e} (\xi + \nabla\phi(e)) \cdot a(\omega, e)(\xi + \nabla\phi(e))$

to *annealed* $\langle\langle \cdot \rangle\rangle \; := \; \frac{1}{Z} \exp\Big(- \sum_{e} V(\xi + \nabla\phi(e))\Big) \Pi_x d\phi(x)$

Naddaf & Spencer '97, Giacomin & Olla & Spohn '01, Conlon & Spencer '11: For $\lambda \;\le\; V''(\eta) \;\le\; 1$

$$\langle\langle \phi(x)\phi(x') \rangle\rangle \;=\; \int_0^\infty \langle\langle \langle G(t, x, x') \rangle\rangle\rangle \, dt \;\approx\; G_{hom}(x - x')$$

where $G(t, x, x')$ parabolic Green's function
for *time-dependent* coefficients $a(t, e) := V''(\nabla\phi(t, e))$
and where $\phi(t, x)$ Glauber dynamics w. r. t. $\langle\langle \cdot \rangle\rangle$

Gradient Gibbs measures II (Biskup & Kotecky '07)

Biskup & Spohn '11:

For $V(\eta) = -\log \int_\lambda^1 \exp\big(-\frac{1}{2}a\eta^2\big)\,\rho(da), \quad \xi = 0$

consider *extended* gradient Gibbs measure

$\langle\langle\langle\cdot\rangle\rangle\rangle := \frac{1}{Z}\exp\big(-\sum_e a(e)\frac{1}{2}(\nabla\phi(e))^2\big)\,\Pi_x d\phi(x)\,\Pi_e da(e).$

Have

$$\langle\langle\phi(x)\phi(x')\rangle\rangle = \langle\langle\langle G(x,x')\rangle\rangle\rangle \approx G_{hom}(x-x')$$

where $G(x,x')$ is elliptic Green's function w. r. t. $\{a(e)\}_e$.

Sketch of Proof

Crucial for our result: higher moments of corrector

Suppose $|\xi| = 1$, $d > 2$ (otherwise $\log T$-terms), $T = \infty$

Have for all $q < \infty$: $\quad \langle |\phi|^q \rangle \ \leq \ C(\lambda, d, q)$

Step 1. For all $q < \infty$: $\quad \langle |\phi|^{2q} |\nabla\phi|^2 \rangle \ \lesssim \ \langle |\phi|^{2q} \rangle$

Step 2. For all $1 \ll q < \infty$

s. t. $p := 2(1+q^{-1}) \approx 2$ satisfies Meyer's estimate we have:

$$\mathrm{var}\left[|\phi|^{q+1}\right] = \langle |\phi|^{2(q+1)} \rangle - \langle |\phi|^{q+1} \rangle^2 \ \lesssim \ \left(1+\langle |\phi|^{2q} \rangle\right)^{\frac{1}{q+1}} \ \langle |\phi|^{2(q+1)} \rangle^{\frac{q}{q+1}}$$

Argument for Step 1 (Caccioppoli in $\langle\cdot\rangle$)

Test $-\nabla\cdot a(\xi+\nabla\phi)=0$ with $\phi|\phi|^{2q-1}$:

$$-\nabla\cdot(\phi|\phi|^{2q-1}\,a(\xi+\nabla\phi)) + 2q|\phi|^{2q}\,\nabla\phi\cdot a(\xi+\nabla\phi) \;=\; 0$$

Stationarity: $\langle|\phi|^{2q}\,\nabla\phi\cdot a(\xi+\nabla\phi)\rangle = 0$

Uniform ellipticity: $\langle|\phi|^{2q}\,|\nabla\phi|^2\rangle \;\lesssim\; \langle|\phi|^{2q}\,|\nabla\phi|\rangle$

Cauchy Schwarz: $\langle|\phi|^{2q}\,|\nabla\phi|^2\rangle \;\lesssim\; \langle|\phi|^{2q}\rangle$

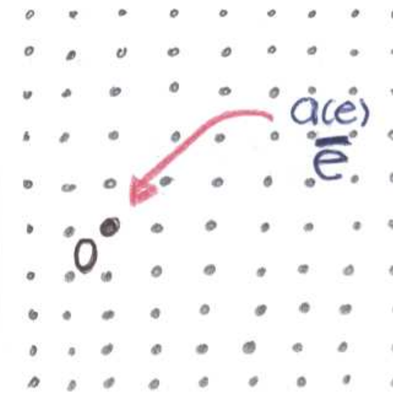
Step 2: Spectral gap and Green's function

Spectral gap estimate

$$\mathrm{var}\left[|\phi|^{q+1}\right] \lesssim \left\langle \sum_e \left(\frac{\partial}{\partial a(e)} |\phi(0)|^{q+1}\right)^2 \right\rangle \sim \left\langle \sum_e |\phi(0)|^{2q} \left|\frac{\partial \phi(0)}{\partial a(e)}\right|^2 \right\rangle$$

Green's function

$$\frac{\partial \phi(0)}{\partial a(e)} = -\nabla G(0,e) \cdot (\xi + \nabla \phi)(e)$$



Get

$$\mathrm{var}\left[|\phi|^{q+1}\right] \lesssim \sum_e \langle |\phi(0)|^{2q} \, |\nabla G(0,e)|^2 \, |(\xi + \nabla \phi)(e)|^2 \, \rangle$$

Step 2: Gain in homogeneity through Step 1

Recall Step 1 $\langle|\phi|^{2q}\,|\nabla\phi|^2\rangle \lesssim \langle|\phi|^{2q}\rangle$

Recall $\mathrm{var}\left[|\phi|^{q+1}\right] \lesssim \sum_e \left\langle|\phi(0)|^{2q}\,|\nabla G(0,e)|^2\,|(\xi+\nabla\phi)(e)|^2\right\rangle$

Hölder $\left\langle|\phi(0)|^{2q}\,|\nabla G(0,e)|^2|(\xi+\nabla\phi)(e)|^2\right\rangle$

$p=2(1+\frac{1}{q})$ $\leq \left\langle|\phi(0)|^{2(q+1)}\,|\nabla G(0,e)|^p\right\rangle^{\frac{2}{p}} \left\langle|(\xi+\nabla\phi)(e)|^{2(q+1)}\right\rangle^{\frac{1}{q+1}}$

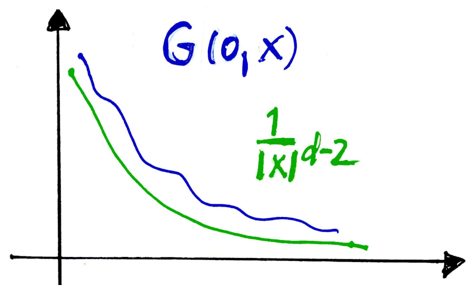
Step 1 $\langle|(\xi+\nabla\phi)(e)|^{2(q+1)}\rangle \lesssim 1+\langle|\phi|^{2q}\,|\nabla\phi|^2\rangle \lesssim 1+\langle|\phi|^{2q}\rangle$

Step 2: Need uniform control of Green's function

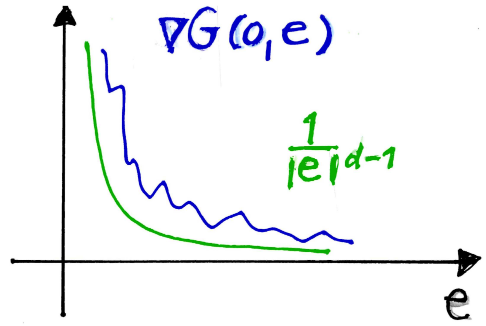
Have $\quad \text{var}\left[|\phi|^{q+1}\right] \lesssim \left(1+\langle|\phi|^{2q}\rangle\right)^{\frac{1}{q+1}} \sum_{e} \langle |\phi(0)|^{2(q+1)} \, |\nabla G(0,e)|^{p} \rangle^{\frac{2}{p}}$

In which sense

$$G(0,x) \sim \frac{1}{|x|^{d-2}}$$



$$|\nabla G(0,e)| \sim \frac{1}{|e|^{d-1}}$$



uniformly in $\{a(e)\}_e \subset [\lambda, 1]$

Step 2: Recall elliptic regularity (discrete version)

Harnack: For $\nabla \cdot (a\nabla u) = 0$ in $\{|x| \le 2R\}$

$$\sup_{|x|\le R} |u(x)| \lesssim \frac{1}{R^d} \sum_{|x|\le 2R} |u(x)|$$

Caccioppoli: For $\nabla \cdot (a\nabla u) = 0$ in $\{|x| \le 2R\}$

$$\sum_{|e|\le R} |\nabla u(e)|^2 \lesssim \frac{1}{R^2} \sum_{|x|\le 2R} |u(x)|^2$$

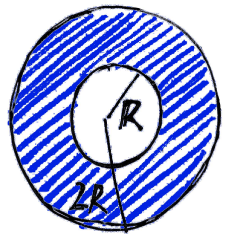
Meyers: $\exists\, p(d,\lambda) > 2$ s. t. for $\nabla \cdot (a\nabla u) = \nabla \cdot f$

$$\sum_e |\nabla u(e)|^p \lesssim \sum_e |f(e)|^p$$

Step 2: Control of Green's function

Get for some $p > 2$

$$\left(\frac{1}{R^d}\sum_{R\le|e|<2R}|\nabla G(0,e)|^p\right)^{\frac{1}{p}} \lesssim \frac{1}{R^{d-1}}$$

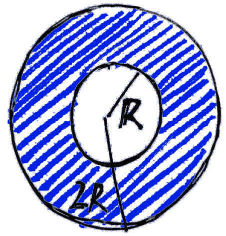


... averaged "quenched estimate"

vs. pointwise "annealed estimate" ...

$$\langle|\nabla G(0,e)|^2\rangle^{\frac{1}{2}} \lesssim \frac{1}{|e|^{d-1}}$$ [Delmotte & Deuschel '05].

Step 2: Apply estimate on Green's function



for some $p > 2$

$$\sum_{R\le|e|<2R} |\nabla G(0,e)|^p \lesssim R^d\left(\frac{1}{R^{d-1}}\right)^p$$

Get

$$\sum_{R \le |e| < 2R} \left\langle |\phi(0)|^{2(q+1)}\, |\nabla G(0,e)|^p \right\rangle^{\frac{2}{p}}$$

$$\le \left(R^d\right)^{1-\frac{2}{p}} \left(\sum_{R \le |e| < 2R} \left\langle |\phi(0)|^{2(q+1)}\, |\nabla G(0,e)|^p \right\rangle \right)^{\frac{2}{p}}$$

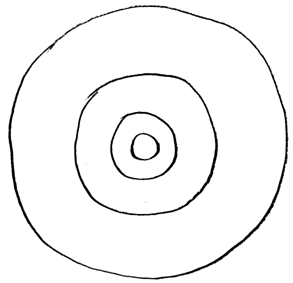
$p = 2(1+\frac{1}{q}) \approx 2$ for $q \gg 1$

$$= \left(R^d\right)^{1-\frac{2}{p}} \left\langle |\phi(0)|^{2(q+1)} \sum_{R \le |e| < 2R} |\nabla G(0,e)|^p \right\rangle^{\frac{2}{p}}$$

$$\lesssim \langle |\phi(0)|^{2(q+1)} \rangle^{\frac{q}{q+1}} \frac{1}{R^{d-2}}$$

Step 2: Conclusion

Dyadic annuli $\sum_e \rightsquigarrow \sum_{\ell=0}^{\infty} \sum_{2^\ell \le |e| < 2^{\ell+1}}$



Get gain in homogeneity in ϕ:

$$\mathrm{var}\left[|\phi|^{q+1}\right] \lesssim \left(1+\langle|\phi|^{2q}\rangle\right)^{\frac{1}{q+1}} \langle|\phi|^{2(q+1)}\rangle^{\frac{q}{q+1}} \underbrace{\sum_{\ell=0}^{\infty} \frac{1}{(2^\ell)^{d-2}}}$$

finite for $d > 2$

logarithm for $d = 2$

Future directions

Allow for (fast decaying) **correlations**
(à la Dobrushin-Shlosman) [ok]

Extension to **continuum** elliptic equations [ok]

From simplified concept of errors to **actual errors** [ok]

Extension to **percolation** clusters [?]

Avoid (some of the) **logarithms** $\ln T$ for $d = 2$ [?]

Extension to **systems** (elasticity) [??]