Hydrodynamic limit for the Ginzburg-Landau $\nabla\phi$ interface model with a conservation law and the Dirichlet boundary condition

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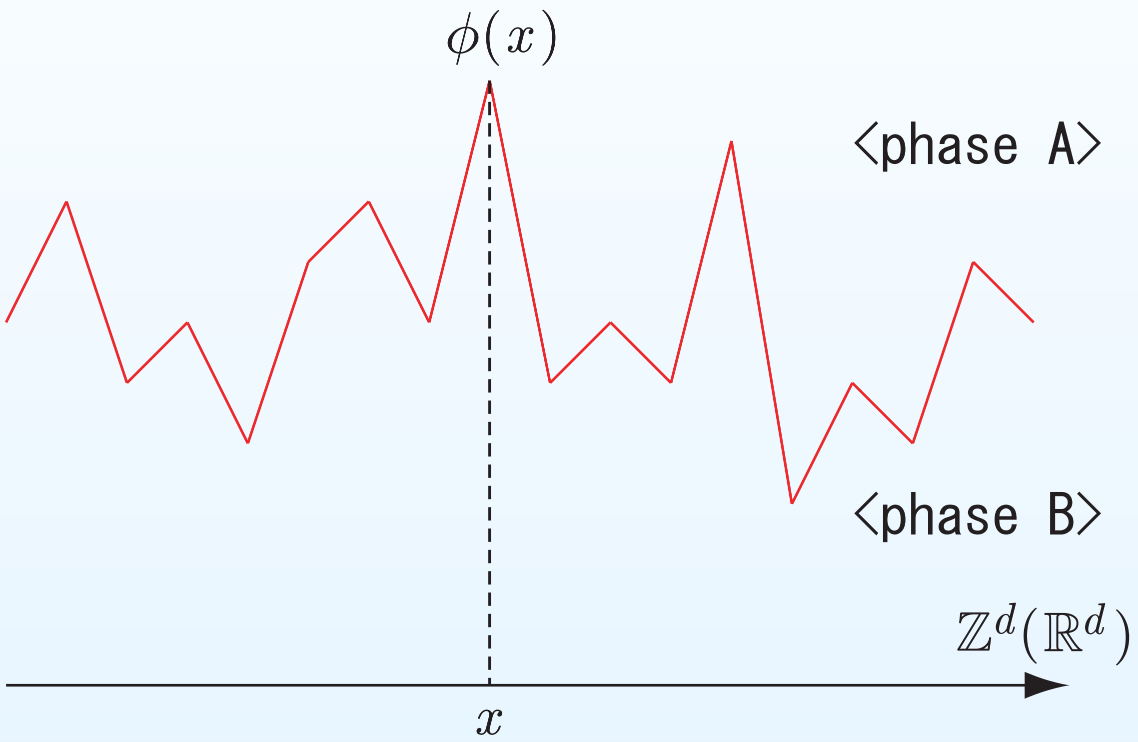
Workshop “Gradient Random Fields”

Model

Microscopic interface

Interface $\phi = \{\phi(x) \in \mathbb{R};\ x \in \mathbb{Z}^d\}$



$\phi(x)$: the height at position x

Energy of miscroscopic interface

Energy of the microscopic interface $\phi = \{\phi(x) \in \mathbb{R}; x \in \mathbb{Z}^d\}$

$$H(\phi) = \frac{1}{2} \sum_{x,y \in \mathbb{Z}^d, |x-y|=1} V(\phi(x) - \phi(y))$$

($V : \mathbb{R} \to \mathbb{R}$ is C^2, symm., $\|V''\|_\infty < \infty$)

Dynamics - Langevin equation

Langevin eq.

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t)dt + \sqrt{2}dw_t(x), \qquad (1)$$

for

$$x \in \Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d \text{ with periodic b.c.}$$
$$x \in D_N = ND \cap \mathbb{Z}^d \text{ with Dirichlet b.c.}$$

- $w = \{w_t(x); x \in \Gamma_N\}$: independent 1D B.m.'s
- $\frac{\partial H}{\partial \phi(x)} = \sum_{y:|x-y|=1} V'(\phi(x) - \phi(y))$

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Hydrodynamic scaling limit (LLN)

Macroscopic interface $h^N(t,\theta)$
$(t \in [0,t], \theta \in [0,1)^d =: \mathbb{T}^d$ or $\theta \in D)$

$$h^N(t, x/N) = N^{-1}\phi_{N^2 t}(x), \quad x \in \Gamma_N$$

Theorem 1 (Funaki-Spohn for Γ_N, N. for D_N with Dirichlet b.c.). *If V is strictly convex, i.e., there exist $c_-, c_+ > 0$ such that*

$$c_- \le V''(\eta) \le c_+, \quad \eta \in \mathbb{R}$$

we have

$$h^N \longrightarrow h : \frac{\partial h}{\partial t} = \operatorname{div} \nabla\sigma(\nabla h) \tag{2}$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}$ is the surface tension introduced via thermodynamic limit.

Total surface tension

The equation (2) is the gradient flow with respect to the energy functional

$$\Sigma(h) = \int \sigma(\nabla h(\theta))\, d\theta \tag{3}$$

in L^2-space. The functional Σ is called ”total surface tension,” which gives the total energy of the interface h.

Remark 1. The assumption “V is strictly convex” can be relaxed. If we have the convexity of σ (see Cotar-Deuschel-Müller and Cotar-Deuschel) and the characterization of Gibbs measures for gradient fields, we can show the hydrodynamic limit. (joint work with J.-D. Deuschel and I. Vignard)

Dynamics with a conservation law

Let us consider

$$d\phi_t(x) = \Delta \left\{ \frac{\partial H}{\partial \phi(\cdot)}(\phi_t) \right\}(x)dt + \sqrt{2}d\tilde{w}_t(x), \qquad (4)$$

for

$$x \in \Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d \text{ with periodic b.c.}$$
$$x \in D_N = ND \cap \mathbb{Z}^d \text{ with Dirichlet b.c.}$$

- $\tilde{w} = \{\tilde{w}_t(x); x \in \Gamma_N\}$: Gaussian process with covariance structure

$$E[\tilde{w}_s(x)\tilde{w}_t(y)] = -\Delta(x,y)s \wedge t$$

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Dynamics with a conservation law

- Δ: (discrete) Laplacian

$$\Delta f(x) = \sum_{y \in \Gamma_N, |x-y|=1} (f(y) - f(x)), \quad x \in \Gamma_N$$

Remark 2. By Itô's formula, it is easy to see

$$\sum_{x \in \Gamma_N} \phi_t(x) \equiv \sum_{x \in \Gamma_N} \phi_0(x) \, (= \text{const.}), \quad t \geq 0, \tag{5}$$

that is, the total sum of the height variable ($\fallingdotseq$ number of particle) is conserved by this time evolution.

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Hydrodynamic scaling limit on the periodic torus

Macroscopic interface $h^N(t,\theta)(t \in [0,t], \theta \in [0,1)^d =: \mathbb{T}^d)$

$$h^N(t, x/N) = N^{-1}\phi_{N^4 t}(x), \quad x \in \Gamma_N$$

Theorem 2 (N. 2002). *If V is strictly convex, i.e., there exist $c_-, c_+ > 0$ such that*

$$c_- \le V''(\eta) \le c_+, \quad \eta \in \mathbb{R}$$

we have

$$h^N \longrightarrow h : \frac{\partial h}{\partial t} = -\Delta \,\mathrm{div}\, \nabla\sigma(\nabla h)$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}$ is the surface tension introduced via thermodynamic limit.

Problem

What happen in the case with Dirichlet b.c.?

Main Result

Hydrodynamic scaling limit on finite domain

Theorem 3. *Let D be a finite, convex domain with Lipschitz boundary. We assume that there exists $h_0 \in H^{-1}(D)$ such that*

$$\sup_{N \geq 1} E\left[\|h^N(0)\|^2_{H^{-1}(D)}\right] < \infty,$$

$$\lim_{N\to\infty} E\|h^N(0) - h_0\|^2_{H^{-1}(D)} = 0.$$

We then have

$$\lim_{N\to\infty} E\|h^N(t) - h(t)\|^2_{H^{-1}(D)} = 0,$$

where h is the weak solution of nonlinear PDE

$$\frac{\partial h}{\partial t} = \Delta \operatorname{div} \nabla\sigma(\nabla h). \tag{6}$$

Limit equation

$h \in C([0,T], H^{-1}(D)) \cap L^2([0,T], H_0^1(D))$ and for test functions $J_1 \in C^\infty([0,T] \times D)$ and $J_2 \in C_0^1(D)$,

$$\begin{aligned}
\int_D h(t,\theta) J_1(t,\theta)\, d\theta
&= \int_D h_0(\theta) J_1(t,\theta)\, d\theta + \int_0^t \int_D h(s,\theta) \frac{d}{ds} J_1(s,\theta)\, d\theta\, ds \\
&\quad + \int_0^t \int_D \nabla u(s,\theta) \cdot \nabla J_1(s,\theta)\, d\theta\, ds,
\end{aligned}$$

$$\int_D u(t,\theta) J_2(\theta)\, d\theta = - \int_D \nabla \sigma(\nabla h(t)) \cdot \nabla J_2(\theta)\, d\theta$$

Rough sketch of the proof

How to show

The proof is by H^{-1}-method in

- Funaki-Spohn, Commun. Math. Phys. ('97)
- N., Probab. J. Math. Univ. Tokyo ('02)
- N., Probab. Theory Relat. Fields ('03)

What we need to do

Following the results stated befote, we have the conclusion once we have

- a priori bounds for stochastic dynamics
- a priori bounds for discretized equation corresponding to (6)
- uniqueness of ergodic stationary measure
  (In [N. 03] this property plays a key role.)
- establish local equilibrium
- derive PDE (6)

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Notations (1)

- $(\mathbb{Z}^d)^*$: all oriented bonds in $\mathbb{Z}^d$, i.e.

$$(\mathbb{Z}^d)^* = \{(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d; |x-y| = 1\}$$

- Γ_N^*: all oriented bonds in Γ_N
- $\mathcal{X}$: all $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ satisfying the following conditions:

1. $\eta(b) = -\eta(-b)$,
   where $-b = (y,x)$ for $b = (x,y)$.
2. For every closed loops $\mathcal{C}$

$$\sum_{b \in \mathcal{C}} \eta(b) = 0$$

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Notations (2)

- ∇: discrete gradient

$$\nabla\phi(b) = \phi(x) - \phi(y), \quad b = (x, y)$$

- $\Lambda_l = \{x \in \mathbb{Z}^d;\ \max |x_i| \le l\}$
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- $\mathcal{X}_{\Lambda^*} = \{(\nabla\phi(b); b \in \Lambda^*);\ \phi \in \mathbb{R}^\Lambda\}$
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A priori bounds for the SDEs

Proposition 4. *There exists constants $K_1, K_2 > 0$ such that*

$$E\|h^N(t)\|^2_{H^{-1}} + K_1 N^{-d} E \int_0^t \sum_{b \in \overline{D_N^*}} \left(\nabla \phi_s^N(b)\right)^2 ds \le E\|h^N(0)\|^2_{H^{-1}} + K_2(1+t), \quad t > 0$$

holds, where

$$\|h^N\|^2_{-1,N} := N^{-d-4} \sum_{x \in D_N} (\phi^N(x) - \langle \phi^N \rangle) \times (-\Delta_{D_N})^{-1} (\phi^N(x) - \langle \phi^N \rangle) + N^{-2d-2} \langle \phi^N \rangle^2.$$

Discretization for PDE

Let us consider a system of ODEs

$$\begin{cases} \frac{\partial}{\partial t}\bar{h}^N(t,x/N) = -\Delta_N u_N(x/N), \quad x \in D_N \\ \qquad u_N = \mathrm{div}_N\{(\nabla\sigma)(\nabla^N \bar{h}^N(t))\}(x/N), \quad x \in D_N \\ \bar{h}^N(t,x/N) = 0, \quad x \notin D_N. \end{cases} \tag{7}$$

and we extend $\bar{h}^N$ to the function from $[0,T] \times \mathbb{R}^d$ by interpolation as follows:

$$\bar{h}^N(t,\theta) = \bar{h}^N(t,x/N), \quad x \in \mathbb{Z}^d.$$

We consider the solution with initial datum

$$\bar{h}_0^N(x/N) = N^d \int_{B(x/N,1/N)} h_0(\theta')\, d\theta', \quad h_0 \in C_0^2(D).$$

A priori bound for the discretized PDE

Proposition 5. *If initial data is smooth enough, then there exists a constant $C := C(T, h_0)$ such that*

$$\sup_N \sup_{0\le t\le T} \left(\|\bar{h}^N(t)\|^2_{-1,N} + \|\nabla^N h^N(t)\|^2_{L^2} \right) \le C,$$

$$\sup_N \sup_{0\le t\le T} \left\| \frac{d}{dt}\bar{h}^N(t) \right\|^2_{-1,N} \le C,$$

$$\sup_N \int_0^T \|u^N(t)\|^p_{L^p}\, dt \le C,$$

$$\sup_N \int_0^T \|\nabla^N u^N(t)\|^p_{L^p}\, dt \le C$$

holds.

Gibbs measures on the gradient field

- $\mu_{\Lambda,\xi}$: finite volume Gibbs measure on $\mathcal{X}_{\Lambda,\xi}$, i.e.,

$$\mu_{\Lambda,\xi}(d\eta) = \frac{1}{Z_{\Lambda,\xi}} \exp(-H(\eta)) d\eta_{\Lambda,\xi},$$

where $Z_{\Lambda,\xi}$ is a normalizing constant.

- μ: Grandcanonical Gibbs measure on $\mathcal{X}$ iff μ satisfies DLR equation

$$\mu(\cdot|\mathscr{F}_{(\mathbb{Z}^d)^* \setminus \overline{\Lambda^*}})(\xi) = \mu_{\Lambda,\xi}(\cdot), \quad \mu\text{-a.s. } \xi,$$

holds for every finite set $\Lambda \subset \mathbb{Z}^d$.

- μ_u: shift-invariant ergodic Gibbs meas. on gradient field $\mathcal{X}$ with mean $u \in \mathbb{R}^d$, i.e.,

$$E^{\mu_u}[\eta((e_i, 0))] = u_i, \quad 1 \le i \le d$$

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Dynamics on the gradient field

For the solution ϕ_t of SDE (1), $\eta_t = \nabla \phi_t$ satisfies

$$d\eta_t(b) = -\nabla \Delta U.(\eta_t)(b)\,dt + \sqrt{2}d\nabla \tilde{w}_t(b), \qquad (8)$$

where

$$U_x(\eta) := \sum_{b:x_b=x} V'(\eta(b))$$

Generator for the SDE on $(\mathbb{Z}^d)^*$

The generator for (8) is given by

$$\mathscr{L} = \sum_{x \in \mathbb{Z}^d} \mathscr{L}_x,$$

$$\mathscr{L}_x = -\partial_x \Delta \partial(x) + \Delta U.(x)\partial_x,$$

$$\partial_x = 2 \sum_{b : x_b = x} \frac{\partial}{\partial \eta(b)}$$

Stationary measures and Gibbs measures

Theorem 6. *Let a measure μ on $\mathscr{X}$ be invariant under spatial shift and tempered, that is,*

$$E^{\mu}[\eta(b)^2] < \infty, \quad b \in \left(\mathbb{Z}^d\right)^* .$$

holds. If μ is a stationary measure corresponding $\mathscr{L}$, i.e.,

$$\int_{\mathcal{X}} \mathscr{L} f(\eta)\mu(d\eta) = 0$$

holds for every $f \in C^2_{\mathrm{loc}}(\mathcal{X})$, μ is then a grandcanonical Gibbs measure.

Connection to the large deviation problem

If $d \leq 3$, the large deviation problem can be shown (it is reported at the workshop held at Warwick). The restriction "$d \leq 3$" is from the luck of infomation on the stationary measures. Once we have Theorem 6, the result can be extened to arbitrary cases.

Proof of Theorem 6 (1)

We shall apply the same method in [Deuschel-N.-Vignard, in preparation], which is based on [Fritz, 1982]. Our goal is the following:

$$\lim_{n\to\infty} n^{-d} I_{\Lambda_n}(\mu|_{\Lambda_n}) = 0,$$

where

- $I_{\Lambda_n}(\nu) = \mathscr{E}_{\Lambda_n}(\sqrt{f}, \sqrt{f}), \quad f = \dfrac{d\nu}{d\mu_{\Lambda_n}}$
- μ_{Λ_n}: finite volume Gibbs measure on Λ_n with free boundary condition
- $\mathscr{E}_{\Lambda_n}$: Dirichlet form for the time evolution with free boundary condition

Once we have the above, we obtain that μ is canonical Gibbs measure. However, in this setting, the canonical Gibbs measure is also grandcanonical, thus we have the conclusion.

Proof of Theorem 6 (2)

From stationarity, we have

$$\int \mathscr{L}\psi_n(\cdot,\xi)(\eta)\mu(d\eta) = 0,$$

where $\psi_n(\eta,\xi) \in C^2_{\mathrm{loc}}(\mathcal{X} \times \mathcal{X})$.

Multiplying $F \in C^2_{\mathrm{loc}}(\mathcal{X})$ and integrating in ξ, we obtain

$$\iint F(\xi)\mathscr{L}\psi_n(\cdot,\xi)(\eta)\mu(d\eta)\nu_{\Lambda_n^*}(d\xi) = 0. \tag{9}$$

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Proof of Theorem 6 (3)

Roughly saying, if we can take F as

$$F(\xi) = \log\left(\frac{d\mu|_{\Lambda_n}}{d\mu_{\Lambda_n}}(\xi)\right)$$

and suitable ψ_n, we can obtain the entropy production and error terms from LHS of (9).