Hydrodynamic limit for the Ginzburg-Landau $\nabla \phi$ interface model with a conservation law and the Dirichlet boundary condition

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Workshop “Gradient Random Fields”
Model

- Microscopic interface
- Energy of microscopic interface
- Dynamics - Langevin equation
- Hydrodynamic scaling limit (LLN)
- Total surface tension
- Dynamics with a conservation law
- Hydrodynamic scaling limit on the periodic torus
- Problem

Main Result

Rough sketch of the proof
Microscopic interface

Interface $\phi = \{\phi(x) \in \mathbb{R}; x \in \mathbb{Z}^d\}$

$\phi(x)$: the height at position $x$

Main Result
Rough sketch of the proof
Energy of microscopic interface

Energy of the microscopic interface \( \phi = \{\phi(x) \in \mathbb{R}; x \in \mathbb{Z}^d\} \)

\[
H(\phi) = \frac{1}{2} \sum_{x, y \in \mathbb{Z}^d, |x - y| = 1} V(\phi(x) - \phi(y))
\]

\((V : \mathbb{R} \rightarrow \mathbb{R} \text{ is } C^2, \text{ symm., } \|V''\|_\infty < \infty)\)
Dynamics - Langevin equation

Langevin eq.

\[
d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t) dt + \sqrt{2}dw_t(x),
\]

for

\[
x \in \Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d \text{ with periodic b.c.}
\]

\[
x \in D_N = ND \cap \mathbb{Z}^d \text{ with Dirichlet b.c.}
\]

\[
\begin{align*}
  w &= \{w_t(x); x \in \Gamma_N\}: \text{independent 1D B.m.'s} \\
  \frac{\partial H}{\partial \phi(x)} &= \sum_{y:|x-y|=1} V'(\phi(x) - \phi(y))
\end{align*}
\]
Dynamics - Langevin equation

Langevin eq.

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- \( w = \{w_t(x); x \in \Gamma_N\} \): independent 1D B.m.'s
- \( \frac{\partial H}{\partial \phi(x)} = \sum_{y:|x-y|=1} V'(\phi(x) - \phi(y)) \)
Hydrodynamic scaling limit (LLN)

Macroscopic interface $h^N(t, \theta)$

$$h^N(t, x/N) = N^{-1} \phi_{N^2}t(x), \quad x \in \Gamma_N$$

Theorem 1 (Funaki-Spohn for $\Gamma_N$, N. for $D_N$ with Dirichlet b.c.). If $V$ is strictly convex, i.e., there exist $c_-, c_+ > 0$ such that

$$c_- \leq V''(\eta) \leq c_+, \quad \eta \in \mathbb{R}$$

we have

$$h^N \rightarrow h : \frac{\partial h}{\partial t} = \text{div} \nabla \sigma(\nabla h)$$

where $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$ is the surface tension introduced via thermodynamic limit.
Total surface tension

The equation (2) is the gradient flow with respect to the energy functional

$$\Sigma(h) = \int \sigma(\nabla h(\theta)) \, d\theta$$

in $L^2$-space. The functional $\Sigma$ is called "total surface tension," which gives the total energy of the interface $h$.

Remark 1. The assumption "$V$ is strictly convex" can be relaxed. If we have the convexity of $\sigma$ (see Cotar-Deuschel-Müller and Cotar-Deuschel) and the characterization of Gibbs measures for gradient fields, we can show the hydrodynamic limit. (joint work with J.-D. Deuschel and I. Vignard)
Dynamics with a conservation law

Let us consider

\[ d\phi_t(x) = \Delta \left\{ \frac{\partial H}{\partial \phi(\cdot)}(\phi_t) \right\} (x) dt + \sqrt{2} d\tilde{w}_t(x), \quad (4) \]

for

\[ x \in \Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d \quad \text{with periodic b.c.} \]
\[ x \in D_N = N D \cap \mathbb{Z}^d \quad \text{with Dirichlet b.c.} \]

\[ \tilde{w} = \{\tilde{w}_t(x); x \in \Gamma_N\}: \text{Gaussian process with covariance structure} \]

\[ E[\tilde{w}_s(x)\tilde{w}_t(y)] = -\Delta(x, y) s \wedge t \]
Dynamics with a conservation law

Let us consider

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d\phi_t(x) = \Delta \left\{ \frac{\partial H}{\partial \phi(\cdot)}(\phi_t) \right\}(x)dt + \sqrt{2}d\tilde{w}_t(x), \tag{4}
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\[E[\tilde{w}_s(x)\tilde{w}_t(y)] = -\Delta(x,y)s \wedge t\]
Dynamics with a conservation law

- \( \Delta \): (discrete) Laplacian

\[
\Delta f(x) = \sum_{y \in \Gamma_N, |x-y|=1} (f(y) - f(x)), \quad x \in \Gamma_N
\]

**Remark 2.** By Itô’s formula, it is easy to see

\[
\sum_{x \in \Gamma_N} \phi_t(x) \equiv \sum_{x \in \Gamma_N} \phi_0(x) \ (= \text{const.}), \quad t \geq 0, \quad (5)
\]

that is, the total sum of the height variable (\(\equiv\) number of particle) is conserved by this time evolution.
Dynamics with a conservation law

- **Δ**: (discrete) Laplacian

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Hydrodynamic scaling limit on the periodic torus

Macroscopic interface $h^N(t, \theta)(t \in [0, t], \theta \in [0, 1)^d =: \mathbb{T}^d)$

$$h^N(t, x/N) = N^{-1} \phi_{N^4t}(x), \quad x \in \Gamma_N$$

**Theorem 2** (N. 2002). If $V$ is strictly convex, i.e., there exist $c_-, c_+ > 0$ such that

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we have

$$h^N \longrightarrow h : \frac{\partial h}{\partial t} = -\Delta \text{div} \nabla \sigma(\nabla h)$$

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Problem

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Main Result
Rough sketch of the proof

What happen in the case with Dirichlet b.c.?
Main Result

- Hydrodynamic scaling limit on finite domain
- Limit equation

Rough sketch of the proof
Hydrodynamic scaling limit on finite domain

**Theorem 3.** Let $D$ be a finite, convex domain with Lipschitz boundary. We assume that there exists $h_0 \in H^{-1}(D)$ such that

\[
\sup_{N \geq 1} E \left[ \| h^N(0) \|^2_{H^{-1}(D)} \right] < \infty,
\]

\[
\lim_{N \to \infty} E \| h^N(0) - h_0 \|^2_{H^{-1}(D)} = 0.
\]

We then have

\[
\lim_{N \to \infty} E \| h^N(t) - h(t) \|^2_{H^{-1}(D)} = 0,
\]

where $h$ is the weak solution of nonlinear PDE

\[
\frac{\partial h}{\partial t} = \Delta \text{div} \nabla \sigma(\nabla h).
\]
Model

Main Result

- Hydrodynamic scaling limit on finite domain
- Limit equation

Rough sketch of the proof

Limit equation

\[
h \in C([0, T], H^{-1}(D)) \cap L^2([0, T], H^1_0(D)) \text{ and for test functions } J_1 \in C^{\infty}([0, T] \times D) \text{ and } J_2 \in C^1_0(D),
\]

\[
\int_D h(t, \theta) J_1(t, \theta) \, d\theta = \int_D h_0(\theta) J_1(t, \theta) \, d\theta + \int_0^t \int_D h(s, \theta) \frac{d}{ds} J_1(s, \theta) \, d\theta \, ds
\]

\[
+ \int_0^t \int_D \nabla u(s, \theta) \cdot \nabla J_1(s, \theta) \, d\theta \, ds,
\]

\[
\int_D u(t, \theta) J_2(\theta) \, d\theta = - \int_D \nabla \sigma(\nabla h(t)) \cdot \nabla J_2(\theta) \, d\theta
\]
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How to show

The proof is by $H^{-1}$-method in

- Funaki-Spohn, Commun. Math. Phys. ('97)
- N., Probab. Theory Relat. Fields ('03)
What we need to do

Following the results stated before, we have the conclusion once we have

- a priori bounds for stochastic dynamics
- a priori bounds for discretized equation corresponding to (6)
- uniqueness of ergodic stationary measure
  (In [N. 03] this property plays a key role.)
- establish local equilibrium
- derive PDE (6)
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Notations (1)

- \((\mathbb{Z}^d)^*\): all oriented bonds in \(\mathbb{Z}^d\), i.e.

\[
(\mathbb{Z}^d)^* = \{ (x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d; |x - y| = 1 \}
\]

- \(\Gamma_N^*\): all oriented bonds in \(\Gamma_N\)
- \(\mathcal{X}\): all \(\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}\) satisfying the following conditions:
  1. \(\eta(b) = -\eta(-b)\),
     where \(-b = (y, x)\) for \(b = (x, y)\).
  2. For every closed loops \(\mathcal{C}\)

\[
\sum_{b \in \mathcal{C}} \eta(b) = 0
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holds.
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Notations (2)

- $\nabla$: discrete gradient

\[ \nabla \phi(b) = \phi(x) - \phi(y), \quad b = (x, y) \]

- $\Lambda_l = \{ x \in \mathbb{Z}^d; \max |x_i| \leq l \}$
- $\Lambda^*_l = \{ (x, y) \in (\mathbb{Z}^d)^*; x, y \in \Lambda_l \}$
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A priori bounds for the SDEs

**Proposition 4.** There exists constants $K_1, K_2 > 0$ such that

$$E\|h^N(t)\|_{H^{-1}}^2 + K_1 N^{-d} E \int_0^t \sum_{b \in D_N^*} (\nabla \phi^N_s(b))^2 \, ds$$

$$\leq E\|h^N(0)\|_{H^{-1}}^2 + K_2 (1 + t), \quad t > 0$$

holds, where

$$\|h^N\|_{-1,N}^2 := N^{-d-4} \sum_{x \in D_N} (\phi^N(x) - \langle \phi^N \rangle)$$

$$\times (-\Delta_{D_N})^{-1}(\phi^N(x) - \langle \phi^N \rangle)$$

$$+ N^{-2d-2} \langle \phi^N \rangle^2.$$
Let us consider a system of ODEs

\[
\begin{align*}
\frac{\partial}{\partial t} \bar{h}^N(t, x/N) &= -\Delta_N u_N(x/N), \quad x \in D_N \\
u_N &= \text{div}_N \{ (\nabla \sigma)(\nabla^N \bar{h}^N(t)) \}(x/N), \quad x \in D_N \\
\bar{h}^N(t, x/N) &= 0, \quad x \notin D_N.
\end{align*}
\]

and we extend \( \bar{h}^N \) to the function from \([0, T] \times \mathbb{R}^d\) by interpolation as follows:

\[
\bar{h}^N(t, \theta) = \bar{h}^N(t, x/N), \quad x \in \mathbb{Z}^d.
\]

We consider the solution with initial datum

\[
\bar{h}^N_0(x/N) = N^d \int_{B(x/N, 1/N)} h_0(\theta') \, d\theta', \quad h_0 \in C^2_0(D).
\]
**A priori bound for the discretized PDE**

**Proposition 5.** If initial data is smooth enough, then there exists a constant $C := C(T, h_0)$ such that

\[
\sup_N \sup_{0 \leq t \leq T} \left( \| \bar{h}^N(t) \|_{-1,N}^2 + \| \nabla^N h^N(t) \|_{L^2}^2 \right) \leq C,
\]

\[
\sup_N \sup_{0 \leq t \leq T} \left\| \frac{d}{dt} \bar{h}^N(t) \right\|_{-1,N}^2 \leq C,
\]

\[
\sup_N \int_0^T \| u^N(t) \|_{L^p}^p \, dt \leq C,
\]

\[
\sup_N \int_0^T \| \nabla^N u^N(t) \|_{L^p}^p \, dt \leq C
\]

holds.
Gibbs measures on the gradient field

- \( \mu_{\Lambda,\xi} \): finite volume Gibbs measure on \( \mathcal{X}_{\Lambda,\xi} \), i.e.,

\[
\mu_{\Lambda,\xi}(d\eta) = \frac{1}{Z_{\Lambda,\xi}} \exp(-H(\eta))d\eta_{\Lambda,\xi},
\]

where \( Z_{\Lambda,\xi} \) is a normalizing constant.

- \( \mu \): Grandcanonical Gibbs measure on \( \mathcal{X} \) iff \( \mu \) satisfies DLR equation

\[
\mu(\cdot|\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\Lambda^*})(\xi) = \mu_{\Lambda,\xi}(\cdot), \quad \mu\text{-a.s. } \xi,
\]

holds for every finite set \( \Lambda \subset \mathbb{Z}^d \).

- \( \mu_u \): shift-invariant ergodic Gibbs meas. on gradient field \( \mathcal{X} \) with mean \( u \in \mathbb{R}^d \), i.e.,

\[
E^{\mu_u}[\eta((e_i, 0))] = u_i, \quad 1 \leq i \leq d
\]
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- $\mu$: Grandcanonical Gibbs measure on $\mathcal{X}$ iff $\mu$ satisfies DLR equation

$$\mu(\cdot|\mathcal{F}(\mathbb{Z}^d)^*\setminus\Lambda^*)\xi = \mu_{\Lambda,\xi}(\cdot), \text{ } \mu\text{-a.s. } \xi,$$

holds for every finite set $\Lambda \subset \mathbb{Z}^d$.

- $\mu_u$: shift-invariant ergodic Gibbs measure on gradient field $\mathcal{X}$ with mean $u \in \mathbb{R}^d$, i.e.,

$$E^{\mu_u}[\eta((e_i,0))] = u_i, \quad 1 \leq i \leq d$$
For the solution \( \phi_t \) of SDE (1), \( \eta_t = \nabla \phi_t \) satisfies

\[
d\eta_t(b) = -\nabla \Delta U(\eta_t)(b) \, dt + \sqrt{2} d\nabla \tilde{w}_t(b), \tag{8}
\]

where

\[
U_x(\eta) := \sum_{b : x_b = x} V'(\eta(b))
\]
Generator for the SDE on \((\mathbb{Z}^d)^*\)

The generator for (8) is given by

\[
\mathcal{L} = \sum_{x \in \mathbb{Z}^d} \mathcal{L}_x,
\]

\[
\mathcal{L}_x = -\partial_x \Delta \partial (x) + \Delta U.(x) \partial_x,
\]

\[
\partial_x = 2 \sum_{b: x_b = x} \frac{\partial}{\partial \eta(b)}
\]
Theorem 6. Let a measure $\mu$ on $\mathcal{X}$ be invariant under spatial shift and tempered, that is,

$$E^\mu[\eta(b)^2] < \infty, \quad b \in (\mathbb{Z}^d)^*.$$ 

holds. If $\mu$ is a stationary measure corresponding $\mathcal{L}$, i.e.,

$$\int_{\mathcal{X}} \mathcal{L} f(\eta) \mu(d\eta) = 0$$

holds for every $f \in C^2_{\text{loc}}(\mathcal{X})$, $\mu$ is then a grandcanonical Gibbs measure.
Connection to the large deviation problem

If $d \leq 3$, the large deviation problem can be shown (it is reported at the workshop held at Warwick). The restriction “$d \leq 3$” is from the luck of information on the stationary measures. Once we have Theorem 6, the result can be extended to arbitrary cases.
Proof of Theorem 6 (1)

We shall apply the same method in [Deuschel-N.-Vignard, in preparation], which is based on [Fritz, 1982]. Our goal is the following:

$$\lim_{n \to \infty} n^{-d} I_{\Lambda_n} (\mu|_{\Lambda_n}) = 0,$$

where

- \( I_{\Lambda_n}(\nu) = \mathcal{E}_{\Lambda_n}(\sqrt{f}, \sqrt{f}) \), \( f = \frac{d\nu}{d\mu_{\Lambda_n}} \)
- \( \mu_{\Lambda_n} \): finite volume Gibbs measure on \( \Lambda_n \) with free boundary condition
- \( \mathcal{E}_{\Lambda_n} \): Dirichlet form for the time evolution with free boundary condition

Once we have the above, we obtain that \( \mu \) is canonical Gibbs measure. However, in this setting, the canonical Gibbs measure is also grandcanonical, thus we have the conclusion.
Proof of Theorem 6 (2)

From stationarity, we have

$$\int \mathcal{L} \psi_n(\cdot, \xi)(\eta) \mu(d\eta) = 0,$$

where $\psi_n(\eta, \xi) \in C^2_{\text{loc}}(\mathcal{X} \times \mathcal{X}).$

Multiplying $F \in C^2_{\text{loc}}(\mathcal{X})$ and integrating in $\xi$, we obtain

$$\iint F(\xi) \mathcal{L} \psi_n(\cdot, \xi)(\eta) \mu(d\eta) \nu_{\Lambda_n^*}(d\xi) = 0. \quad (9)$$
From stationarity, we have

\[ \int \mathcal{L} \psi_n (\cdot, \xi)(\eta) \mu(d\eta) = 0, \]

where \( \psi_n (\eta, \xi) \in C^2_{\text{loc}} (\mathcal{X} \times \mathcal{X}) \).

Multiplying \( F \in C^2_{\text{loc}} (\mathcal{X}) \) and integrating in \( \xi \), we obtain

\[ \iint F(\xi) \mathcal{L} \psi_n (\cdot, \xi)(\eta) \mu(d\eta) \nu_{\Lambda_n^*}(d\xi) = 0. \]
Proof of Theorem 6 (3)

Roughly saying, if we can take $F$ as

$$F(\xi) = \log \left( \frac{d\mu|_{\Lambda_n}}{d\mu_{\Lambda_n}}(\xi) \right)$$

and suitable $\psi_n$, we can obtain the entropy production and error terms from LHS of (9).