Christof Külske

Discrete approximations to massless models

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(1) Discretizations of two types for rotator models

(2) Gibbs under local Transforms? - two criteria

(3) Discretizations of rotator measures are Gibbs for small $\frac{\beta}{q^2}$

(4) Discretizations of rotator measures are non-Gibbs for large $\frac{\beta}{q^2}$

(5) Comparison with Fröhlich-Spencer for clock model

with: van Enter, Opoku

effective interface = unbounded spin variables: $\varphi = (\varphi_i)_{i \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$

classical bounded spins = rotators $\varphi = (\varphi_i)_{i \in \mathbb{Z}^d} \in (S^1)^{\mathbb{Z}^d}$

Hamiltonian $\equiv$ energy function

$$H(\varphi) = \frac{1}{2} \sum_{i,j} p(i-j) V(\varphi_i - \varphi_j)$$

pair-potential $V(t) = V(-t)$

$p(i-j)$ = finite range random walk kernel

$$H(\varphi) = \frac{1}{2} \sum_{i,j} p(i-j) U(\varphi_i - \varphi_j)$$

pair-potential $U(t) = U(-t)$

We will consider Gibbs measures, which are defined for (here translation-invariant) absolutely summable interactions Φ (that is, $\sum_{A;\ 0\in A} ||\Phi_A|| < \infty$) via the DLR equations, expressing that given an external configuration η_{Λ^c}, the probability density of configurations in a volume Λ is given by the Gibbs expression

$$\frac{d\mu_\Lambda^{\eta_{\Lambda^c}}}{d\alpha_\Lambda}(\sigma_\Lambda) = \frac{\exp\left(-H_\Lambda(\sigma_\Lambda\eta_{\Lambda^c})\right)}{Z_\Lambda^{\eta_{\Lambda^c}}}, \quad \text{where} \quad H_\Lambda(\sigma_\Lambda\eta_{\Lambda^c}) = \sum_{A;\ A\cap\Lambda\neq\emptyset} \beta\Phi_A(\sigma_\Lambda\eta_{\Lambda^c}),$$

and α_Λ is the product of α over the sites in Λ.

In the standard nearest-neighbour models, (the plane rotor or XY-model), as well as in the clock models, where the spins take discrete values, we have

$$-H_\Lambda(\sigma_\Lambda\eta_{\Lambda^c}) = \beta \sum_{<i,j>\in\Lambda} \sigma_i \cdot \sigma_j + \beta \sum_{<i\in\Lambda, j\in\Lambda^c>} \sigma_i \cdot \eta_j.$$

Local discretisation map $T : S^1 \mapsto \{1, \ldots, q\}$
map spin to the midpoints of segments

AIM: Compare $T\mu$ where $\mu \in \mathcal{G}_{\Phi,\alpha}$ to the Gibbs measures $\mu' \in \mathcal{G}_{\Phi,T\alpha}$
where $T\alpha$ is the product.

THEOREM EFS If there is a renormalized interaction for one translation-invariant Gibbs measure, then the renormalized measure of any other translation-invariant measure is a Gibbs measure for the same interaction.

If there is an Φ' such that $T\mu \in \mathcal{G}_{\Phi',T\alpha}$ for a $\mu \in \mathcal{G}_{\Phi,\alpha}$,
then $T\mathcal{G}_{\Phi,\alpha} \subset \mathcal{G}_{\Phi',T\alpha}$.

EFS93: A.C.D. van Enter, R. Fernández, A.D. Sokal: Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, J. Stat. Phys. 72, 879-1167 (1993).

Compare the fuzzy Potts model.

$$\mu_\Lambda(\sigma) = \frac{1}{Z_\Lambda} \exp\Big(2\beta \sum_{i\sim j} 1_{\sigma_i=\sigma_j}\Big)$$

$\sigma_i \in \{1,\dots,q\}$ local state space is decomposed into classes

apply the sitewise transformation: $T : \{1,\dots,q\} \to \{1,\dots,q'\}$, $q' < q$

Questions: Gibbsianness on Lattice, Tree, Meanfield?
(Maes, Haeggstroem, Kuelske)

Known: $T\mu$ can be non-Gibbs at low temperatures

First layer spin variables: $\sigma = (\sigma_i)_{i \in G} \in S^G$, (S, d) metric space

Interaction potential only on first layer: $\Phi = (\Phi_A(\sigma_A))_{A \subset G}$

Local a priori measure on the first layer: $\alpha(d\sigma_i)$

Hamiltonian $\equiv$ Formal energy function: $H(\sigma) = \sum_{A: A \subset G} \Phi_A(\sigma)$

Second layer spin variables: $\eta = (\eta_i)_{i \in G} \in S'^G$, (S', d') metric space

Joint spin variables: $\xi = (\sigma_i, \eta_i)_{i \in G} \in (S \times S')^G$

Joint local a priori measure: $K(d\sigma_i, d\eta_i) = \alpha(d\sigma_i) K(d\eta_i | \sigma_i)$

with a stochastic (or deterministic) transformation $K(d\eta_i | \sigma_i)$

Let $\mu(d\sigma)$ be a Gibbs measure for the first layer

Aim: Study the second layer measure

$$\mu'(d\eta) = \int_{S^G} \mu(d\sigma) \prod_{i \in G} K(d\eta_i | \sigma_i)$$

η are noisy observations of σ's

η are gene activities caused by the genes σ

η are time-evolved initial spins σ

η are coarse-grained images of σ

- van Enter, Fernandez, den Hollander, Redig (CMP 2002):
  Ising under spinflip
- Külske, Redig (PTRF 2006):
  Unbounded Continuous variables under Diffusions
- Külske, LeNy (CMP 2007):
  Mean-Field Ising - symmetry breaking in bad configurations
- Külske, Opoku (EJP, JMP 2008) Goodness of Gibbsianness,
  Lattice vs. Meanfield
- van Enter, Ruszel (JMP 2008, SPA 2009):
  Bounded Continuous variables (circle) under Diffusions
- Enter, Kuelske, Opoku, Ruszel (2010):
  Gibbs-non-Gibbs properties for n-vector lattice and mean-field models

Initial system: Nearest neighbor Ising model $\mu_{t=0} := \mu^{+}_{\beta,h}$

The dynamics:

symmetric independent spin-flips:

$$\mu_t(\eta_\Lambda) = \int \mu_{t=0}(d\sigma_\Lambda) \prod_{x\in\Lambda} p_t(\sigma_x, \eta_x)$$

transition kernel for rate-1 flips: $p_t(+,+) = \frac{1}{2}(1 + e^{-2t})$

$$\Big(p_t(+,+) = p_t(-,-) = 1 - p_t(+,-) = 1 - p_t(-,+)\Big)$$

$\Rightarrow$ trivial infinite-time limiting measure (locally):

$$\lim_{t\uparrow\infty} \mu_t = \bigotimes_{x\in\mathbb{Z}^d} \frac{1}{2}\Big(\delta_+ + \delta_-\Big)$$

$\mu_{\beta,h=0,t}$ **fails to be Gibbs for β large, t large** due to "hidden phase transitions"

Constrained (quenched) first layer Model: Study Gibbs specification

$$\gamma_\Lambda[\eta_\Lambda](d\sigma_\Lambda|\bar\sigma) = \frac{\exp\Big(-\sum_{A\cap\Lambda\neq\emptyset}\Phi_A(\sigma_\Lambda\bar\sigma_{\Lambda^c})\Big)\prod_{i\in\Lambda}\alpha_{\eta_i}(d\sigma_i)}{\int_{E^\Lambda}\exp\Big(-\sum_{A\cap\Lambda\neq\emptyset}\Phi_A(\tilde\sigma_\Lambda\bar\sigma_{\Lambda^c})\Big)\prod_{i\in\Lambda}\alpha_{\eta_i}(d\tilde\sigma_i)}$$

where $\alpha_{\eta_i} = K(d\sigma_i|\eta_i)$

Sufficient for μ' to be Gibbs:

$\gamma^\Phi[\eta]$ satisfies Dobrushin-condition

uniformly in second layer configurations η

”Information of conditioning passes through first layer in a local way”

If: 1st Layer is Gibbs (but not necessarily in uniqueness regime) & Constrained first layer in uniqueness regime

Then: 2nd Layer is Gibbs (but not necessarily in uniqueness regime)

$$\left\| \gamma'_i(\cdot|\eta_{i^c}) - \gamma'_i(\cdot|\bar{\eta}_{i^c}) \right\| \leq \sum_j Q_{i,j} d'(\eta_j, \bar{\eta}_j)$$

Definition: Posterior-metric associated to K on second layer local spin space

$$d'(\eta_j, \bar{\eta}_j) := \|\alpha_{\eta_j} - \alpha_{\bar{\eta}_j}\|$$

is a measure of relevance of a local variation of the second layer on the first layer

Dobrushin Uniqueness condition (1968):

$$C_{i,j} := \sup_{\sigma=\sigma' \text{ on } j^c} \|\gamma_i(\,\cdot\,|\sigma_{i^c}) - \gamma_i(\,\cdot\,|\sigma'_{i^c})\|_i$$
$$\equiv \text{ Dobrushin-Matrix}$$

$$\text{Dobrushin-constant} \equiv c \equiv \sup_{i\in G} \sum_{j\in G} C_{i,j} = \|C\|_\infty < 1$$

$$\|\nu_1 - \nu_2\| := \sup_{f:|f|\leq 1} |\nu_1(f) - \nu_2(f)| = \frac{\text{Variational distance}}{2}$$

***THEOREM* 1.** *(Külske, Opoku EJP, Ph.D. thesis Opoku Groningen) Suppose* $\sup_i \sum_j \bar{C}_{ij} < 1$ *where*

$\bar{C}_{ij}$ *bounds the first layer constaint model uniformly*

1. *Then μ' is a Gibbs measure for a specification γ'.*
2. *γ' satisfies the continuity estimate*

$$\left\| \gamma'_i(d\eta_i|\eta_{i^c}) - \gamma'_i(d\eta_i|\bar{\eta}_{i^c}) \right\| \leq \sum_{j \in G \setminus i} Q_{i,j} d'(\eta_j, \bar{\eta}_j).$$

where

$$Q_{i,j} = 4e^{2\sum_{A \ni i} \|\Phi_A\|_\infty} \Big(\sum_{k \in G \setminus i} \delta_k \Big(\sum_{A \supset \{i,k\}} \Phi_A \Big) \bar{D}_{kj} \Big) e^{\sum_{A \ni j} \delta_j(\Phi_A)}$$

with $\bar{D} = \sum_{n=0}^{\infty} \bar{C}^n$.

- Local state spaces: $E = E' = S^{q-1},\ q \geq 2$,
- Hamiltonian:
$$H(\sigma) = -\sum_{i,j\in\mathbb{Z}^d} J_{ij}\sigma_i \cdot \sigma_j$$
with $\sup_i \sum_j |J_{ij}| < \infty$
- Joint single-site a priori probability measure:
$$K(d\sigma_i, d\eta_i) = K_t(d\sigma_i, d\eta_i) = k_t(\sigma_i, \eta_i)\alpha_0(d\sigma_i)\alpha_0(d\eta_i),$$
- $\alpha_0 \equiv$ equidistribution on S^{q-1} and $k_t(\sigma_i, \eta_i) = e^{\Delta t}(\sigma_i, \eta_i)$ heat kernel on the sphere
- Image or time-evolved measure:
$$\mu_t(d\eta) = \int \mu(d\sigma) \prod_i k_t(\sigma_i, \eta_i)\alpha_0(d\eta_i).$$
- Infinite-time local limiting measure = product over the equidistributions on the spheres

short times imply strongly concentrated α_{η_i}

THEOREM 2. *Assume that*

$$\sqrt{2}\left(\sup_i \sum_{j\in G} e^{|J_{ij}|}|J_{ij}|\right)\left(1-e^{-(q-1)t}\right)^{\frac{1}{2}} < 1,$$

then

1. *the measure μ_t is Gibbs for a specification γ_t, and*
2. *γ_t satisfies the continuity estimate*

$$\left\|\gamma_{i,t}(d\eta_i|\eta_{i^c}) - \gamma_{i,t}(d\eta_i|\bar{\eta}_{i^c})\right\| \leq \sum_{j\in G\setminus i} \bar{Q}_{i,j}(t) d(\eta_j, \bar{\eta}_j)$$

with

$$\bar{Q}_{i,j}(t) = \frac{1}{2}\min\left\{\sqrt{\frac{\pi}{t}} Q_{i,j}(t), e^{4\sum_l |J_{jl}|} - 1\right\}$$

$$Q_{i,j}(t) = 8e^{4\sup_{i\in G}\sum_{j\in G}|J_{ij}|} \sum_{k\in G\setminus i} |J_{ik}|\bar{D}_{kj}(t),$$

Spatial decay given by $\bar{D}(t) = \sum_{n=0}^{\infty}\left(1-e^{-(q-1)t}\right)^{\frac{n}{2}} A^n$

with $A_{ij} = e^{|J_{ij}|}|J_{ij}|$

Possible to adopt previous Theorem on Gibbsianness of local transforms to local discretizations, but better use another version:

Recall: Gibbsianness of renormalized model follows if constrained model in the first layer is in Dobrushin uniqueness regime *uniformly in the chosen constraint*

1D-Rotator: The discretized measure $T\mu$ is Gibbs if

$$\sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d \setminus \{i\}} \bar{C}_{ij} < 1,$$

where

$$2\,\bar{C}_{ij} = \begin{cases} \sup_{\substack{\eta_j, \bar{\eta}_j \in S^1; \\ T(\eta_j) = T(\bar{\eta}_j), l \in S'}} \int_{S_l^1} \alpha(d\sigma_i) \left| \frac{e^{\beta\,\sigma_i \cdot \eta_j}}{\int_{S_l^1} \alpha(d\widehat{\sigma}_i)\, e^{\beta\,\widehat{\sigma}_i \cdot \eta_j}} - \frac{e^{\beta\,\sigma_i \cdot \bar{\eta}_j}}{\int_{S_l^1} \alpha(d\widehat{\sigma}_i)\, e^{\beta\,\widehat{\sigma}_i \cdot \bar{\eta}_j}} \right|, & \text{if} |i - j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Decomposition $S = \bigcup_{s' \in S'} S_{s'}$
Note: S' finite or countably infinite
Discretization $T(s) := s'$ for $S_{s'} \ni s$.

G vertex set of underlying graph
Family of metrics $(d_{ij})_{j \in G \setminus \{i\}}$ on the local spin space at the site i

$$d_{ij}(\sigma_i, \tau_i) := \sup_{\substack{\zeta, \bar{\zeta} \\ \zeta_{j^c} = \bar{\zeta}_{j^c}; T(\zeta_j) = T(\bar{\zeta}_j)}} \Big| H_i(\sigma_i \zeta_{i^c}) - H_i(\sigma_i \bar{\zeta}_{i^c}) - \Big(H_i(\tau_i \zeta_{i^c}) - H_i(\tau_i \bar{\zeta}_{i^c}) \Big) \Big|,$$

change of energy difference between spin configurations σ_i, τ_i
caused by variations at j

j*-diameter* at i:
$\text{diam}_{ij}(A) = \sup_{s,t \in A} d_{ij}(s,t)$ where A runs over the sets in the decomposition

***THEOREM* 3.** *(EnKuOP11) Let μ be a Gibbs measure of the specification with Gibbsian potential Φ with an arbitrary a priori measure α, on a graph with vertex set G. Let T denote the local coarse-graining map where we assume that $\alpha(S_{s'}) > 0$ for all labels $s' \in S'$.*
Suppose that

$$\sup_{i \in G} \sum_{j \in G \setminus i} \sup_{s'} \textit{diam}_{ij}(S_{s'}) < 4.$$

Then the transformed measure $T(\mu)$ is Gibbs

1**-dimensional sphere:** We have for n.n. i and j by Cauchy-Schwartz that

$$d_{ij}(\sigma_i, \tau_i) = \beta \sup_{\zeta_j, \bar{\zeta}_j; T(\zeta_j)=T(\bar{\zeta}_j)} \left| (\sigma_i - \tau_i) \cdot (\zeta_j - \bar{\zeta}_j) \right|$$
$$\leq \beta \|\sigma_i - \tau_i\|_2 2 \sin \frac{\pi}{q}$$

and so $\text{diam}_{ij} S_{s'} = \beta \times (2 \sin \frac{\pi}{q})^2$. This gives the criterion

$$2d\beta(\sin \frac{\pi}{q})^2 < 1$$

for Gibbsianness of the coarse-grained model.
(standard estimate would give a worse condition without the square)

q**-dimensional sphere:** Criterion for Gibbsianness:

$$2d\beta(\sin \psi)^2 < 1.$$

$\psi :=$ one half of the maximal angle under which a set $S_{s'}$ appears as seen from the origin

This argument provides an independent rigorous route to the existence of a Kosterlitz-Thouless (slow decay of correlations) phase in a discrete-spin model.

This is known to happen in the clock model by Fröhlich and Spencer CMP 81, at intermediate temperatures.

At very low temperatures $\beta \geq q^2$ the clock model has discrete symmetry breaking.

***THEOREM* 4.** *Rotators on the circle. For each $d \geq 3$ there is a q_0 such that for $q \geq q_0$ there is an interaction Φ' with a discrete -clock - rotation invariance such that there are uncountably many translation-invariant ergodic states in the set of Gibbs measures $\mathcal{G}_{\Phi'}$.*

Proof:

Take β such there is continuous symmetry breaking for continuous model.

Take q large enough.

Then the renormalized Hamiltonian for one translation invariant measures exists.

It is independent of which one.

Theorem

Consider the Gibbs measures of the continuous S^1 rotor model at β discretized with q. Then, at β sufficiently large ($\geq q^2$), the model is non-Gibbs.

Sketch of Proof:

Condition on checkerboard north south configuration.

The conditioned model has an two groundstates: East and West.
The cost to flip from one to the other is of the order of $\frac{\beta}{q^2}$