Scaling limits for dynamic models of Young diagrams *

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*Partly with Makiko Sasada, Martin Sauer and Bin Xie,
"Gradient Random Fields", BIRS (Banff), June 1st, 2011.

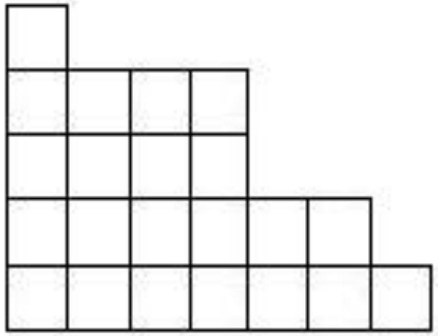
Motivation

- Scaling limits for random Young diagrams (LLN).
  - 2D: Vershik '96 discussed under several types of statistics and derived Vershik curves in the limit.
  - 3D: Cerf-Kenyon '01 derived the limit surface Wulff shape characterized by a certain variational formula (under uniform statistics).

- Our goal is to establish the corresponding dynamic theory.

- Our model describes a motion of (decreasing) interfaces, called SOS dynamics.

2D



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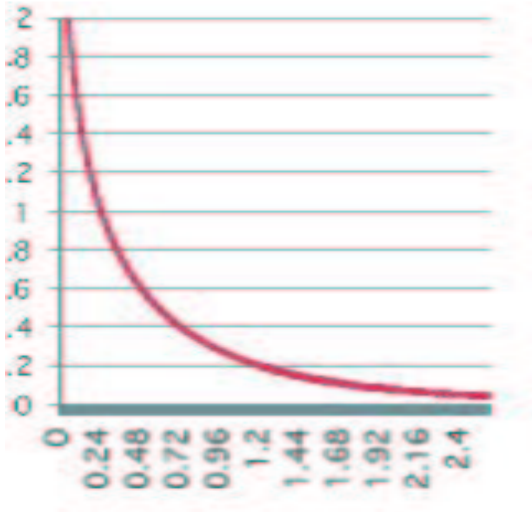
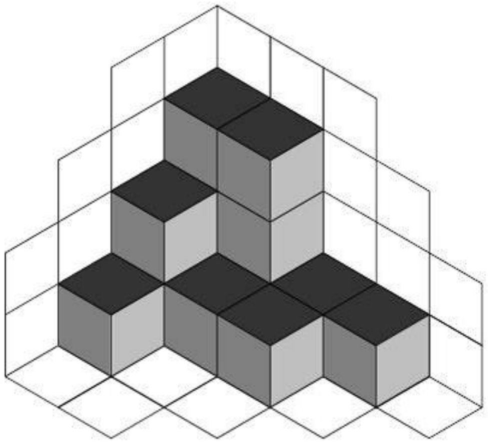


3D



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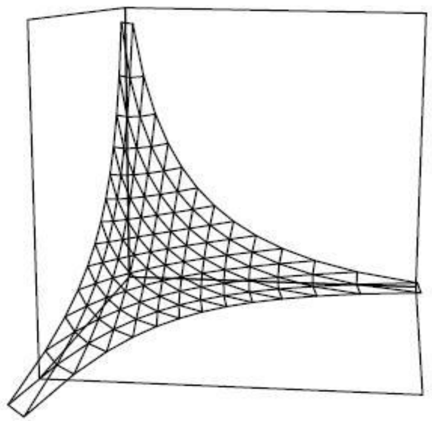
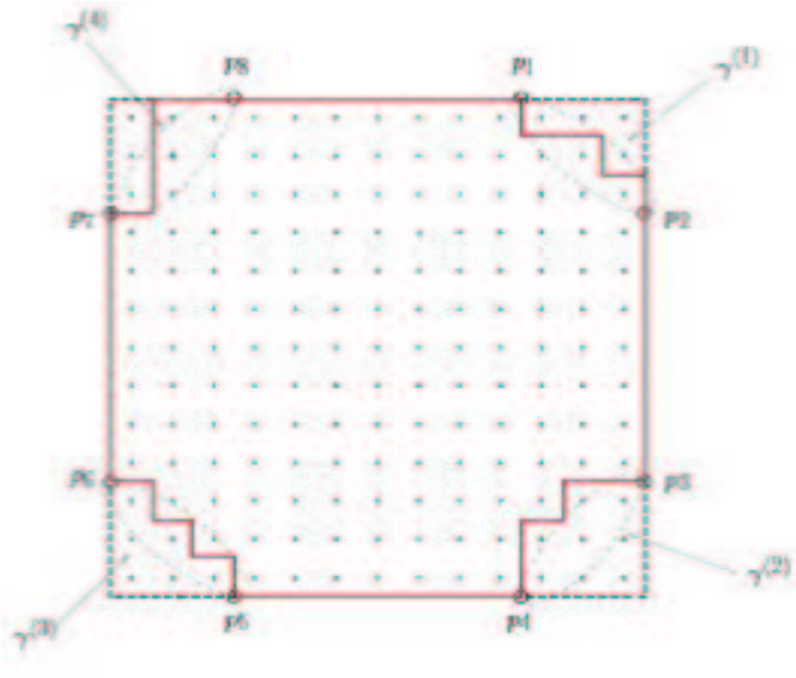


Zero-temperature Stochastic Ising model



(taken from Caputo-Martinelli-Simenhaus-Toninelli '10)

Plan of talk

1. Ensembles of 2D Young diagrams

2. Non-conservative systems

   2.1. Static results (for grandcanonical ensembles)
   LLN (Vershik curves), CLT

   2.2. Dynamic results

   2.2.1. Dynamics of gradient fields (WAZRP, WASEP with stochastic reservoirs at boundary)

   2.2.2. Hydrodynamic limits (LLN)

   2.2.3. Non-equilibrium fluctuations (CLT, SPDEs)

3. Conservative systems

3.1. Static results (for canonical ensembles of gradients)

3.1.1. Equivalence of ensembles under inhomogeneous conditioning (Local equilibrium)

3.1.2. Related Young diagrams

3.2. Hydrodynamic limits

Surface diffusion: conservative dynamics (conjecture)

— Dynamics associated with canonical ensembles

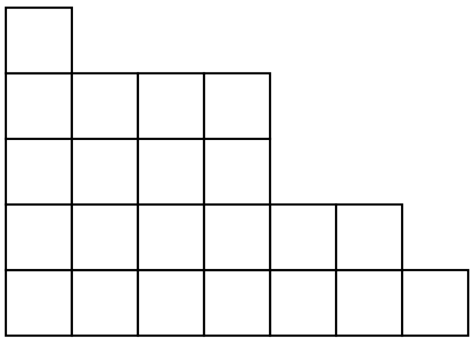
4. 3D case

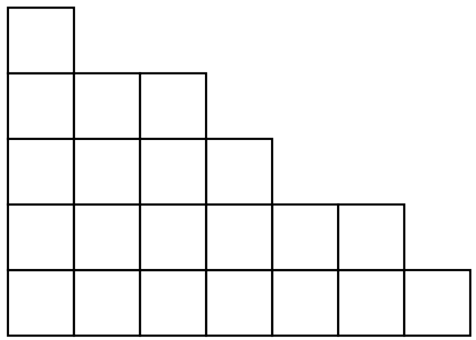
Honeycomb dimers dynamics

1. Ensembles of 2D Young diagrams

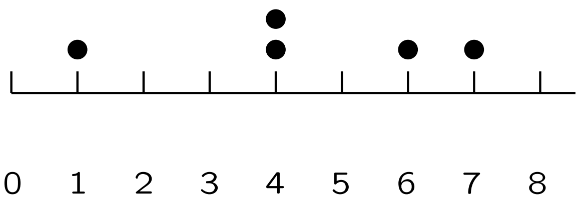
Uniform (Bose)-case: Restricted Uniform (Fermi)-case:

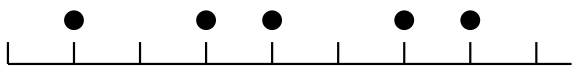
height $\psi : [0, \infty) \to \mathbb{Z}_+$





height difference $\eta : \mathbb{N} \to \mathbb{Z}_+$





- Uniform (Bose)-case:

$$\mathcal{P}_n = \{\psi; \text{Young diagram with area } n\}, \quad \mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$$

- Restricted Uniform (Fermi)-case:

$$\mathcal{Q}_n = \{\psi \in \mathcal{P}_n; \text{height difference} \in \{0,1\}\}, \quad \mathcal{Q} = \bigcup_{n=0}^{\infty} \mathcal{Q}_n$$

$$n(\psi) := n \quad \text{if } \psi \in \mathcal{P}_n \quad (\text{i.e. } n(\psi) = \text{area of } \psi)$$

- canonical ensembles:

  Uniform statistics (U-case)

  $$\mu_U^n := \text{uniform prob. meas. on } \mathcal{P}_n$$

  Restricted uniform statistics (RU-case)

  $$\mu_R^n := \text{uniform prob. meas. on } \mathcal{Q}_n$$

- grandcanonical ensembles (superposition of CE):

  $$0 < \varepsilon < 1: \text{ parameter}$$

  U-case $\mu_U^\varepsilon(\psi) := \frac{1}{Z_U(\varepsilon)}\varepsilon^{n(\psi)}, \ \ \psi \in \mathcal{P}$

  RU-case $\mu_R^\varepsilon(\psi) := \frac{1}{Z_R(\varepsilon)}\varepsilon^{n(\psi)}, \ \ \psi \in \mathcal{Q}$

2. Non-conservative systems

2.1. Static results (for grandcanonical ensembles)

(a) LLN (Vershik curves)

- For $N > 0$, choose $\varepsilon \equiv \varepsilon(N) = \varepsilon_U(N), \varepsilon_R(N)$ s.t.

$$E^{\mu_U^\varepsilon}[n(\psi)] = N^2, \quad E^{\mu_R^\varepsilon}[n(\psi)] = N^2.$$

(i.e., the averaged areas of YD $= N^2$). Then,

$$\varepsilon_U(N) = 1 - \frac{\alpha}{N} + \cdots, \quad \alpha = \frac{\pi}{\sqrt{6}},$$
$$\varepsilon_R(N) = 1 - \frac{\beta}{N} + \cdots, \quad \beta = \frac{\pi}{\sqrt{12}}.$$

(cf. Hardy-Ramanujan's formula: $\sharp \mathcal{P}_n \sim \frac{1}{4\sqrt{3}n} e^{2\alpha\sqrt{n}}$)

- Scaling for Young diagrams: For $\psi \in \mathcal{P}$,

$$\tilde{\psi}^N(u) := \frac{1}{N}\psi(Nu), \quad u > 0.$$

(i.e., the averaged areas of scaled YD $= 1$).

Proposition 1. *(Vershik, '96, LLN under* $\mu_U^{\varepsilon(N)}, \mu_R^{\varepsilon(N)}$*)*

$$\tilde{\psi}^N(u) \underset{N\to\infty}{\longrightarrow} \psi_U(u) \text{ in prob. under } \mu_U^{\varepsilon(N)},$$

$$\tilde{\psi}^N(u) \underset{N\to\infty}{\longrightarrow} \psi_R(u) \text{ in prob. under } \mu_R^{\varepsilon(N)},$$

where

$$\psi_U(u) = -\frac{1}{\alpha}\log\left(1 - e^{-\alpha u}\right),$$

$$\psi_R(u) = \frac{1}{\beta}\log\left(1 + e^{-\beta u}\right), \quad u \geq 0.$$

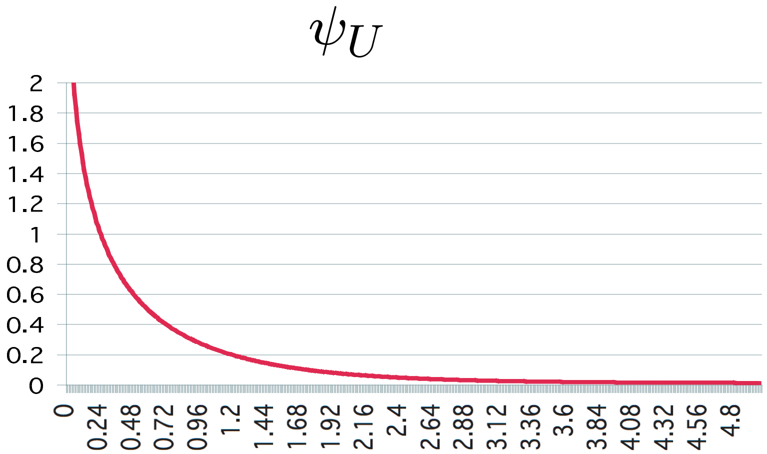
The limit shapes are called Vershik curves.

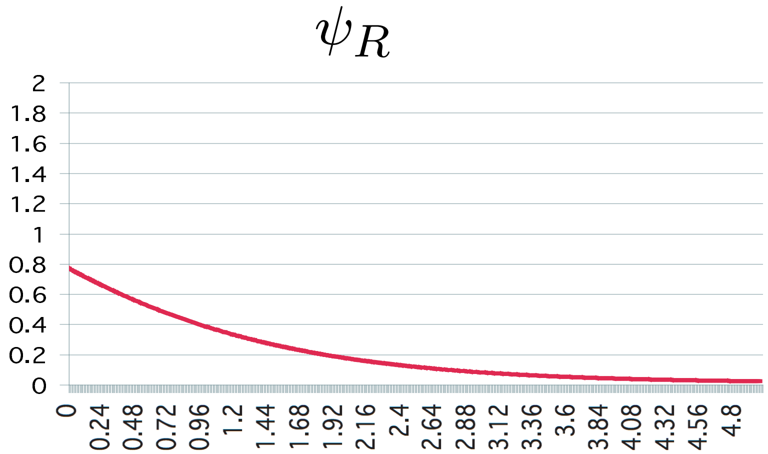
Remark 1.

(1) *Similar results hold under canonical ensembles* $\mu_U^{N^2}, \mu_R^{N^2}$.

(2) $y = \psi_U(u) \Leftrightarrow e^{-\alpha u} + e^{-\alpha y} = 1, \quad y = \psi_R(u) \Leftrightarrow e^{\beta y} - e^{-\beta u} = 1.$

Vershik curves





(b) CLT

- Known results
  - Pittel, '97: U-case
  - Yakubovich, '99: RU-case
  - Vershik-Yakubovich, '01:
    U-case with constraint on heights
  - Beltoft-Boutillier-Enriquez, '10:
    U-case in a rectangular box
  - Beltoft, '10: thesis

- CLT under canonical ensembles can be reduced from that under grandcanonical ensembles by removing the effect of fluctuations of area.

- Fluctuations

$$\Psi_U^N(u) := \sqrt{N}\big(\tilde{\psi}^N(u) - \psi_U(u)\big)$$
$$\Psi_R^N(u) := \sqrt{N}\big(\tilde{\psi}^N(u) - \psi_R(u)\big), \quad u \geq 0$$

Proposition 2. *(CLT under grandcanonical ensembles)*

$$\Psi_U^N(u) \underset{N\to\infty}{\Longrightarrow} \Psi_U(u) \text{ weakly under } \mu_U^{\varepsilon(N)},$$

$$\Psi_R^N(u) \underset{N\to\infty}{\Longrightarrow} \Psi_R(u) \text{ weakly under } \mu_R^{\varepsilon(N)},$$

where Ψ_U, Ψ_R *are mean* 0 *Gaussian processes with covariance structures*

$$C_U(u,v) = \frac{1}{\alpha}\min\{\rho_U(u), \rho_U(v)\},$$

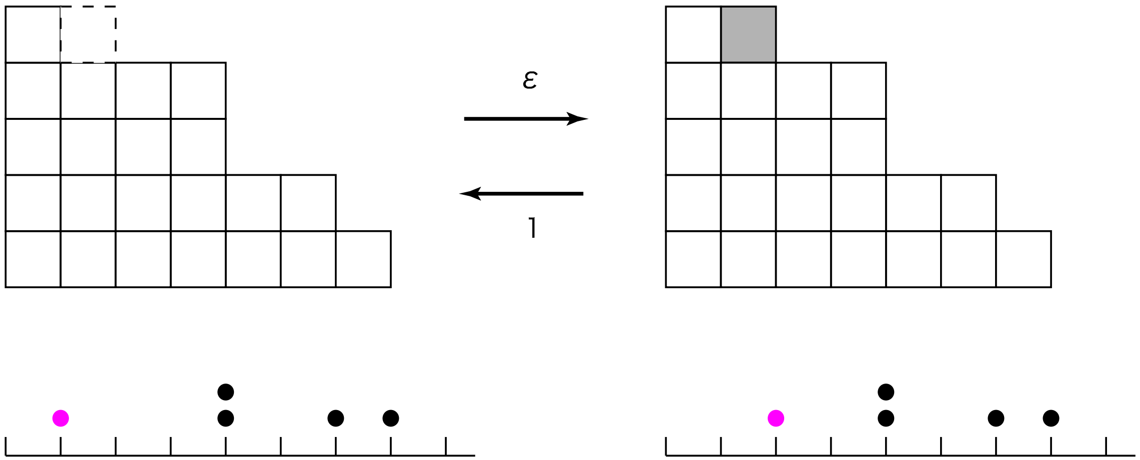
$$C_R(u,v) = \frac{1}{\beta}\min\{\rho_R(u), \rho_R(v)\}, \quad u,v>0,$$

and $\rho_U = -\psi_U'$, $\rho_R = -\psi_R'$ *are slopes of Vershik curves, respectively.*

2.2. Dynamic results

2.2.1. Dynamics of gradient fields (WAZRP, WASEP with stochastic reservoirs at boundary)

- Dynamics associated with grandcanonical ensembles



- Young diagrams $\Longleftrightarrow$ Height differences (Gradient fields)

U-case $\quad \xi(k) := \psi(k-1) - \psi(k) \in \mathbb{Z}_+, \quad k \in \mathbb{N}$

RU-case $\quad \eta(k) := \psi(k-1) - \psi(k) \in \{0, 1\}, \quad k \in \mathbb{N}$

Dynamics of height differences:

- U-case: $\xi_t(k) \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $\quad \xi_t(0) = \infty$
  Weakly asymmetric zero-range process with weakly asymmetric stochastic reservoir at $k = 0$
- RU-case: $\eta_t(k) \in \{0, 1\}$, $k \in \mathbb{N}$, $\quad \eta_t(0) = \infty$
  Weakly asymmetric simple exclusion process with weakly asymmetric stochastic reservoir at $k = 0$

2.2.2. Hydrodynamic limits (LLN) *

Height differences ξ_t or η_t
$\Rightarrow$ Evolving height functions $\psi_t(u), u > 0$

Diffusive scaling in space and time:

$$\tilde{\psi}^N(t,u) := \frac{1}{N}\psi_{N^2t}(Nu), \quad u > 0.$$

*jointly with Makiko Sasada CMP'10

Theorem 3. *(U-case) If* $\tilde{\psi}_U^N(0,u) \underset{N\to\infty}{\longrightarrow} \psi_0(u)$*, then*

$$\tilde{\psi}_U^N(t,u) \underset{N\to\infty}{\longrightarrow} \psi_U(t,u) \text{ in prob.}$$

The limit $\psi_U(t,u)$ *is a solution of nonlinear PDE:*

$$\partial_t \psi = \{\psi'/(1-\psi')\}' + \alpha\psi'/(1-\psi'), \quad u > 0,$$
$$\psi(0,\cdot) = \psi_0(\cdot),$$
$$\psi(t,0+) = \infty,\ \psi(t,\infty) = 0,$$

where $\partial_t \psi = \partial\psi/\partial t, \psi' = \partial\psi/\partial u \ (< 0)$.

Remark 2. *Vershik curve* ψ_U *is a unique stationary sol of this PDE.*

Theorem 4. *(RU-case) If* $\tilde{\psi}_R^N(0,u) \underset{N\to\infty}{\longrightarrow} \psi_0(u)$*, then*

$$\tilde{\psi}_R^N(t,u) \underset{N\to\infty}{\longrightarrow} \psi_R(t,u) \text{ in prob.}$$

The limit $\psi_R(t,u)$ *is a solution of nonlinear PDE:*

$$\partial_t \psi = \psi'' + \beta\psi'(1+\psi'), \quad u > 0,$$
$$\psi(0,\cdot) = \psi_0(\cdot),$$
$$\psi'(t,0+) = -\frac{1}{2},\ \psi(t,\infty) = 0.$$

Remark 3. *Vershik curve* ψ_R *is a unique stationary sol of this PDE.*

- The boundary condition at 0 follows from the pointwise ergodicity:

$$\lim_{N\to\infty} P\left[\left|\frac{1}{T_2-T_1}\int_{T_1}^{T_2} \eta_{N^2 s}(1)ds - \frac{1}{2}\right| > \delta\right] = 0,$$

for every $\delta > 0$ and $0 \le T_1 < T_2$.

2.2.3. Non-equilibrium fluctuations (CLT, SPDEs) *

$$\Psi_U^N(t,u) := \sqrt{N}\big(\tilde{\psi}_U^N(t,u) - \psi_U(t,u)\big)$$
$$\Psi_R^N(t,u) := \sqrt{N}\big(\tilde{\psi}_R^N(t,u) - \psi_R(t,u)\big)$$

Theorem 5. *(U-case)* $\Psi_U^N(t,u) \underset{N\to\infty}{\Longrightarrow} \Psi_U(t,u)$ *weakly.*

The limit $\Psi_U(t,u)$ *is a solution of SPDE:*

$$\partial_t \Psi(t,u) = \left(\frac{\Psi'(t,u)}{(1+\rho_U(t,u))^2}\right)' + \alpha \frac{\Psi'(t,u)}{(1+\rho_U(t,u))^2} + \sqrt{\frac{2\rho_U(t,u)}{1+\rho_U(t,u)}}\dot{W}(t,u)$$

where $\rho_U(t,u) = -\psi_U'(t,u)$ *and* $\dot{W}(t,u)$ *is the space-time white noise on* $[0,\infty) \times \mathbb{R}_+$.

*jointly with Makiko Sasada, Martin Sauer and Bin Xie, '11

Theorem 6. *(RU-case)* $\Psi_R^N(t,u) \underset{N\to\infty}{\Longrightarrow} \Psi_R(t,u)$ *weakly.*

The limit $\Psi_R(t,u)$ *is a solution of SPDE:*

$$\partial_t \Psi(t,u) = \Psi''(t,u) + \beta(1 - 2\rho_R(t,u))\Psi'(t,u) + \sqrt{2\rho_R(t,u)(1-\rho_R(t,u))}\,\dot{W}(t,u),$$

$$\Psi'(t,0+) = 0,$$

where $\rho_R(t,u) = -\psi_R'(t,u)$.

Invariant measures of SPDEs

• U-case: Since $\rho_U(t,u) \underset{t\to\infty}{\longrightarrow} \rho_U(u) := -\psi'_U(u)$, the SPDE in equilibrium has the form:

$$\partial_t \Psi = -g_U(u)\, Q_U \Psi + \sqrt{2 g_U(u)}\dot{W}(t,u)$$

where

$$g_U(u) = \frac{\rho_U(u)}{1+\rho_U(u)},$$

$$Q_U = -\frac{\partial}{\partial u}\left(\frac{1}{\rho_U(u)(1+\rho_U(u))}\frac{\partial}{\partial u}\right), \quad u > 0.$$

Thus the invariant measure of $\Psi_U(t,u)$ is $N(0, Q_U^{-1})$. Since $C_U(u,v)$ is the Green kernel of Q_U^{-1} (by checking $Q_U C_U(\cdot, v) = \delta_v(\cdot)$), this gives another proof of static result, Proposition 2 in U-case.

• RU-case: Since $\rho_R(t,u) \underset{t\to\infty}{\longrightarrow} \rho_R(u) := -\psi'_R(u)$, the SPDE in equilibrium has the form:

$$\partial_t \Psi = -g_R(u)\, Q_R \Psi + \sqrt{2g_R(u)}\dot{W}(t,u)$$

where

$$g_R(u) = \rho_R(u)(1-\rho_R(u)),$$
$$Q_R = -\frac{\partial}{\partial u}\left(\frac{1}{\rho_R(u)(1-\rho_R(u))}\frac{\partial}{\partial u}\right) \quad \text{on } L^2(\mathbb{R}_+, du),$$

with Neumann condition at $u=0$. Thus the invariant measure of $\Psi_R(t,u)$ is $N(0, Q_R^{-1})$. Since $C_R(u,v)$ is the Green kernel of Q_R^{-1} (by checking $Q_R C_R(\cdot, v) = \delta_v(\cdot)$ and Neumann condition at $u=0$), this gives another proof of static result, Proposition 2 in RU-case.

3. Conservative systems

3.1. Static results (for canonical ensembles of gradients)

3.1.1. Equivalence of ensembles under inhomogeneous conditioning (Local equilibrium)

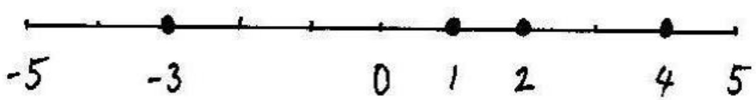
• $\eta = (\eta_k)_{k\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$: particle configuration on $\mathbb{Z}$

$$K_{\Lambda_\ell}(\eta) := \sum_{k\in\Lambda_\ell} \eta_k, \quad M_{\Lambda_\ell}(\eta) := \sum_{k\in\Lambda_\ell} k\eta_k.$$

$\Lambda_\ell = \{-\ell, \cdots, \ell\}$

• Canonical ensemble = uniform probability measures $\nu_{\Lambda_\ell,K,M}$ on $\Sigma_{\Lambda_\ell,K,M} = \{\eta \in \{0,1\}^{\Lambda_\ell}; K_{\Lambda_\ell}(\eta) = K, M_{\Lambda_\ell}(\eta) = M\}$

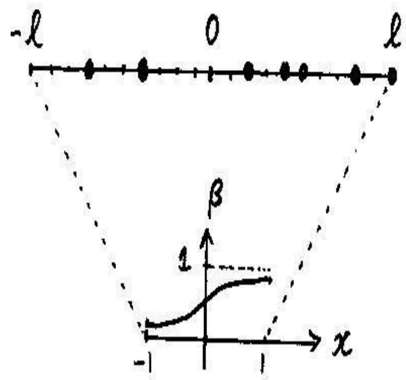
• Grandcanonical ensemble = Bernoulli measures ν_α on $\{0,1\}^{\mathbb{Z}}$ with mean $\alpha, \alpha \in (0,1)$



$\ell = 5, K = 4, M = 4$

Theorem 7. $K = K_\ell, M = M_\ell, k_j = k_{\ell,j}, 1 \le j \le p$, *s.t.*

$$\lim_{\ell\to\infty} \frac{K}{2\ell+1} = \rho \in (0,1),$$
$$\lim_{\ell\to\infty} \frac{M}{(2\ell+1)^2} = m \in \big(-\frac{1}{2}\rho(1-\rho), \frac{1}{2}\rho(1-\rho)\big),$$
$$\lim_{\ell\to\infty} \frac{k_j}{\ell} = x_j \in (-1,1), \quad (\{x_j\} \text{ are distinct})$$



Then, for $^\forall f_j, 1 \le j \le p$ *local functions,*

$$\lim_{\ell\to\infty} E_{\nu_{\Lambda_\ell,K,M}}[\prod_{j=1}^{p} \tau_{k_j} f_j] = \prod_{j=1}^{p} E_{\nu_{\beta(x_j)}}[f_j],$$

where τ_k *are shifts by* k *and*

$$\beta(x) \equiv \beta(x;a,b) = \frac{e^{bx}a}{e^{bx}a + (1-a)},$$

with $a \in (0,1)$ *and* $b \in \mathbb{R}$ *determined from* ρ *and* m *by*

(1) $$\frac{1}{2}\int_{-1}^{1} \beta(x;a,b)dx = \rho, \quad \frac{1}{4}\int_{-1}^{1} x\beta(x;a,b)dx = m.$$

Remark 4. (i) *(Local equilibrium) Theorem 1 implies:*

$$\lim_{\ell \to \infty, \frac{k}{\ell} \to x} \nu_{\Lambda_\ell, K, M} \circ \tau_k^{-1} = \nu_{\beta(x)},$$

and asymptotic independence for distinct x*.*

(ii) *The relation (1) defines a diffeomorphism:*

$$(a, b) \in (0,1) \times \mathbb{R} \mapsto (\rho, m) \in D = \left\{ 0 < \rho < 1, |m| < \frac{1}{2}\rho(1-\rho) \right\}$$

This part is due to arXiv:1103.5823

Proof • If $\beta(\cdot) = \beta(\cdot; a, b)$ for some a, b, then

$$\nu^{\Lambda_\ell}_{\beta(\cdot)}(\cdot|\Sigma_{\Lambda_\ell,K,M}) = \nu_{\Lambda_\ell,K,M}(\cdot),$$

where $\nu^{\Lambda_\ell}_{\beta(\cdot)}$ = distri. of indep. $\{\eta_k\}_{k\in\Lambda_\ell}$ s.t. $E[\eta_k] = \beta(k/\ell)$.

• ($p = 1$ for simplicity) From the above observation,

$E_{\nu_{\Lambda_\ell,K,M}}[\tau_k f] - E_{\nu_{\beta(x)}}[f]$

$$= \sum_{\xi\in\{0,1\}^{\Gamma+k}} \{f(\xi) - E_{\nu_{\beta(x)}}[f]\} \frac{\nu^{\Lambda_\ell}_{\beta(\cdot;a,b)}(\eta|_{\Gamma+k} = \xi, K_{\Lambda_\ell}(\eta) = K, M_{\Lambda_\ell}(\eta) = M)}{\nu^{\Lambda_\ell}_{\beta(\cdot;a,b)}(K_{\Lambda_\ell}(\eta) = K, M_{\Lambda_\ell}(\eta) = M)},$$

for all local function f with support $\Gamma \Subset \mathbb{Z}$.

• We show the local limit theorem for $(K_{\Lambda_\ell}(\eta), M_{\Lambda_\ell}(\eta))$ under $\nu^{\Lambda_\ell}_{\beta(\cdot)}$. The sum of independent r.v.'s $M_{\Lambda_\ell} = \sum_{k\in\Lambda_\ell} k\eta_k$ has a growing weight k, and therefore $\{k\eta_k\}_k$ doesn't satisfy "good" moment conditions required for the classical local limit theorem (cf. [Petrov]). $\square$

3.1.2. Related Young diagrams

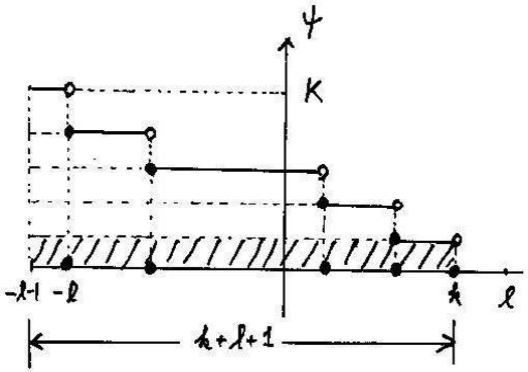
- Young diagrams in RU-case (i.e. height difference $\in \{0,1\}$)

  height $= K$, side length $= 2\ell+1$, area $= A$

  $\psi^\ell(u), u \in [-\ell-1, \ell]$: height function of Young diagram

- Corresponding particle picture:
  $\eta_k := \psi^\ell(k-1) - \psi^\ell(k)$: height difference, $\quad \eta = (\eta_k)_{k \in \Lambda_\ell}$

  $K = K_{\Lambda_\ell}(\eta)$: height at $u = -\ell - 1$
  $A = \sum_{k \in \Lambda_\ell} (k+\ell+1)\eta_k = (\ell+1)K_{\Lambda_\ell}(\eta) + M_{\Lambda_\ell}(\eta)$



• Scaling: $\tilde{\psi}^\ell(x) := \frac{1}{\ell}\psi^\ell(\ell x), \quad x \in [-1,1]$

Corollary 8. *Under the same conditions as Theorem 1,*

$$\lim_{\ell\to\infty} \nu_{\Lambda_\ell,K,M}\left(\sup_{x\in[-1,1]} \left|\tilde{\psi}^\ell(x) - \psi(x)\right| > \delta\right) = 0, \quad \delta > 0,$$

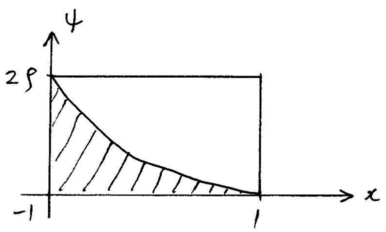
where $\psi(x) = \int_x^1 \beta(y)dy, x \in [-1,1]$*.*

The limit ψ has a slope $\psi'(x) = -\beta(x)$ and satisfies

$$\psi(-1) = 2\rho, \quad \psi(1) = 0, \quad \int_{-1}^1 \psi(x)dx = 2\rho + 4m,$$

$$\psi'' + c\psi'(1+\psi') = 0, \quad (-\psi' : \text{stationary sol of viscous Burgers' eq})$$

with $c = -b$ ($c = \pi/\sqrt{12}$ for Vershik curve).

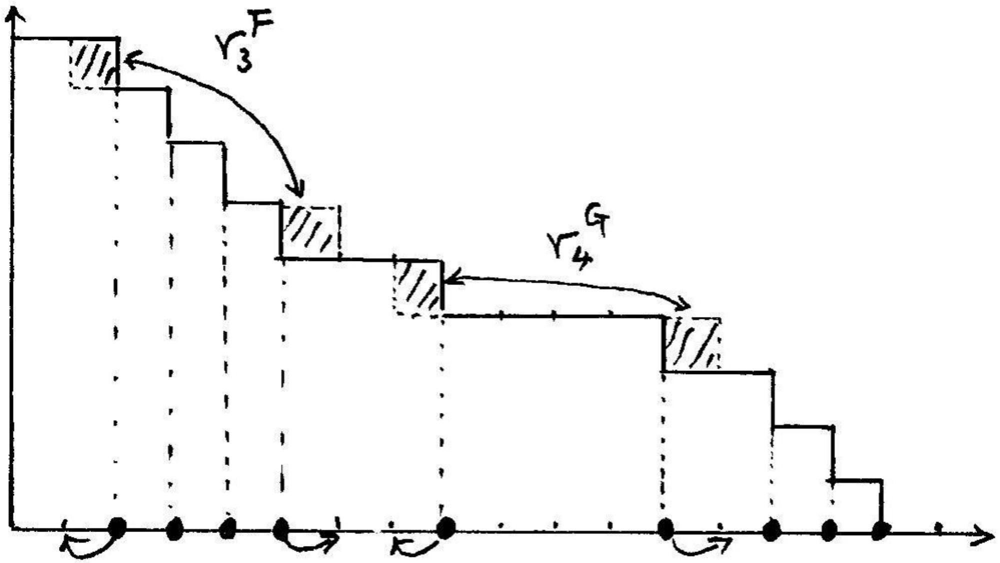


• Beltoft-Boutillier-Enriquez ('10) : U-case, Grandcanonical ensembles, in a rectangular box

3.2. Hydrodynamic limits

• Surface diffusion: conservative dynamics (conjecture)

• Dynamics associated with the RU-canonical ensembles:



- Dynamics associated with the RU-canonical ensembles:
  - The dynamics preserve the area of YD, i.e., creation and annihilation of unit squares take place simultaneously.
  - Or, a unit square moves on the surface of YD until it finds another stable position keeping height differences $\in \{0, 1\}$.
  - The jump rate of a square falling down a stair with length r and its reversed transition is $c_r^F > 0$.
  - The jump rate of a square sliding over a flat piece of length r and its reversed transition is $c_r^G > 0$.

- We consider the associated particle system on a torus.
  $\eta(k) \in \{0, 1\}, \quad k \in \mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$

- Scaling ($t \mapsto N^4 t$):

$$\xi_t^N(du) = \frac{1}{N} \sum_{k \in \mathbb{N}} \eta_{N^4 t}(k) \delta_{k/N}(du), \qquad u \in \mathbb{T} = [0,1].$$

- Expected result: $\xi_t^N(du) \to \rho(t,u)du$

Cahn-Hilliard type nonlinear PDE:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial^2}{\partial u^2} \left\{ D(\rho) \frac{\partial^2 \rho}{\partial u^2} \right\}, \qquad u \in \mathbb{T},$$

$$\begin{aligned} D(\rho) =& \frac{1}{\rho(1-\rho)} \inf_{g:\text{tame}} \frac{1}{4} \sum_{r=1}^{\infty} \langle c_{(0,r)} \{\pi^{(0,r)}(\Gamma_g + \frac{1}{2} \sum_k k^2 \eta(k))\}^2 \\ &+ c_{(-r,0)} \{\pi^{(-r,0)}(\Gamma_g + \frac{1}{2} \sum_k k^2 \eta(k))\}^2 \rangle_\rho, \qquad \Gamma_g = \sum_k \tau_k g \end{aligned}$$

$c_{(0,\pm r)}$: jump rates determined by c_r^F, c_r^G

$\pi^{(0,\pm r)}$: transition operators

• Laplacian replacement (Fluctuation-dissipation relation) for the current:

$$W = -D(\rho)(\eta_1 - 2\eta_0 + \eta_{-1}) + L^{\exists}F$$

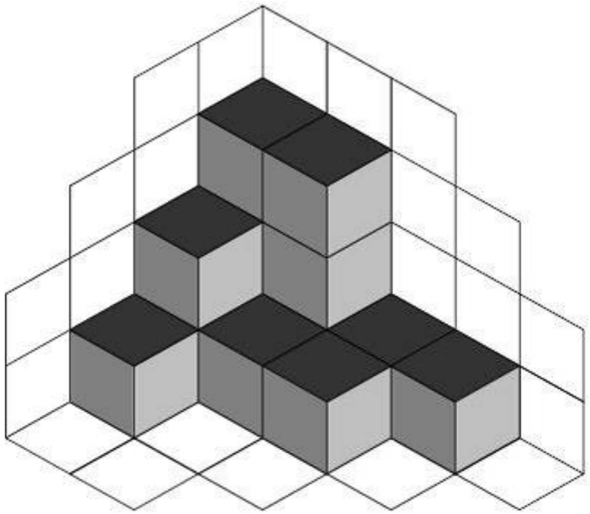
where

$$W = \sum_{r=1}^{\infty}(r+1)W_r$$

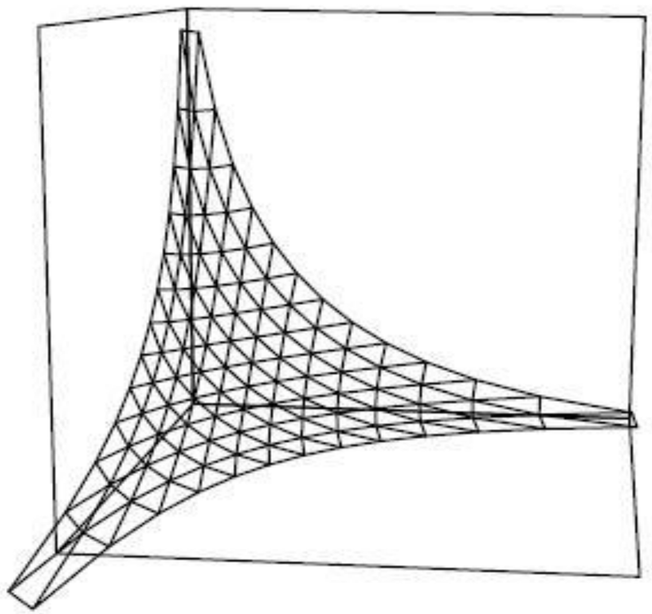
$$W_r = c_r^F\left(1_{\{\text{outward jump}\}} - 1_{\{\text{inward jump}\}}\right) + c_r^G\left(1_{\{\text{outward jump}\}} - 1_{\{\text{inward jump}\}}\right)$$

4. 3D case

Limit shapes of scaled surfaces of 3D Young diagrams under uniform ensemble are studied by Cerf-Kenyon ’01

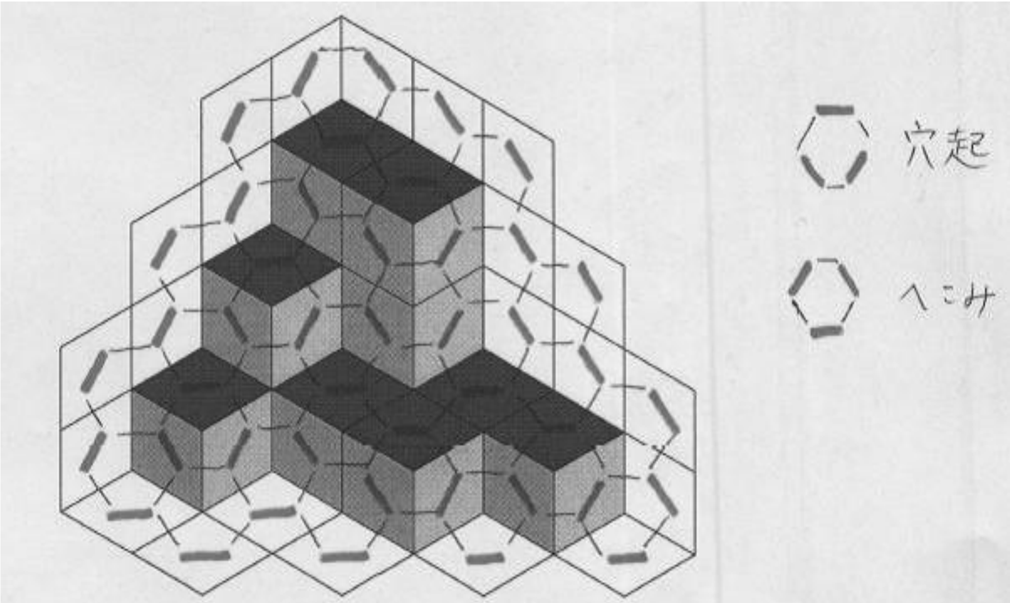


limit surface



taken from Cerf-Kenyon

Under the projection to the plane $\{x + y + z = 0\}$, 3D Young diagrams can be transformed into lozenge tiling or dimer configurations on a honeycomb lattice.

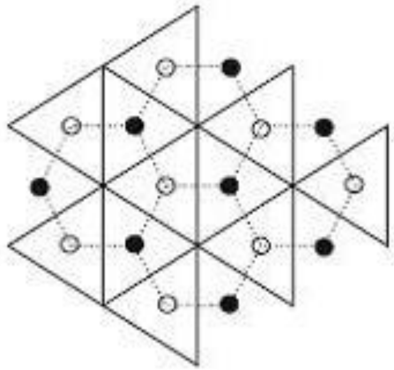


protuberance

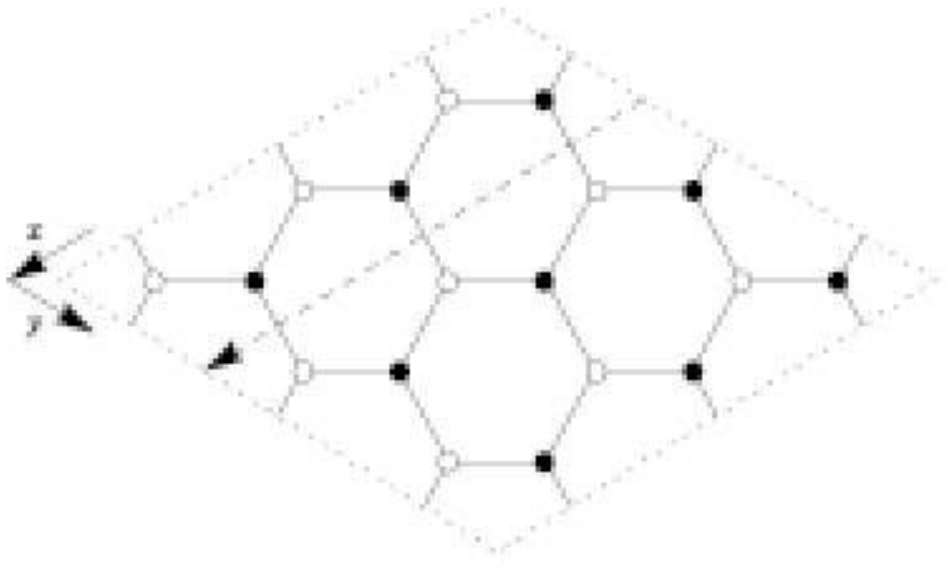
cave

Honeycomb lattice G_∞

Dual lattice
(triangular lattice)



Torus $H_N = G_\infty / N\mathbb{Z}^2$



picture of H_3

Dynamics of dimers on H_N

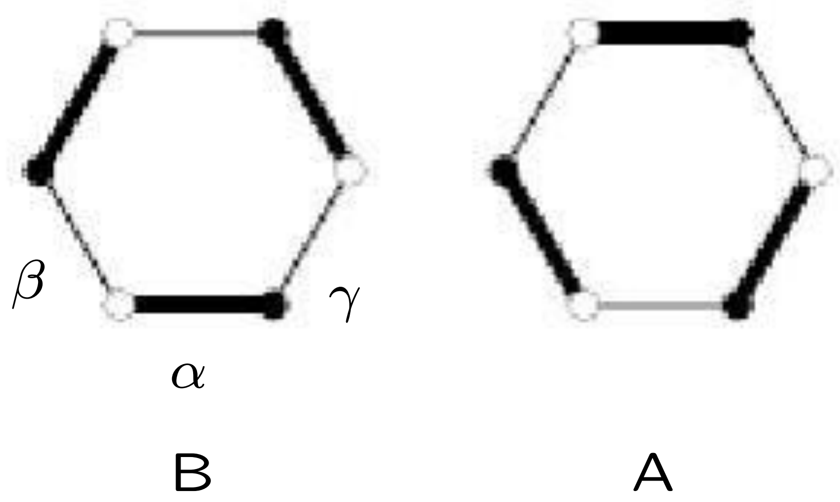
H_N^*: dual lattice of H_N (triangular lattice)
  $i \in H_N^*$ represents a hexagon
$H_N^B = \{$all undirected bonds of $H_N\}$
$\mathcal{X}_N = \{\eta : H_N^B \to \{0,1\}$, dimer covers of $H_N\}$
  i.e., $\{b = \{u, v\} \in H_N^B; \eta_b = 1\}$ covers H_N disjointly.

Generator of simple dimer process on H_N

$f : \mathcal{X}_N \to \mathbb{R}$

$$Lf(\eta) = \sum_{i \in H_N^*} \left[1_{\{\eta_i = A\}} + 1_{\{\eta_i = B\}} \right] \left\{ f(\eta^i) - f(\eta) \right\},$$

where η_i = restriction of η on the hexagon i, η^i is obtained from η by replacing η_i: $A \leftrightarrow B$.



Remark. If we consider on G_∞, for the grandcanonical ensemble μ_U^ε to be invariant, the rate of B $\to$ A (creation) is ε while the rate of A $\to$ B (annihilation) is 1 as in 2D case.

Hydrodynamic limit

H: continuum torus of lozenge, η_t: L-process on $\mathcal{X}_N$.
Macroscopic empirical distribution of δ-bonds ($\delta = \beta$ or γ)

$$\xi^{\delta,N}(\eta, dx) = \frac{1}{N^2} \sum_{b \in H_N^B : \delta-\text{type}} \eta_b \delta_{\frac{1}{N}x_b}(dx), \quad x = (x_\beta, x_\gamma) \in H,$$

$$\xi_t^{\delta,N}(dx) = \xi^{\delta,N}(\eta_{N^2 t}, dx).$$

Expected result: $\xi_t^{\delta,N} \to \xi_t^\delta$ and the limit is the solution of

$$\frac{\partial \xi_t^\delta}{\partial t} = \frac{\partial}{\partial x_\delta}\left\{ \sum_{\delta_1,\delta_2 \in \{\beta,\gamma\}} D_{\delta_1\delta_2}(\xi_t^\beta, \xi_t^\gamma) \frac{\partial \xi^{\delta_2}}{\partial x_{\delta_1}} \right\},$$

where

$$D_{\delta_1\delta_2}(s,t) = \frac{1}{2\chi_{\delta_1\delta_2}} \inf_{g \in \mathcal{C}_0, a_1+a_2=1} \langle c_0\{\pi_0(a_1 \sum_i i_\beta \tau_i \eta_{b_\beta} + a_2 \sum_i i_\gamma \tau_i \eta_{b_\gamma} - \Gamma_g)\}^2\rangle,$$

$$\chi_{\delta_1,\delta_2}(s,t) = \sum_i \langle \eta_{b_{\delta_1}}; \eta_{b_{\delta_2}+i_\beta e_\beta + i_\gamma e_\gamma}\rangle,$$

and $\langle\cdot\rangle = \langle\cdot\rangle_{s,t}$: Gibbs measures (Kenyon, Okounkov, Sheffield '06).

The correlation function decays slowly (quadratically):

$$
\begin{aligned}
&\langle \eta_{b_{\delta_1}}; \eta_{b_{\delta_2}+i_\beta e_\beta + i_\gamma e_\gamma} \rangle \\
&\quad \equiv \langle \eta_{b_{\delta_1}} \eta_{b_{\delta_2}+i_\beta e_\beta + i_\gamma e_\gamma} \rangle - \langle \eta_{b_{\delta_1}} \rangle \langle \eta_{b_{\delta_2}} \rangle \\
&\qquad \sim \frac{\text{const}}{|(i_\beta, i_\gamma)|^2}.
\end{aligned}
$$

In particular, χ does not converge absolutely. However, CLT is shown by Kenyon '08, Boutillier '07 as Naddaf-Spencer '97 did for $\nabla\phi$-interface model (Recall C^2-property of the surface tension is not known for $\nabla\phi$-interface model).

End of slides. Click [END] to finish the presentation.