Interacting Fermions Approach to 2D Critical Models

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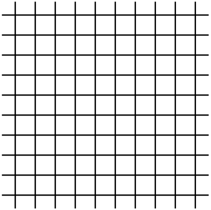
- (Brief) Introduction on Ising model and Onsager's exact solution.
- Definitions of the Eight-Vertex and Ashkin-Teller models.
- Qualitative discussion of the critical properties.
- List of rigorous results.
- Open problems.

Configuration: Place $+1$ or -1 at each site of Λ

$$\sigma = \{\sigma_{\mathbf{x}} = \pm 1 \mid \mathbf{x} \in \Lambda\}$$



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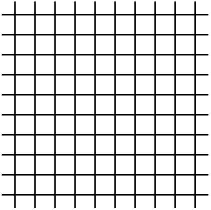

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$$H(\sigma) = -J \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j}$$

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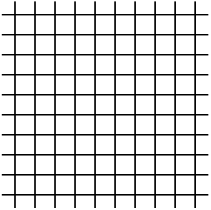

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Probability: given the *inverse temperature* $\beta \geq 0$

$$P(\sigma) = \frac{1}{Z(\Lambda,\beta)} e^{-\beta H(\sigma)} \qquad Z(\Lambda,\beta) = \sum_{\sigma} e^{-\beta H(\sigma)}$$

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Energy Density

$$G(\mathbf{x}-\mathbf{y}) = \langle O_{\mathbf{x}} O_{\mathbf{y}} \rangle - \langle O_{\mathbf{x}} \rangle \langle O_{\mathbf{y}} \rangle \qquad O_{\mathbf{x}} = \sum_{j=0,1} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j}$$

Onsager's Exact Solution [1944]

- free energy

$$\beta f(\beta) = \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \log\left[\Big(1 - \sinh 2\beta J\Big)^2 + \alpha(\mathbf{k}) \sinh(2\beta J)\right]$$

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- correlations for $|\mathbf{x} - \mathbf{y}| \to \infty$

$$G(\mathbf{x} - \mathbf{y}) \leq Ce^{-\mu(\beta)|\mathbf{x}-\mathbf{y}|} \qquad \text{if} \quad \beta \neq \beta_c$$

$$G(\mathbf{x} - \mathbf{y}) \sim \frac{C}{|\mathbf{x} - \mathbf{y}|^2} \qquad \text{if} \quad \beta = \beta_c$$

Grassmann Algebra [=Fermions] $\psi_1, \ldots, \psi_n$ such that

$$\psi_i \psi_j = -\psi_j \psi_i$$

for $i_1 < i_2 < \cdots < i_q$

$$\int d\psi_j \; \psi_{i_1} \cdots \psi_{i_p} \psi_j \psi_{i_{p+2}} \cdots \psi_{i_q} = (-1)^p \psi_{i_1} \cdots \psi_{i_p} \psi_{i_{p+2}} \cdots \psi_{i_q}$$

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Grassmann Gaussian Integral Two sets of Grassmann variables, $\psi_1, \cdots, \psi_n$ and $\bar{\psi}_1, \cdots, \bar{\psi}_n$

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Onsager's construction [...actually Kasteleyn's] Two sets of $2|\Lambda|$ Grassmann variables,

$$Z(\Lambda, \beta) = \det M = \int D\psi D\bar{\psi} \; \exp\Big\{ \sum_{\alpha,\beta=1,2} \sum_{i,j\in\Lambda} \bar{\psi}_{\alpha,i} M^{\alpha\beta}_{ij} \psi_{\beta,j} \Big\}$$

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Ising model= system of free fermions

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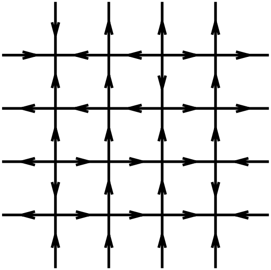

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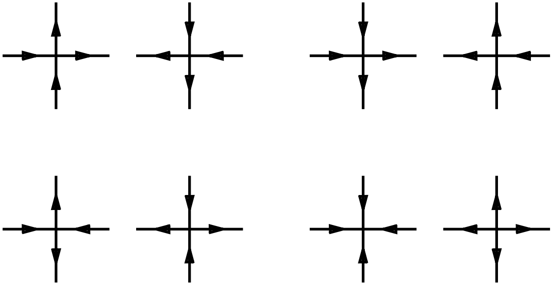

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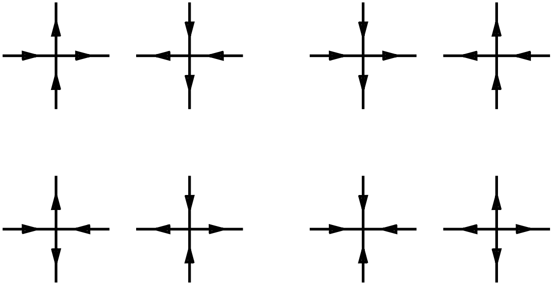

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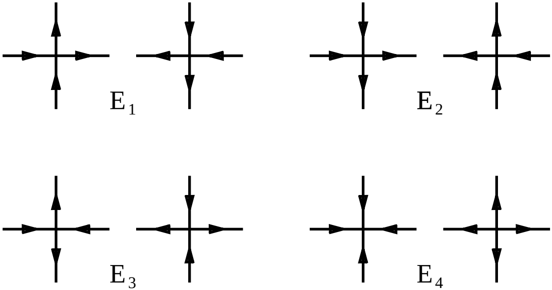

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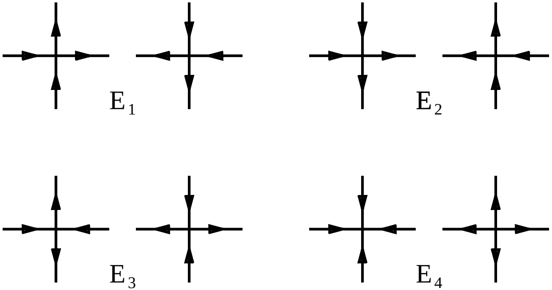

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Total Energy

$$H(\omega) = E_1 n_1(\omega) + E_2 n_2(\omega) + E_3 n_3(\omega) + E_4 n_4(\omega)$$

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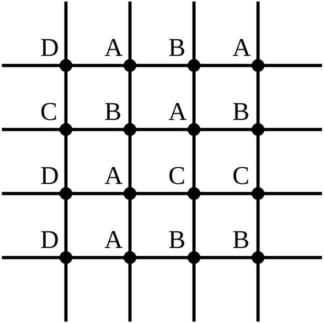

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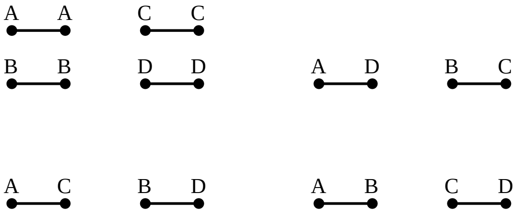

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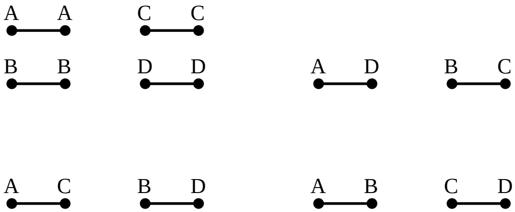

Assign four different energies

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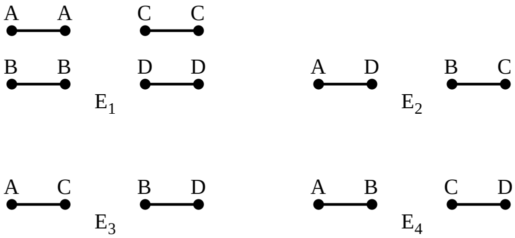

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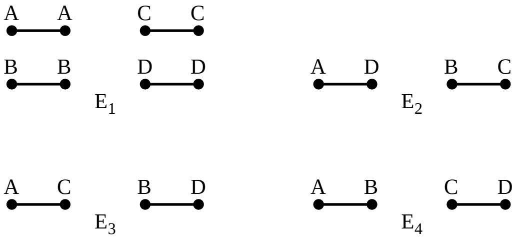

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- Without loss of generality, assume $E_1 + E_2 + E_3 + E_4 = 0$
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  i.e. the independent parameters are 3.
- 8V and AT belong to a bigger class, the *double Ising Models*:
  (more intuitive qualitative analysis)

Wu (1971),
Kadanoff and Wegner (1971)
Fan (1972)

A configuration (σ, σ') is the product of two configurations of spins $\sigma = \{\sigma_{\mathsf{x}} = \pm 1\}_{\mathsf{x} \in \Lambda}$ and $\sigma' = \{\sigma'_{\mathsf{x}} = \pm 1\}_{\mathsf{x} \in \Lambda}$.

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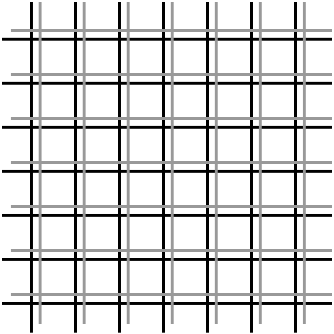

The **energy** of (σ, σ') is function of J, J' and J_4

$$H(\sigma, \sigma') = -J \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} - J' \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j} - J_4 V(\sigma, \sigma')$$

where V quartic in σ and σ':

$$V(\sigma, \sigma') = \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sum_{\substack{\mathbf{x}'\in\Lambda \\ j'=0,1}} v_{j-j'}(\mathbf{x}-\mathbf{x}')\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{x}'}\sigma'_{\mathbf{x}'+\mathbf{e}_{j'}}$$

for $v_j(\mathbf{x})$ a lattice function such that $|v_j(\mathbf{x})| \le c e^{-\kappa|\mathbf{x}|}$.

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for $v_j(\mathbf{x})$ a lattice function such that $|v_j(\mathbf{x})| \leq ce^{-\kappa|\mathbf{x}|}$.

Probability of a configuration (σ, σ')

$$P(\sigma,\sigma') = \frac{1}{Z} e^{-\beta H(\sigma,\sigma')} \qquad\qquad Z = \sum_{\sigma,\sigma'} e^{-\beta H(\sigma,\sigma')}$$

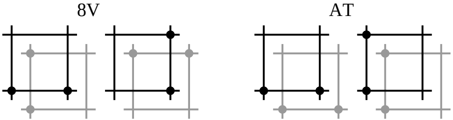
The 8V and AT models are equivalent to a doubled Ising model if:

$$E_1 = -J - J' - J_4 \qquad E_2 = J + J' - J_4$$

$$E_3 = J - J' + J_4 \qquad E_4 = -J + J' + J_4$$

$$\text{8V}: \qquad V(\sigma,\sigma') = \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma_{\mathbf{x}+\mathbf{e}_0}\sigma_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)}\sigma'_{\mathbf{x}+\mathbf{e}_1}\sigma'_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)}$$

$$\text{AT}: \qquad V(\sigma,\sigma') = \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j}$$



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Energy Density - Crossover

$$G_\varepsilon(\mathbf{x}-\mathbf{y}) = \langle O^\varepsilon_{\mathbf{x}} O^\varepsilon_{\mathbf{y}}\rangle - \langle O^\varepsilon_{\mathbf{x}}\rangle\langle O^\varepsilon_{\mathbf{y}}\rangle$$

where

$$O^+_{\mathbf{x}} = \sum_{j=0,1} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} + \sum_{j=0,1} \sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j} \qquad O^-_{\mathbf{x}} = \sum_{j=0,1} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} - \sum_{j=0,1} \sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j}$$

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(inverse) **critical temperature** β_c s.t. $\mu(\beta_c) = 0$, then:

- algebraic decay of correlations

$$G_\varepsilon(\mathbf{x}-\mathbf{y}) \sim \frac{C}{1+|\mathbf{x}-\mathbf{y}|^{2x_\varepsilon}} \;, \qquad\qquad |C(\beta)| = \infty$$

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Finally we have four critical exponents:

$$x_+ \qquad x_- \qquad \nu \qquad \alpha$$

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$$H(\sigma, \sigma') = -J \sum_{\substack{\mathbf{x} \in \Lambda \\ j=0,1}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} - J' \sum_{\substack{\mathbf{x} \in \Lambda \\ j=0,1}} \sigma'_{\mathbf{x}} \sigma'_{\mathbf{x}+\mathbf{e}_j}$$

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- for $J \neq J'$, $J_4 = 0$: two critical temperatures,

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- for $0 < |J_4| << |J - J'|$: two critical temperatures, for $\lambda = J_4/J$ and $\lambda' = J_4/J'$

$$\beta_c = \frac{1}{2J}\ln(\sqrt{2}+1) + \mathrm{O}(\lambda,\lambda') \qquad \beta'_c = \frac{1}{2J'}\ln(\sqrt{2}+1) + \mathrm{O}(\lambda,\lambda')$$

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[universality]

Assume for definiteness $J > 0$

$$H(\sigma, \sigma') = -J \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} - J \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j}$$

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- for $0 < |J_4| << J$: one critical temperature, for $\lambda = J_4/J$

  $$\beta_c = \frac{1}{2J} \ln(\sqrt{2} + 1) + \mathrm{O}(\lambda)$$

  critical exponents

  $$x_+ = 1 + X_+(\lambda) \qquad x_- = 1 + X_-(\lambda)$$

[non-universality]

For $|J' - J| \to 0$,

$$|\beta_{1,c} - \beta_{2,c}| \sim |J - J'|^{x_T}$$

A 5° index, the **transition index** x_T. Then we have 5 critical exponents:

$$x_+(\lambda) \qquad x_-(\lambda) \qquad \nu(\lambda) \qquad \alpha(\lambda) \qquad x_T(\lambda)$$

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- When critical indexes are model-**independent**, they can be compared with experiments.

	u. class	ν	ν_{th}	α	α_{th}
Rb_2C0F_4	Ising	.99±.04	1		0(log)
K_2C0F_4	Ising	.97±.04	1		0(log)
^{4}He/graphite	Potts-3			.36±.03	.33...
H_2/graphite	Potts-3			.36±.05	.33...
H/Ni (111)	Potts-4			.68±.07	.66...
PVA	SAW	.79±.01	.75		
PMMA	θ-SAW	.56±.01	.57...		
3-MP-NE	3D Ising	.625±.003	.630±.002		
SF6	3D Ising			.11±.03	.110±.003
^{4}He	3D XY	.6702±.0002	.669±.001		

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- In 8V and AT critical exponents are model-**dependent**: still a weak form of universality is retained: some *universal* formulas have been conjectured for these *nonuniversal* indexes.

Kadanoff and Wegner (1971)
Luther and Peschel (1975)

$$d\nu = 2 - \alpha \qquad \nu = \frac{1}{2 - x_+}$$

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Widom scaling relations: valid at criticality for *any model* in *any dimension* < 4; they don't *characterize* classes of models

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Extended scaling relation: characterize models with scaling limit given by *Thirring Model*

Thirring model (Thirring 1955) is a toy model of interacting, 2-dimensional, fermion, quantum field theory. The Action is

$$\int d\mathbf{x}\ \bar{\psi}_{\mathbf{x}}\,\partial\!\!\!/\psi_{\mathbf{x}} + \lambda \int d\mathbf{x}\ \ (\bar{\psi}_{\mathbf{x}}\psi_{\mathbf{x}})^2$$

for

$$\psi_{\mathbf{x}} = (\psi_{1,\mathbf{x}}, \psi_{2,\mathbf{x}}) \qquad \bar{\psi}_{\mathbf{x}} = \begin{pmatrix} \bar{\psi}_{1,\mathbf{x}} \\ \bar{\psi}_{2,\mathbf{x}} \end{pmatrix} \qquad \partial\!\!\!/ = 2 \times 2\mathrm{matrix}$$

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From the *formal* explicit solution of the Thirring model (Klaiber 1967, Hagen 1967)

$$x_+^{Th} = \frac{1 - \frac{\lambda}{4\pi}}{1 + \frac{\lambda}{4\pi}} \qquad x_-^{Th} = \frac{1 + \frac{\lambda}{4\pi}}{1 - \frac{\lambda}{4\pi}}$$

Rigorous Results

Lieb (1967), Sutherland (1967)

- $f(\beta)$, β_c and α for 6V.
  By-product: $f(\beta_c)$ and α for AT

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No exact solution for x_+, x_-, x_T; no exact solution for other Double Ising models.

Spencer, Pinson and Spencer (2000)

Ising model with finite range (even) perturbation:

$$H(\sigma) = -J \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} - J_4 V(\sigma)$$

If $\varepsilon = J_4/J$

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Method of the proof:

- Functional integral representation of the Ising model

$$Z(\Lambda,\beta) = \int D\psi D\bar{\psi}\ \exp\left\{\sum \bar{\psi} M \psi + \lambda \sum (\bar{\psi}\partial\psi)^2\right\} \qquad \lambda \sim J_4/J$$

- Renormalization group approach for computing x_+.

  based on RG approach for fermion system Feldman, Knörrer, Trubowitz, (1998)

Mastropietro (2004)

Double Ising: for $J' = J$ and J_4/J small enough

- convergent power series for $\beta_c(J_4/J)$
- convergent power series for $\nu(J_4/J)$ and $x_+(J_4/J)$.

Giuliani and Mastropietro (2005)

Double Ising: for $J \neq J'$ and J_4/J, J_4/J' small enough

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- convergent power series for $x_T(J_4/J)$ [*First time the index x_T was introduced*]

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Above power series are convergent but no explicit formulas: not useful for extended scaling formula.

Benfatto, Falco, Mastropietro (2007), (2009)

Thirring model for $|\lambda|$ small enough:

- Existence of the theory (in the sense of the Osterwalder-Schrader)
- Proof of Hagen and Klaiber's formula for correlations.

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Double Ising model: for J_4/J small enough

- proof of the universal formulas

$$2\nu = 2 - \alpha \qquad \nu = \frac{1}{2 - x_+} \qquad x_+ \, x_- = 1$$

- a new scaling relation for the index x_T

$$x_T = \frac{2 - x_+}{2 - x_-}$$

Similar results for the XYZ quantum chain

Multi-scale decomposition:

$$Z = \int dP(\psi)\, e^{\lambda V(\psi)} = E\left[e^{\lambda V(\psi)}\right]$$
$$= \lim_{h \to -\infty} E_h \circ E_{h+1} \cdots E_{-1} \circ E_0 \left[e^{\lambda V(\psi_h + \cdots + \psi_{-1} + \psi_0)}\right]$$

where $\psi_h, \ldots, \psi_{-1}, \psi_0$ are i.r.v. and

$$E_j[\psi_{j,\mathbf{x}}\psi_{j,\mathbf{y}}] = \Gamma_j(\mathbf{x}-\mathbf{y}) \qquad \text{with} \quad |\partial^m \Gamma_j(\mathbf{x})| \le \gamma^{mj} C e^{c\gamma^j |x|}$$

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Define

$$e^{\lambda_{-1}V(\varphi)+R_{-1}(\varphi)} = E_0\left[e^{\lambda V(\varphi+\psi_0)}\right]$$
$$e^{\lambda_{-2}V(\varphi)+R_{-2}(\varphi)} = E_{-1}\left[e^{\lambda_{-1}V(\varphi+\psi_{-1})+R_{-1}(\varphi+\psi_{-1})}\right]$$
$$\ldots$$
$$e^{\lambda_j V(\varphi)+R_j(\varphi)} = E_{j+1}\left[e^{\lambda_{j+1}V(\varphi+\psi_{j+1})+R_{j+1}(\varphi+\psi_{j+1})}\right]$$
$$\ldots$$

In correspondence there is a sequence of effective couplings

$$\lambda_h, \lambda_{h+1}, \ldots, \lambda_{-1}, \lambda$$

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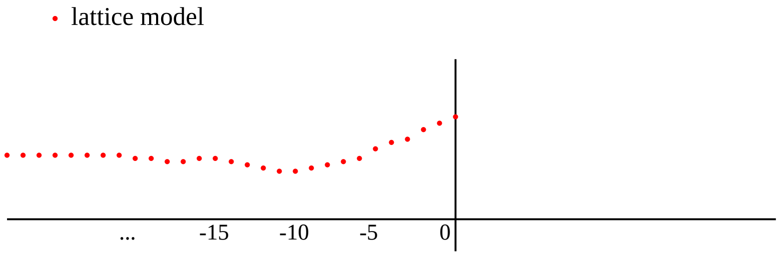

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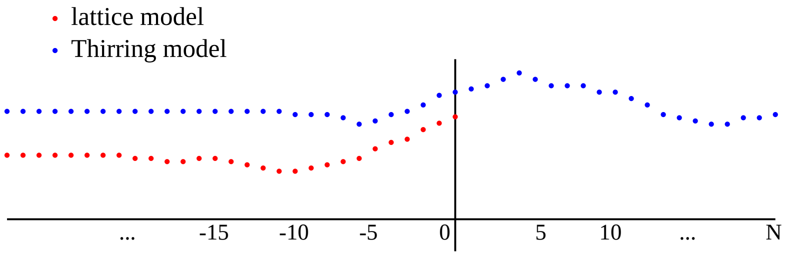

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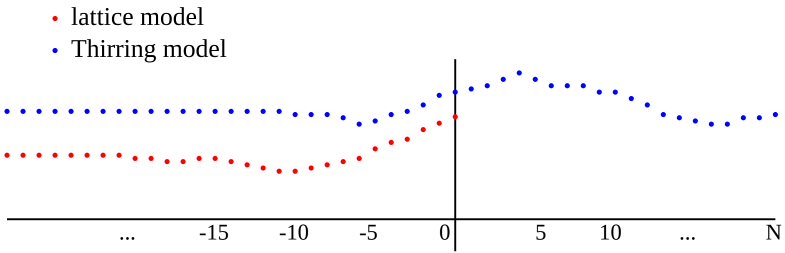

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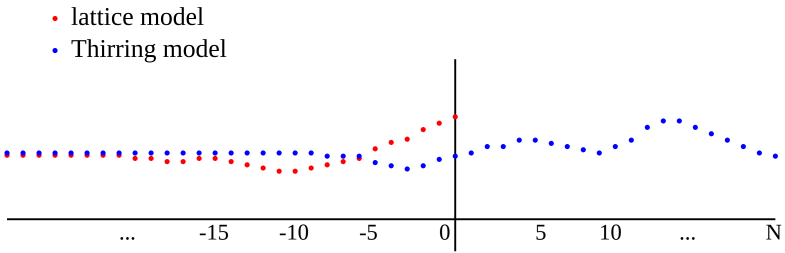

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Therefore

$$x_\varepsilon(\lambda) = x_\varepsilon^{Th}(\lambda^{Th}) \qquad \varepsilon = \pm$$

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- Connection with real laboratory experiments [Se/Ni (100)]? No **experimental** verification of Kadanoff law.

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Ising	free fermions	free fermions
Ising + n.n.n.	interacting fermions	free fermions
8V, AT, XYZ	interacting fermions	Thirring

in preparation:

$(1+1)$D Hubbard	interacting fermions	SU(2) Thirring

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- $q-$States Potts / Completely Packed Loop /...
- Spin-Spin Correlation in Ising / Other Kadanoff Formula