Pinning and depinning of interfaces in random media

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joint work with Nicolas Dirr and Michael Scheutzow

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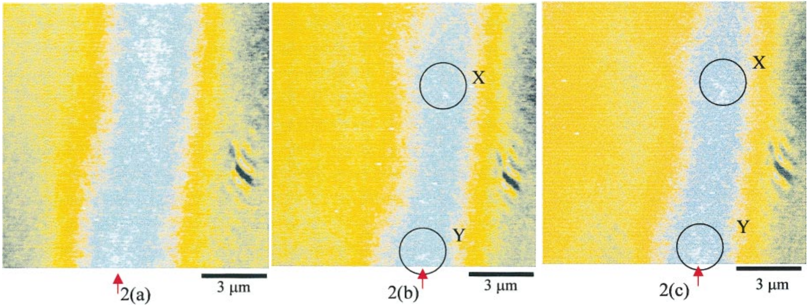
An experimental observation

Pinning of a ferroelectric domain wall

increasing applied field ⟶

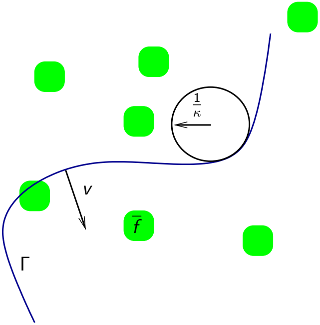


From: T. J. Yang et. al., Direct Observation of Pinning and Bowing of a Single Ferroelectric Domain Wall, *PRL*, 1999

Forced mean curvature flow

Consider an interface moving by forced mean curvature flow:

$$v_\nu(x) = \kappa(x) + \overline{f}(x), \quad x \in \Gamma \subset \mathbf{R}^{n+1}.$$



v_ν: Normal velocity of the interface

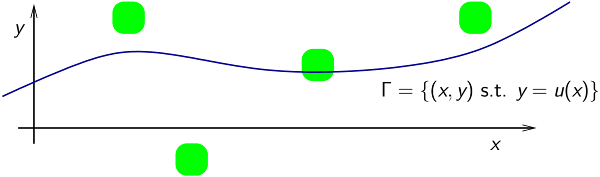
κ: Mean curvature of the interface

$\overline{f}$: Force

Can formally be thought of as a viscous gradient flow from an energy functional

$$\mathcal{H}^n(\Gamma) + \int_{\mathbf{R}^{n+1}\cap E} \overline{f}(x)\,\mathrm{d}x, \quad \Gamma = \partial E.$$

The interface as the graph of a function



$$v_\nu(x) = \kappa(x) + \bar{f}(x), \quad x \in \Gamma \subset \mathbf{R}^{n+1}$$

If $\Gamma(t) = \{(x, y) \text{ s.t. } y = u(x, t)\}$, $u \colon \mathbf{R}^n \to \mathbf{R}$, then this is equivalent to

$$u_t(x) = \sqrt{1 + |\nabla u(x)|^2}\,\frac{1}{n}\operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}}\right) + \sqrt{1 + |\nabla u(x)|^2}\,\bar{f}(x, u(x))$$

Formal approximation for small gradient:

$$u_t(x, t) = \Delta u(x, t) + \bar{f}(x, u(x, t))$$

This describes the time evolution of a nearly flat interface subject to line tension in a quenched environment.

What are we interested in?

Split up the forcing into a heterogeneous part and an external, constant, load F so that

$$\overline{f}(x, y) = -f(x, y) + F,$$

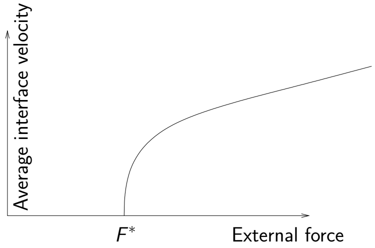
and get

$$u_t(x, t) = \Delta u(x, t) - f(x, u(x, t)) + F.$$

Question

What is the overall behavior of the solution u depending on F?

- Hysteresis: There exists a stationary solution up to a critical F^*
- Ballistic movement: $\overline{v} \;\; = \frac{u(t)}{t} \to const.$
- Critical behavior: $|\overline{v}| = |F - F^*|^{\alpha}$



The periodic case

$$u_t(x,t) = \Delta u(x,t) - f(x, u(x,t)) + F \tag{1}$$

$$u\colon T^n \times \mathbf{R}^+ \to \mathbf{R}, \quad f \in C^2(T^n \times \mathbf{R}, \mathbf{R}), \quad f(x,y) = f(x, y+1), \quad \int_{T^n \times [0,1]} f = 0$$

Thm (Dirr-Yip, 2006):

- *There exists* $F^* \geq 0$ *s.t.* (1) *admits a stationary solution for all* $F \leq F^*$.
- *For* $F > F^*$ *there exists a unique time-space periodic ('pulsating wave') solution (i.e.,* $u(x, t+T) = u(x, t) + 1$*).*
- *If critical stationary solutions (i.e., stationary solutions at* $F = F^*$*) are non-degenerate, then* $|\overline{v}| = \frac{1}{T} = |F - F^*|^{1/2} + o(|F - F^*|^{1/2})$

Existence of pulsating wave solutions can also be shown for MCF-graph case, forcing small in C^1 (Dirr-Karali-Yip, 2008).

Overview: MCF in heterogeneous media

- Caffarelli-De la Llave (Thermodynamic limit of Ising model with heterogeneous interaction)
- Lions-Souganidis (Homogenization, heterogeneity in the coefficient)
- Cardaliaguet-Lions-Souganidis (Homogenization, periodic forcing)
- Bhattacharya-Craciun (Homogenization, periodic forcing)
- Bhattacharya-D. (Phase transformations, elasticity)

Random environment

$$u_t(x,t,\omega) = \Delta u(x,t,\omega) - f(x,u(x,t,\omega),\omega) + F, \qquad (2)$$

$$u\colon \mathbf{R}^n \times \mathbf{R}^+ \times \Omega \to \mathbf{R}, \quad f\colon \mathbf{R}^n \times \mathbf{R} \times \Omega \to \mathbf{R}, \quad u(x,0) = 0.$$

Specific form of f.

Short range interaction: physicists call this 'Quenched Edwards-Wilkinson Model.'

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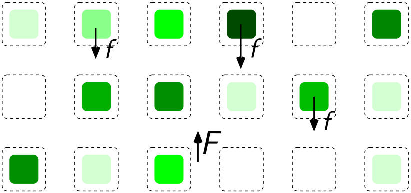

Pinning sites on lattice "(Lattice)"

$$f^{\mathrm{L}}(x,y,\omega) = \sum_{i\in\mathbf{Z}^n,\, j\in\mathbf{Z}+1/2} f_{ij}(\omega)\varphi(x-i, y-j), \quad \varphi \in C^\infty(\mathbf{R}^n \times \mathbf{R}, [0,\infty)),$$

$$\varphi(x,y) = 0 \text{ if } ||(x,y)||_\infty > r_1, \text{ with } r_1 < 1/2, \quad \varphi(x,y) = 1 \text{ if } ||(x,y)||_\infty \le r_0.$$

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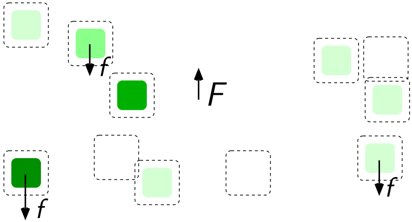

Poisson process "(Poisson)"

$$f^{\mathrm{P}}(x,y,\omega) = \sum_{k\in\mathbf{N}} f_k(\omega)\varphi(x - x_k(\omega), y - y_k(\omega)), \quad \varphi \in C^\infty(\mathbf{R}^n \times \mathbf{R}, [0,\infty)),$$

$$\varphi(x,y) = 0 \text{ if } \|(x,y)\|_\infty > r_1, \quad \varphi(x,y) = 1 \text{ if } \|(x,y)\|_\infty \le r_0, \quad y_k > r_1.$$

Existence of a stationary solution

Do solutions of the evolution equation become pinned by the obstacles for sufficiently small driving force, even though there are arbitrarily large areas with arbitrarily weak obstacles?

Existence of a stationary solution, $n = 1$

Do solutions of the evolution equation become pinned by the obstacles for sufficiently small driving force, even though there are arbitrarily large areas with arbitrarily weak obstacles?

Theorem (Dirr-D.-Scheutzow, 2009):

Case (Lattice): *Let $f_{ij} \geq 0$ be so that*

$$\mathbf{P}(\{f_{ij} > q\}) > p$$

for some $q, p > 0$. Then, there exists $F^{**} > 0$ and $v\colon \mathbf{R} \to \mathbf{R}$, $v > 0$ so that, a.s., for all $F < F^{**}$,

$$0 > Kv - f^{\mathsf{L}}(x, v(x), \omega) + F.$$

Here, K is either the Laplacian or the mean curvature operator.

This implies that v is a supersolution to the stationary equation, and thus provides a barrier that a solution starting with zero initial condition can not penetrate (comparison principle for viscosity solutions).

Existence of a stationary solution, $n \geq 1$

Do solutions of the evolution equation become pinned by the obstacles for sufficiently small driving force, even though there are arbitrarily large areas with arbitrarily weak obstacles?

Theorem (Dirr-D.-Scheutzow, 2009):

Case (Poisson): *Let (x_k, y_k) be distributed according to a $n+1$-d Poisson process on $\mathbf{R}^n \times [r_1, \infty)$ with intensity λ, f_k be iid strictly positive and independent of (x_k, y_k). Then there exists $F^{**} > 0$ and $v \colon \mathbf{R} \to \mathbf{R}$, $v > 0$ so that, a.s., for all $F < F^{**}$,*

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A percolation problem

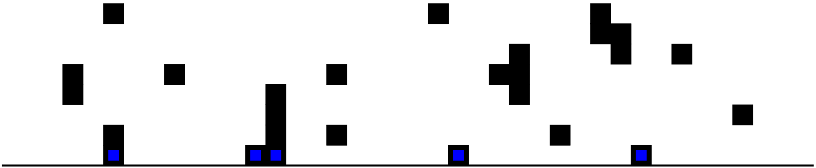
Let $\mathscr{Z} = \mathbf{Z}^n \times \mathbf{N}$.
We consider site percolation on $\mathscr{Z}$: let $p \in (0, 1)$.
Each site is declared *open* with probability p, independent for all sites.

Theorem (Dirr-D.-Grimmett-Holroyd-Scheutzow):

There exists $p_c < 1$ such that if $p > p_c$, then a random non-negative discrete 1-Lipschitz function w: $\mathbf{Z}^n \to \mathbf{N}$ exists with $(x, w(x))$ a.s. open for all $x \in \mathbf{Z}^n$.

Idea:

Blocking argument. Define Λ-path: Finite sequence of distinct sites x_i from a to b so that $x_i - x_{i-1} \in \{\pm e_{n+1}\} \cup \{-e_{n+1} \pm e_j : j = 1, \ldots, n\}$.
Admissible if going up only to closed sites.
Which sites on the positive side are reachable from anywhere below?



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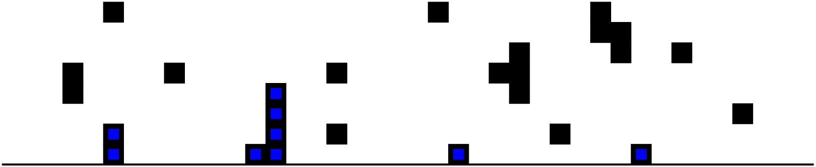

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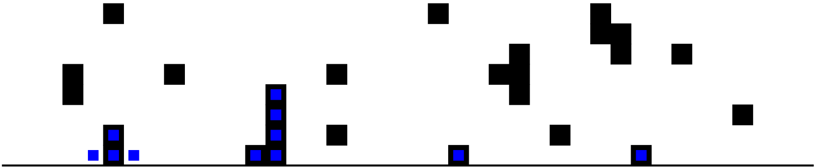

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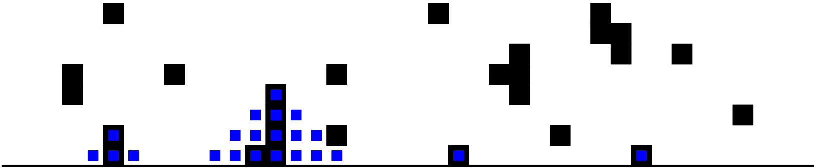

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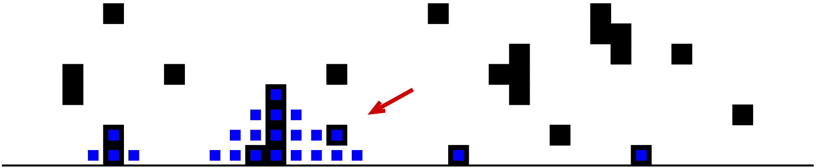

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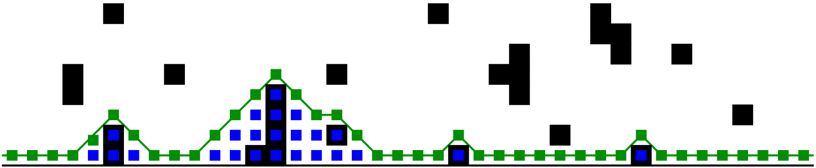
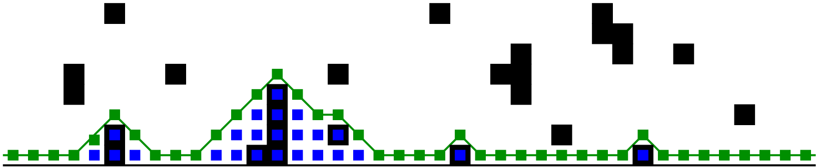


Proof of Lipschitz-Percolation Theorem



- Define $G := \{b \in \mathcal{Z} :$ there ex. path to b from some $a \in \mathbf{Z}^n \times \{\dots, -1, 0\}\}$.
- We have $\mathbf{P}(he_{n+1} \in G) \leq C(cq)^h$, thus there are only finitely many sites in G above each $x \in \mathbf{Z}^n$.
- Define $w(x) := \min\{t > 0 : (x, t) \notin G\}$.
- Properties of w follow from the definition of admissible paths.

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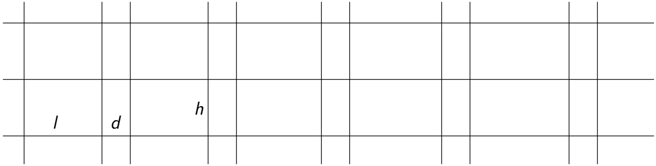
Proof of Pinning-Theorem in n+1 dimensions

- Rescale so that each box of size $l \times h$ contains an obstacle at x_k, y_k of strength f_0 with probability p_c.



- Construct supersolution
  - inside obstacles: parabolas: $\Delta v_{\text{in}} = F_1 < \frac{f_0}{2}$.
  - outside obstacles: $\min_k\{v(x - x_k)\}$, where $\Delta v_{\text{out}} = -F_2$ on $B_{r_l}(0) \setminus B_{r_0}(0)$, $v = 0$ on $\partial B_{\rho_1}(0)$, $\nabla v \cdot \nu = 0$ on $\partial B_{\rho_1}(0)$
  - gluing function v_{glue} with gradient supported on gaps of size d, $v_{\text{glue}} = y_k$.
  - scaling:

$$CF_1 > F_2(h^{-1/n} + d)^n \quad \text{and} \quad F_2 \geq \frac{h}{d^2}.$$

- Works for lattice model if $n = 1$ and Poisson model for any n.
- Works also for MCF.

`arXiv:0911.4254v2 [math.AP]`

Depinning

Can we exclude pinning for unbounded obstacles, if the probability of finding a large obstacle is sufficiently small and the driving force is sufficiently high?

Depinning (only $n = 1$, only Lattice case)

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Theorem (Dirr-Coville-Luckhaus, 2009):
Nonexistence of a stationary solution

Let f_{ij} be so that $\mathbf{P}(\{f_{ij} > q\}) < \alpha \exp(-\lambda q)$ for some $\alpha, \lambda > 0$. Then there exists $F^{***} > 0$ so that a.s. no stationary solution $v > 0$ for equation (2) at $F > F^{***}$ exists.

Proof by asserting that every possible stationary solution of (2) with Dirichlet boundary conditions $u(-L) = 0, u(L) = 0$ becomes large as $L \to \infty$. (The pinning sites are not strong enough to keep the solution flat.)

Depinning (only $n = 1$, only Lattice case) (cont.)

Theorem (D.-Scheutzow, 2011):
Ballistic propagation

Let $u(x, t, \omega)$ solve $u_t(x, t) = u_{xx}(x, t) - f^{\mathsf{L}}(x, u(x, t), \omega) + F$, with zero initial condition, $x \in \mathbf{R}$. Aussume that $\beta := \exp\{\lambda f_{00}\} < \infty$, f_{ij} iid. Then there exists V: $[0, \infty) \to [0, \infty)$, non-decreasing, not identically zero, depending only on λ, β, and r_1, such that

$$\mathbf{E}\, \frac{1}{t} \int_0^1 u(\xi, t)\, \mathrm{d}\xi \geq V(F) \quad \text{for all } t \geq 0.$$

There is an explicit formula for a possible choice of $V(F)$. In particular, the expected value of the velocity is strictly positive for $F > F^{***}$.

Idea of proof: Every solution of a discretized initial value problem (in space!) $0 = (\hat{u}_{i-1} + \hat{u}_{i+1} - 2\hat{u}_i - f_i(\hat{u}_i(t), \omega) + F)^+ - a_i$, for any initial condition for $\hat{u}_0$, $\hat{u}_{-1}$, for a_i small in a suitable average sense, must become negative for some i a.s..

Proof of depinning

Central Lemma:

Let $\overline{f}_{ij} : \Omega \to [0, \infty)$, $i, j \in \mathbf{Z}$ be random variables s.t. $\overline{f}_i : \Omega \times \mathbf{Z} \to [0, \infty)$ defined as $\overline{f}_i(\omega, j) := \overline{f}_{ij}(\omega)$ are independent. Assume that there ex. $\overline{\beta} > 0, \lambda > 0$ s.t. $\overline{\beta} := \sup_{k,l \in Z} \mathbf{E} \exp(\lambda \overline{f}_{kl}) < \infty$. Then there ex. Ω_0 of full measure such that for any function w: $\Omega \times \mathbf{Z} \to \mathbf{Z}$ that is bounded from below and any $\omega \in \Omega_0$ we have

$$\limsup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \left(w_{i-1} + w_{i+1} - 2w_i - \overline{f}_i(\omega, w_i) + F \right)^+ \geq \overline{V}(F),$$

where $\overline{V}(F) := \sup_{\mu > \lambda} \frac{1}{\mu} \left(\lambda F - \log \left(\frac{1}{1-e^{-\lambda}} - \frac{1}{1-e^{\lambda-\mu}} \right) - \log \overline{\beta} \right) \geq 0$.

Proof: Let $\mu > \lambda$ and define

$$Y_k := \sum_{\substack{\text{all paths } w \text{ of length } k \\ \text{starting at presc. values at } i \in \{-1, 0\}}} \exp(\lambda(w_k - w_{k-1}) - \mu s_k),$$

$s_k := \sum_{i=0}^{k-1} (\Delta_1 w - \overline{f}_i(\omega, w_i) + F)^+$. A calculation shows that for $\gamma = \overline{\beta} \exp(-\lambda F) \left(\frac{1}{1-e^{-\lambda}} - \frac{1}{1-e^{\lambda-\mu}} \right)$, Y_k / γ^k is a non-negative supermartingale.

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Proof (cont): Thus there ex. a set Ω_0 of full measure such that $\sup_{k \in N_0} Y_k / \gamma^k$ is finite. We then have

$$\limsup_{k \to \infty} \frac{1}{k} \sup(\lambda(w_k - w_{k-1}) - \mu s_k) \leq \limsup_{k \to \infty} \frac{1}{k} \log Y_k \leq \log \gamma.$$

So, $\quad \lambda \limsup_{k \to \infty} \frac{w_k - w_{k-1}}{k} < \log \gamma + \mu V(F) = 0$ on $\left\{ \limsup_{k \to \infty} \frac{s_k}{k} < V(F) \right\} \cap \Omega_0$

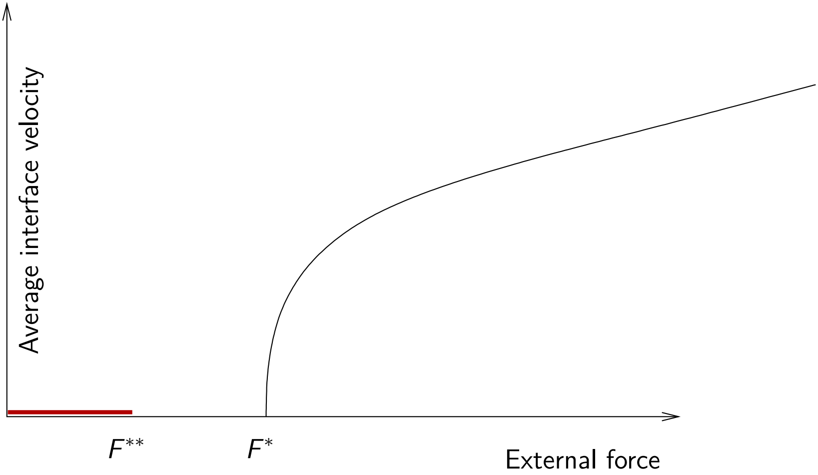
Steps in the proof of the theorem

- Assume $u(x,t)$ is a solution of the evolution equation (a slightly modified evolution equation yielding a subsolution, actually)
- Discretize in x to obtain $\hat{u}$ as seen in Coville-Dirr-Luckhaus
- The discrete Laplacian is bounded from below by the integrated effect of u_t, $f(x,u(x))$, and F.
- Assume the statement of the theorem is false, i.e., $\frac{1}{t}\mathbf{E}\int_0^1 u(\xi,t)\,\mathrm{d}\xi < V(F)$ for some t
- By the ergodic theorem, we have at some $t_0 \le t$ that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\hat{u}_{i-1}+\hat{u}_{i+1}-2\hat{u}_i-\overline{f}_i(\hat{u}_i)+F)^+ < \overline{V}(F)$
- Discretize again by rounding to the nearest integer, obtaining a path $w_i\colon \mathbf{Z}\to\mathbf{Z}$ that is bounded from below. Apply the Lemma with $\overline{f}_i$ chosen appropriately (to dominate pointwise in ω the effect of going through inclusions, this yields a slightly slower but still exponential tail)
- On the set Ω_0, this is a contradiction to the lemma
- Remark: As a corollary, we also get $\limsup_{t\to\infty} u(x,t,\omega)/t \ge V(F)$ for any x and almost surely in ω.

arXiv:1102.5691v1 [math.PR]

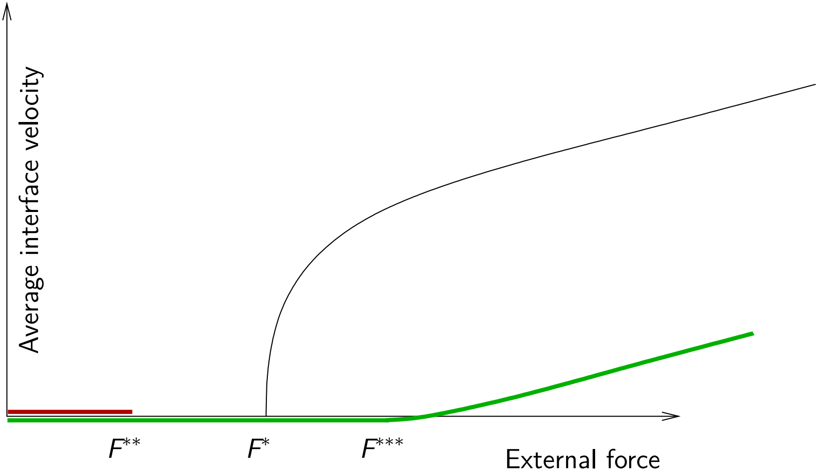
Summary of the results

$n \geq 1$, obstacles scattered by Poisson process, any strength



Summary of the results (cont.)

$n = 1$, obstacle on a lattice, obstacles with exponential tails



Many open questions

- Almost sure liminf statement for depinning
  (i.e., $\liminf_{t\to\infty} u(x,t,\omega)/t \geq V(F)$ a.s.)
- Nonexistence/positive velocity in higher dimensions
- More general random fields, in particular pinning if $f \not\geq 0$
- Nonlocal operators

- Growth of correlations and Hölder seminorm near critical F^*
- Behavior at $F = F^*$

Thank you for your attention.