Anderson localization/delocalization transition for a supersymmetric sigma model

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- the general setting: Anderson localization, random matrices and sigma models
- a toy model for quantum diffusion

Disordered conductors

Anderson localization: disorder-induced localization of conducting electrons

the model

- quantum system $\rightarrow$ lattice field model
- Hamiltonian $H = H^*$: matrix on Λ = cube in $\mathbb{Z}^d$
- ψ eigenvector of H: $\sum_{j\in\Lambda} |\psi_j|^2 = 1 \rightarrow |\psi_j|^2 \propto$ prob. of finding the electron at lattice point j

then:

- $|\psi_j|^2 \simeq const\ \forall j \Rightarrow$ extended state (conductor)
- $|\psi_j|^2 \neq 0$ only near $j = j_0 \Rightarrow$ localized state (insulator)

The problem

study statistical properties of large matrices with random distributed elements:

$$H^* = H, \;\; H_{ij} \; i,j \in \Lambda \subseteq \mathbb{Z}^d, \;\; P(H) \text{ probability distribution}$$

limit $|\Lambda| \to \infty$

- **eigenvalues** $\lambda_1, ...\lambda_{|\Lambda|}$
  - correlation functions
  - largest eigenvalue...
- **eigenvectors** ψ_λ
  - localized: $\psi_\lambda = (0, 1, 0, \ldots, 0)$
  - extended: $\psi_\lambda = \frac{1}{\sqrt{|\Lambda|}}(1, 1, 1, \ldots, 1)$

Models for quantum diffusion:

a) Random Schrödinger

$H_\Lambda = -\Delta + \lambda V,$ Δ discrete Laplacian on Λ, $V_{ij} = \delta_{ij} V_j$ i.i.d. random var., $j \in \Lambda$

$\lambda =$ strenght of the disorder

$$H_N = \begin{pmatrix} 2+\lambda V_1 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2+\lambda V_2 & -1 & 0 & 0 & & & \\ 0 & -1 & \ddots & & & & & \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & & & 0 \\ \vdots & & & & & & \ddots & \\ -1 & 0 & \dots & 0 & & & -1 & 2+\lambda V_N \end{pmatrix}$$

Limit cases

- $\lambda = 0$: $H = -\Delta$ (extended states)
- $\lambda >> 1$: $H \sim \lambda V$ diagonal matrix (localized states)

General case: $\Lambda \to \mathbb{Z}^d$, λ fixed

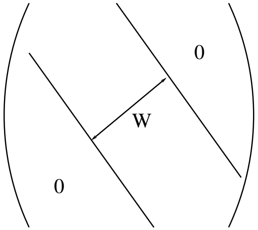
- $d = 1 \to \forall\lambda > 0$ localization (proved)
- $d = 2 \to \forall\lambda > 0$ localization (proved for large disorder)
- $d = 3 \to \begin{cases} \lambda \text{ large} & \text{localized (proved)} \\ \lambda \text{ small} & \text{extended (conjecture)} \end{cases}$

very hard problem!

2) Random band matrix: $H^* = H$, H_{ij} $i, j \in \Lambda \subset \mathbb{Z}^d$

- H_{ij} ind. gaussian rand. var. with $\langle H_{ij} \rangle = 0$
- $\langle |H_{ij}|^2 \rangle = J_{ij}$ with $0 \le J_{ij} \le e^{-|i-j|/W}$

H =



band width $= W$

Limit cases

- $W = |\Lambda|$: GUE (extended states)
- $W = 0$: diagonal disorder (localized states) $\Rightarrow W \simeq \lambda^{-1}$

General case: $|\Lambda| \to \infty$, W fixed

expect same behavior as RS with $W \simeq \lambda^{-1}$:

- $d = 1 \to \forall\, W \geq 0$ localization ("proved")
- $d = 2 \to \forall\, W \geq 0$ localization
- $d = 3 \to \begin{cases} W \text{ small} & \text{localized} \\ W \text{ large} & \text{extended} \end{cases}$

$d = 3 \to$ rigorous estimates for the density of states
(necessary but not enough)

a bit easier (more average) but still a hard problem!

Criteria for quantum diffusion

Green's Function: $G_\epsilon(E;x,y) = (H - E + i\varepsilon)^{-1}(x,y) \quad E \in \mathbb{R}, \varepsilon > 0$

$\rightarrow \langle |G_\epsilon(E;x,y)|^2 \rangle_H = \int \; dH \;\, P(H) \; |G_\epsilon(E;x,y)|^2$

1. $|x - y| >> 1$

- $\langle |G_\epsilon(E;x,y)|^2 \rangle_H \leq \frac{const}{\varepsilon} e^{-|x-y|/\ell} \Rightarrow$ localized
- $\langle |G_\epsilon(E;x,y)|^2 \rangle_H \geq \frac{const}{|x-y|^\alpha} \Rightarrow$ extended

2. $x = y$**,** $\varepsilon|\Lambda| = 1$

- $\langle |G_\epsilon(E;x,x)|^2 \rangle_H \geq \frac{const}{\varepsilon} \Rightarrow$ localized
- $\langle |G_\epsilon(E;x,x)|^2 \rangle_H \leq \; const \Rightarrow$ extended

Technique: supersymmetric approach

1. change of representation → new expression where saddle analysis is possible

2. rigorous saddle analysis
   - integral along the saddle (symmetries, convexity bounds)
   - fluctuations around the saddle (cluster expansion, small probability)

1. Change of representation

algebraic operations involving **ordinary** (bosonic) and **anticommuting** (fermionic) variables

$$\langle |G_\epsilon(E;x,y)|^2 \rangle_H \xrightarrow{\text{SUSY}} \int d\mu(\{Q_j\})\mathcal{O}(Q_x,Q_y)$$

$$H_{ij} \longrightarrow Q_j = 4\times 4 \text{ supermatrix}$$

$$|\Lambda|^2 \text{ variables} \longrightarrow 4|\Lambda| \text{ variables}$$

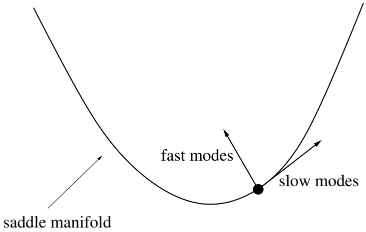
$$H_{ij} \text{ independent var.} \longrightarrow Q_j \text{ strongly correlated}$$

1. advantages: less variables; saddle analysis is possible
2. problems: integrate out fermionic variables; complex measure (no probability estimates); saddle manifold non compact

2. Saddle analysis: analytic tools

new integration variables

- slow modes along the saddle manifold → **non linear sigma model (NLSM)**
- fast modes away from the saddle manifold



NLSM is believed to contain the low energy physics

non linear sigma model

$$d\mu(Q) \;\rightarrow\; d\mu^{saddle}(Q) = [\prod_{j\in\Lambda} dQ_j\ \delta(Q_j^2 - Id)]\ \ e^{-F(\nabla Q)}e^{-\varepsilon M(Q)}$$

features

- saddle is non compact
- no mass: $\varepsilon = \frac{1}{|\Lambda|} \rightarrow 0$ as $|\Lambda| \rightarrow \infty$
- internal symmetries (from SUSY structure)

main problem: obtain the correct ε behavior

hard to exploit the symmetries $\rightarrow$ try something “easier”

A nice SUSY model for quantum diffusion

vector model (no matrices), Zirnbauer (1991) $\rightarrow$ expected to have same features of exact SUSY NLSM model for random band matrix

main advantages

- after integrating out anticommuting variables measure is **positive**
- symmetries are simpler to exploit

$\Rightarrow$ good candidate to develop techniques to treat quantum diffusion

The model

same symmetric group as for the NLSM:

- supermatrix $Q_j \to$ supervector $v_j = (x_j, y_j, z_j, \xi_j, \eta_j)$,

$$x, y, z \in \mathbb{R}, \qquad \xi, \eta \text{ grassmann variables}$$

$$(v, v') = -zz' + xx' + yy' + \xi\eta' - \eta\xi'$$

- saddle constraint: $(Q_j)^2 = Id \longrightarrow (v_j, v_j) = -1$

$$\Rightarrow z_j = \sqrt{1 + x_j^2 + y_j^2 + 2\xi_j\eta_j}$$

- kinetic term: $F(\nabla Q) \to (\nabla v, \nabla v)$

- mass: $M(Q) \to z - 1$

Change of coord. + integrating out the Grassman var.

$$d\mu(v) \longrightarrow d\mu(t) = [\prod_j dt_j e^{-t_j}] \; e^{-\mathcal{B}(t)} \; \det{}^{1/2}[M_\Lambda^\varepsilon(t)] \quad t_j \in \mathbb{R},\; j \in \Lambda$$

- $\mathcal{B}(t) \;=\; \beta \sum_{<j,j'>} (\cosh(t_j - t_{j'}) - 1) \;+\; \varepsilon \sum_{j\in\Lambda} (\cosh\; t_j - 1),$

  $\varepsilon > 0 =$ "mass", $\;\beta > 0$
- $M_\Lambda^\varepsilon(t) > 0$ **positive quadratic form**:

$$\sum_{ij\in\Lambda} f_i M_\Lambda^\varepsilon(t)_{ij} f_j \;=\; \beta \sum_{<j,j'>} (f_j - f_{j'})^2 \; e^{t_j + t_{j'}} + \varepsilon \sum_{j\in\Lambda} f_j^2 \; e^{t_j} > 0$$

The observable:

current-current correlation $\rightarrow$

$$\mathcal{O} = D_{xy}^{-1} = e^{t_x} M_{xy}^{-1} e^{t_y}$$

Main result: phase transition in $d = 3$

- β large: $\left\langle\ D_{xy}^{-1}\ \right\rangle \ \simeq\ (-\beta\Delta + \varepsilon)_{xy}^{-1}\ \sim \frac{1}{|x-y|}$
  "as a quadratic form"

  $\longrightarrow$ **extended states**

- β small: $\left\langle\ D_{xy}^{-1}\ \right\rangle\ \leq\ \frac{1}{\varepsilon} e^{-m_\beta |x-y|}$ pointwise

  $\longrightarrow$ **localized states**

large β

Main result (M.D. T. Spencer, M. Zirnbauer)

If $\beta >> 1$ and $d = 3$ the t field does not fluctuate:

$$\langle\, (\cosh t_x - t_y)^m \,\rangle \leq 2 \qquad \forall x, y \in \Lambda$$

for $0 \leq m \leq \beta^{1/8}$ uniformly in the volume Λ and ε.

Proof:

- bound on nn fluctuations $|x - y| = 1$: Ward identites
- conditional bound on large scale fluctuations: Ward identites
- unconditional bound on large scale fluctuations: previous bounds plus induction on scales ('simple' renormalization group)

Ward identities

$$\text{SUSY} \;\Rightarrow\; 1 \;=\; \left\langle \cosh^m(t_x - t_y) \left(1 - \frac{m}{\beta} C_{xy}\right) \right\rangle$$

where $0 \;<\; C_{xy} \;:= e^{t_x + t_y} \left[(\delta_x - \delta_y)\; M^{-1}(t)\; (\delta_x - \delta_y)\right]$

if $C_{xy} \leq 1$ **for all** t **configurations** then

$$1 = \left\langle \cosh^m(t_x - t_y) \left(1 - \frac{m}{\beta} C_{xy}\right) \right\rangle \geq \langle \cosh^m(t_x - t_y) \rangle \left(1 - \frac{m}{\beta}\right)$$

$$\Rightarrow \quad \langle \cosh^m(t_x - t_y) \rangle \;\leq\; \frac{1}{1 - \frac{m}{\beta}}$$

- if $|x-y|=1$ $C_{xy} \leq 1$ for all t configurations:

$$\Rightarrow \quad \langle \cosh^m(t_x - t_y) \rangle \leq \frac{1}{1-\frac{m}{\beta}}$$

- if $|x-y|>1$ **no uniform bound on** C_{xy}!:
  - $C_{xy} < 1$ if lower scale fluctuations are bounded:

$$\Rightarrow \quad \langle \chi_{xy} \cosh^m(t_x - t_y) \rangle \leq \frac{1}{1-\frac{m}{\beta}}$$

  conditioning must respect SUSY!

  - unconditional bound: induction on scales

β small

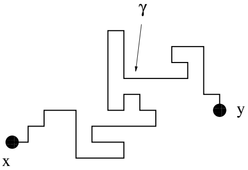
Main result (M.D., T. Spencer)

If $\beta << 1$ then for **any dimension** $d \geq 1$

$$\langle \; D_{xy}^{-1} \; \rangle \;\leq\; \frac{1}{\varepsilon}\, e^{-m|x-y|} \text{ for all } x, y \text{ unif. in } \Lambda \text{ and } \varepsilon$$

Proof

a. reduce the problem to integral along a path γ_{xy} connecting x to y



b. the integral along γ is 1d and can be computed "almost" explicitely. The sum over paths is controlled by β small.

Conclusions

advantages of the SUSY technique

- Ward identities + induction on scales allow to obtain bounds (no multiscale analysis or cluster expansion): "easy" renormalization group
- method gives information both in the extended states and localized states region

open problems

- generalize this technique to the band matrix model (the fermionic term is more complicated, the measure is no longer real)