Strong Central Limit Theorems in PDE with random coefficients and Euclidean Field Theory

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Uniformly Convex Euclidean Field Theory

Let $V : \mathbf{R}^d \to \mathbf{R}$ be a C^2 uniformly convex function, so for some $\lambda, \Lambda > 0$

$$\lambda I_d \ \le \ V''(z) \ \le \ \Lambda I_d \ , \quad \text{for } z \in \mathbf{R}^d \ .$$

Let Ω be the space of all functions $\phi : \mathbf{Z}^d \to \mathbf{R}$ and $\mathcal{F}$ the Borel algebra generated by finite dimensional rectangles $\{\phi(\cdot) \in \Omega : \ |\phi(x_i) - a_i| < r_i, \ i = 1, .., N\}$. Translation operators $\tau_x, \ x \in \mathbf{Z}^d$, act on Ω by $\tau_x \phi(z) = \phi(x + z), \ z \in \mathbf{Z}^d$. For any $d \ge 1$ and $m > 0$ there is a unique translation invariant probability measure P on $(\Omega, \mathcal{F})$ with pdf formally given by

$$\exp\left[- \sum_{x \in \mathbf{Z}^d} V(\nabla \phi(x)) + m^2 \phi(x)^2 \right] \prod_{x \in \mathbf{Z}^d} d\phi(x) / \text{normalization} \ .$$

For $d \ge 3$ the limiting measure $m \to 0$ on fields $\phi : \mathbf{Z}^d \to \mathbf{R}$ exists, and for $d = 1, 2$ the limiting measure $m \to 0$ on gradients $\omega(\cdot) = \nabla \phi(\cdot)$ of fields $\phi : \mathbf{Z}^d \to \mathbf{R}$ exists (Funaki-Spohn 1997).

Correlation Functions

The 2 point correlation function for the field theory is the function $x \to \langle \phi(x)\phi(0) \rangle,\ x \in \mathbf{Z}^d$, where $\langle \cdot \rangle$ denotes expectation with respect to the measure on $(\Omega, \mathcal{F})$. In the Gaussian case (V quadratic) one has

$$(a) \quad \langle \phi(x)\phi(0) \rangle \; = \; G_\eta(x)/2 \text{ with } \eta = m^2, \quad x \in \mathbf{Z}^d,$$

where $G_\eta(\cdot)$ is the fundamental solution to a constant coefficient elliptic equation

$$\eta G_\eta(y) + \nabla^* \mathbf{a}_{\rm hom} \nabla G_\eta(y) \; = \delta(y) \; , \quad y \in \mathbf{Z}^d \; ,$$

with $\mathbf{a}_{\rm hom} = V''(\cdot)/2 = \text{constant}$. More generally if $f : \mathbf{Z}^d \to \mathbf{R}$ and $(\cdot, \cdot)$ denotes the Euclidean inner product on $L^2(\mathbf{Z}^d)$ then

$$(b) \quad \log \{ \langle \, \exp [(f, \phi)] \, \rangle \} \; = \; \frac{1}{4} \sum_{y, y' \in \mathbf{Z}^d} f(y) G_\eta(y - y') f(y') \; .$$

Note that (b) implies (a) by taking $f(y) = \mu[\delta(y - x) - \delta(y)],\ y \in \mathbf{Z}^d$, in (b) and letting $\mu \to 0$.

Homogenization of massless field theories

In the general massless $m = 0$ non-Gaussian case one has (Naddaf-Spencer 1997) for $\mathbf{C}^\infty$ vector fields $h : \mathbf{Z}^d \to \mathbf{R}^d$ of compact support the limit,

$$\lim_{R\to\infty} \frac{1}{R^d} \sum_{y,y'\in\mathbf{Z}^d} h(y/R)\{\langle\nabla\phi(y)\nabla\phi(y')\rangle - \nabla\nabla^* G_0(y-y')\}h(y'/R) \;=\; 0$$

for some $\mathbf{a}_{\rm hom}$ depending on $V(\cdot)$ which satisfies the uniformly elliptic condition $\lambda I_d \;\le\; \mathbf{a}_{\rm hom} \;\le\; \Lambda I_d$. The proof of this uses the Helffer-Sjöstrand formula (1994):

$$\langle (F_1(\cdot) - \langle F_1\rangle)\, F_2(\cdot)\rangle \;=\; \langle dF_1(\cdot)\cdot[d^* d + \nabla^* V''(\nabla\phi(\cdot))\nabla + 2m^2]^{-1} dF_2(\cdot)\rangle\;,$$

which expresses the 2 point correlation function as the expectation of the solution of a PDE with random coefficients. The result follows by adapting methods to prove homogenization of elliptic PDE with random coefficients.

Calderon-Zygmund operators

The $d \times d$ matrix function $\nabla\nabla^* G_0(y),\ y \in \mathbf{Z}^d$, induces an integral operator T on $\ell^2(\mathbf{Z}^d, \mathbf{R}^d)$ by

$$Th(y) \ = \ \sum_{y' \in \mathbf{Z}^d} \nabla\nabla^* G_0(y - y')h(y') \ , \quad h \in \ell^2(\mathbf{Z}^d, \mathbf{R}^d) \ ,$$

with norm $\|T\|_2 \le 1$. This follows from the Fourier space representation for $\hat{T}$ on $L^2([-\pi, \pi]^d)$,

$$\hat{T}\hat{h}(\xi) \ = \ \frac{e(\xi)e^*(\xi)}{e^*(\xi)e(\xi)}\hat{h}(\xi) \ , \quad e(\xi) = [e^{-i\xi_1} - 1, ..., e^{-i\xi_d} - 1] \ .$$

Calderon-Zygmund Theorem: For $1 < p < \infty$, the operator T is bounded on $\ell^p(\mathbf{Z}^d, \mathbf{R}^d)$ and $\lim_{p \to 2} \|T\|_p = 1$.
Weighted norm inequalities: Let $w : \mathbf{Z}^d \to \mathbf{R}$ be a Muckenhoupt A_2 weight. The operator T is bounded on $\ell^2_w(\mathbf{Z}^d, \mathbf{R}^d)$ and $\lim_{w \to 1} \|T\|_{2,w} = 1$ (Pattakos-Volberg 2010).

Poincaré inequalities

The Poincaré inequality in its simplest form is as follows: Suppose $\langle\cdot\rangle$ denotes expectation with respect to a measure on $\mathbf{R}^n$. Then for all C^1 functions $F:\mathbf{R}^n\to\mathbf{R}$ with gradient $dF(\cdot)$, there is a constant C such that

$$\mathrm{var}[F] \ = \ \langle\ [F(\cdot)-\langle F\rangle]^2\ \rangle \ \le \ C\langle |dF(\cdot)|^2\ \rangle\ .$$

Poincaré for EFT (Brascamp-Lieb 1976): $F(\cdot)$ is a function of fields $\phi:\mathbf{Z}^d\to\mathbf{R}$. Then

$$\mathrm{var}[F] \ \le \ \langle\ dF(\cdot,\phi(\cdot))[-\lambda\Delta+2m^2]^{-1}dF(\cdot,\phi(\cdot))\ \rangle$$
$$= \ \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{\langle\ |\hat{dF}(\xi,\phi(\cdot))|^2\ \rangle}{\lambda e^*(\xi)e(\xi)+2m^2}\,d\xi\ .$$

Proof: Follows from HS formula. Also follows from the Poincaré inequality for functions of time dependent fields $\phi(x,t)$ where $t\to\phi(\cdot,t)$ is the diffusion process with invariant measure given by the EFT. Proof of Poincaré in this case follows from the Clark-Ocone formula (Gourcy-Wu 2006).

Dimension $d = 2$ massless EFT

Taking $f(y) = \mu[\delta(y-x) - \delta(y)],\ y \in \mathbf{Z}^d$, we see that

$$\log\left\{\langle\, e^{\mu[\phi(0)-\phi(x)]}\,\rangle\right\} \;=\; \frac{\mu^2}{2}\langle\, [\phi(0)-\phi(x)]^2\,\rangle \;=\; \mu^2 \lim_{\eta\to 0}[G_\eta(0)-G_\eta(x)]$$

in the Gaussian case, where

$$\lim_{\eta\to 0}[G_\eta(0) - G_\eta(x)] \;=\; C\log|x| + O(1/|x|) \quad \text{as } |x| \to \infty\ .$$

In the non-Gaussian case we have the following:
Theorem 1 (Conlon-Spencer 2011): Assume $\lambda/\Lambda > 1/2$. Then there is a constant C depending only on $V(\cdot)$ such that

$$\left|\log\left\{\langle\, e^{\mu[\phi(0)-\phi(x)]}\,\rangle\right\} - \frac{\mu^2}{2}\langle\, [\phi(0)-\phi(x)]^2\,\rangle\right| \;\le\; C\mu^3\ .$$

Proof: This follows from a third moment inequality i.e. on $\langle\, [\phi(0)-\phi(x)]^3\,\rangle_{x,\mu}$ for a non-translation invariant measure depending on x, μ. Gaussian case is trivial $C = 0$.

Second moment inequalities for $d \geq 1$

Theorem 2 (Conlon-Spencer 2011): Let $G_0(\cdot)$ be the Green's function corresponding to the Naddaf-Spencer homogenization matrix $\mathbf{a}_{\rm hom}$. Then there exists $\alpha > 0$ depending only on $d, \lambda/\Lambda$ and C depending only on $V(\cdot)$ such that

$$|\langle\phi(x)\phi(0)\rangle - G_0(x)/2| \ \leq \ C/[|x|+1]^{d-2+\alpha} \quad \text{for } d \geq 3,$$

$$|\langle\nabla\phi(x)\phi(0)\rangle - \nabla G_0(x)/2| \ \leq \ C/[|x|+1]^{d-1+\alpha} \quad \text{for } d \geq 2,$$

$$|\langle\nabla\phi(x)\nabla\phi(0)\rangle - \nabla\nabla^* G_0(x)/2| \ \leq \ C/[|x|+1]^{d+\alpha} \quad \text{for } d \geq 1.$$

Corollary: For $d = 2$ there is the inequality

$$\left|\langle\ [\phi(0)-\phi(x)]^2\ \rangle - 2\lim_{\eta\to 0}[G_\eta(0)-G_\eta(x)]\right| \ \leq C/[|x|+1]^\alpha\ .$$

Proof: Write $\phi(0)-\phi(x) = (\ h(\cdot), \nabla\phi(\cdot))$ where $h(y) \sim C/|y|$ for $|y|$ large.

Third moment inequalities for $d \geq 1$

Theorem A: Suppose $d \geq 1$ and $\lambda/\Lambda > 1/2$. Then there is a positive constant $C(\lambda, \Lambda)$ depending only on λ, Λ such that for any $h_1, h_2, h_3 \in \ell^2(\mathbf{Z}^d, \mathbf{R}^d)$ and $x \in \mathbf{Z}^d,\ \mu \in \mathbf{R}$,

$$|\langle \prod_{j=1}^{3} \left[(h_j(\cdot), \nabla\phi(\cdot)) - \langle (h_j, \nabla\phi) \rangle_{x,\mu} \right] \rangle_{x,\mu}| \ \leq \ C(\lambda, \Lambda) \|h_1\| \|h_2\| \|h_3\| \sup_{\xi \in \mathbf{R}^d} |V'''(\xi)|$$

where

$$\langle \, F(\phi(\cdot)) \, \rangle_{x,\mu} \ = \ \langle e^{\mu[\phi(0) - \phi(x)]} \ F(\phi(\cdot)) \, \rangle \Big/ \text{normalization} \ .$$

Proof: Apply HS formula twice to obtain a representation of the third moment as the expectation of a product of three Green's functions of elliptic PDE with random coefficients. The condition $\lambda/\Lambda > 1/2$ seems to be related to the lack of symmetry in $h_j(\cdot),\ j = 1, 2, 3$, of this expression caused by the Laplacian $d^* d$ term in the HS formula.

Third moment inequality for $d = 2$

To prove Theorem 1 we need to show that there is a constant C depending only on $V(\cdot)$ such that

$$(c) \quad \left|\langle\, [\phi(0) - \phi(x)]^3 \,\rangle_{x,\mu}\right| \;\le\; C\,.$$

Hence we would like to take $h_j(y) = h(y),\ j = 1, 2, 3$, where $h(y) \sim 1/|y|$ as $|y| \to \infty$, so $h(\cdot)$ barely misses being in $\ell^2(\mathbf{Z}^d, \mathbf{R}^d)$. Note however that $h(\cdot) \in \ell^2_w(\mathbf{Z}^d, \mathbf{R}^d)$ for any weight $w : \mathbf{Z}^d \to \mathbf{R}$ of the form $w(y) = w_\alpha(y) = [1 + |y|]^\alpha$ with $\alpha < 0$. The Calderon-Zygmund operator T has norm $\|T\|_{w_\alpha} \le 1 + C|\alpha|$. Hence the ℓ^2 converging perturbation expansion which implies Theorem A also converges in $\ell^2_{w_\alpha}$ for small $|\alpha|$. This implies (c).

We cannot replace $\ell^2_w(\mathbf{Z}^d, \mathbf{R}^d)$ with $w(\cdot)$ close in the Muckenhoupt A_2 sense to 1 in this argument by $\ell^p(\mathbf{Z}^d, \mathbf{R}^d)$ with $p > 2$ close to 1, even though $h(\cdot) \in \ell^p(\mathbf{Z}^d, \mathbf{R}^d)$ for any $p > 2$. The reason is that we need to carry through a spectral decomposition of d^*d and this does not work for $p \neq 2$.

Elliptic PDE with random coefficients

$(\Omega, \mathcal{F}, P)$ a probability space and $\tau_x : \Omega \to \Omega,\ x \in \mathbf{Z}^d$, translation operators which act ergodically on Ω. Let $\mathbf{a} : \Omega \to \mathbf{R}^{d(d+1)/2}$ be a measurable mapping to symmetric $d \times d$ matrices such that $\lambda I_d \le \mathbf{a}(\cdot) \le \Lambda I_d$. Let $u(x, \eta, \omega)$ be the solution to the discrete elliptic equation

$$\eta u(x,\eta,\omega) + \nabla^* \mathbf{a}(\tau_x \omega) \nabla u(x,\eta,\omega) = h(x), \quad x \in \mathbf{Z}^d, \omega \in \Omega.$$

Translation invariance implies $\langle\, u(x,\eta,\cdot)\,\rangle = \sum_{y \in \mathbf{Z}^d} G_{\mathbf{a},\eta}(x-y) h(y)$, where $G_{\mathbf{a},\eta}(x),\ x \in \mathbf{Z}^d$, is the averaged Green's function.

Homogenization (Kozlov 1978): Let $f : \mathbf{R}^d \to \mathbf{R}$ be a C_c^∞ function and set $h(x) = \varepsilon^2 f(\varepsilon x),\ x \in \mathbf{Z}^d$. Then $u(x/\varepsilon, \varepsilon^2 \eta, \omega)$ converges with probability 1 to a function $u(x, \eta)$ which is the solution to the constant coefficient equation

$$\eta u(x,\eta) + \nabla^* \mathbf{a}_{\rm hom} \nabla u(x,\eta) = f(x), \quad x \in \mathbf{R}^d.$$

Rate of Convergence in Homogenization

Can show (Yurinskii 1986) that

$$(d) \quad |\langle\, u(x/\varepsilon, \varepsilon^2\eta, \cdot)\,\rangle - u(x,\eta)| \le C\varepsilon^\alpha \text{ for some } \alpha \text{ depending on } \lambda/\Lambda\ ,$$

provided the operators $\tau_x : \Omega \to \Omega,\ x \in \mathbf{Z}^d$, have strong mixing properties, for example if variables $\mathbf{a}(\tau_x\cdot),\ x \in \mathbf{Z}^d$, independent. If $(\Omega, \mathcal{F}, P)$ is an EFT can use (Naddaf-Spencer 1998) the Poincaré inequality to prove (d). This method has been recently extended to the i.i.d. case yielding optimal α (Gloria-Otto 2009). Observe from

$$\langle\, u(x/\varepsilon, \varepsilon^2\eta, \cdot)\,\rangle \;=\; \int_{\varepsilon\mathbf{Z}^d} \varepsilon^{2-d} G_{\mathbf{a},\varepsilon^2\eta}\left(\frac{x-z}{\varepsilon}\right) f(z)\, dz\ ,$$

that (d) follows from the inequality

$$(e)\ |\varepsilon^{2-d} G_{\mathbf{a},\varepsilon^2\eta}(x/\varepsilon) - G_{\mathbf{a}_{\rm hom},\eta}(x)| \le \frac{C\varepsilon^\alpha}{[|x|+\varepsilon]^{d-2+\alpha}} e^{-\gamma\sqrt{\eta/\Lambda}|x|}\ ,\ x \in \varepsilon\mathbf{Z}^d\ .$$

Note that it is sufficient to establish (e) for $\varepsilon = 1$ uniformly as $\eta \to 0$.

Averaged Green's function inequalities for $d \ge 1$

Theorem 3 (Conlon-Spencer 2011): Let $\tilde{\mathbf{a}} : \mathbf{R}^d \to \mathbf{R}^{d(d+1)/2}$ be a C^1 function on $\mathbf{R}^d$ with values in the space of symmetric $d \times d$ matrices which satisfies $\lambda I_d \le \tilde{\mathbf{a}}(\cdot) \le \Lambda I_d$ and $\|D\tilde{\mathbf{a}}(\cdot)\|_\infty < \infty$, and set $\mathbf{a}(\omega) = \mathbf{a}(\nabla\phi(\cdot)) = \tilde{\mathbf{a}}(\nabla\phi(0))$. Then there exists $\alpha, \gamma > 0$ depending only on $d, \lambda/\Lambda$ and C depending only on $\tilde{\mathbf{a}}(\cdot)$ such that

$$|G_{\mathbf{a},\eta}(x) - G_{\mathbf{a}_{\rm hom},\eta}(x)| \le \frac{C}{[|x|+1]^{d-2+\alpha}} e^{-\gamma\sqrt{\eta/\Lambda}|x|} , \quad \text{for } d \ge 3 ,$$

$$|\nabla G_{\mathbf{a},\eta}(x) - \nabla G_{\mathbf{a}_{\rm hom},\eta}(x)| \le \frac{C}{[|x|+1]^{d-1+\alpha}} e^{-\gamma\sqrt{\eta/\Lambda}|x|} , \quad \text{for } d \ge 2 ,$$

$$|\nabla\nabla G_{\mathbf{a},\eta}(x) - \nabla\nabla G_{\mathbf{a}_{\rm hom},\eta}(x)| \le \frac{C}{[|x|+1]^{d+\alpha}} e^{-\gamma\sqrt{\eta/\Lambda}|x|} , \quad \text{for } d \ge 1 .$$

Proof: Use a representation for the Fourier transform $\hat{G}_{\mathbf{a},\eta}(\xi),\ \xi \in [-\pi,\pi]^d$, of $G_{\mathbf{a},\eta}(\cdot)$ from (Conlon-Naddaf 2000) plus show functions related to $\hat{G}_{\mathbf{a},\eta}(\cdot)$ are in certain $L^p([-\pi,\pi]^d)$ spaces.

Elliptic PDE on the probability space

For a function $\psi : \Omega \to \mathbf{C}$ define the ξ derivative of $\psi(\cdot)$ in the j direction $\partial_{j,\xi}$ by

$$\partial_{j,\xi}\psi(\omega) = e^{-i\xi_j}\psi(\tau_{\mathbf{e}_j}\omega) - \psi(\omega) \ , \quad \partial_\xi = [\partial_{1,\xi}, ..., \partial_{d,\xi}] \ .$$

Let $\mathbf{b}(\cdot) \in L^2(\Omega, \mathbf{C}^d)$ and $\Phi(\xi, \eta, \omega)$ the solution to the PDE

$$\eta\Phi(\xi,\eta,\omega) + \partial_\xi^* \mathbf{a}(\omega)\partial_\xi\Phi(\xi,\eta,\omega) \ = \ \partial_\xi^*\mathbf{b}(\omega) \ .$$

Then $\|\partial_\xi\Phi(\xi,\eta,\cdot)\| \le \|\mathbf{b}(\cdot)\|/\lambda$ which implies $\|P\partial_\xi\Phi(\xi,\eta,\cdot)\| \le \|\mathbf{b}(\cdot)\|/\lambda$ where $P : L^2(\Omega, \mathbf{C}^d) \to L^2(\Omega, \mathbf{C}^d)$ is the projection orthogonal to the constant function. Hence if $g(\cdot) \in L^p(\mathbf{Z}^d)$ there is an inequality

$$(f) \quad \|P\sum_{x \in \mathbf{Z}^d} g(x)\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)\| \ \le \ C_p\|g\|_p \ ,$$

which holds for $p = 1$ with $C_1 = 1/\lambda$.

Naddaf-Spencer argument

Let $(\Omega, \mathcal{F}, P)$ be a massless EFT. Then Poincaré for EFT implies

$$\|P\sum_{x\in\mathbf{Z}^d} g(x)\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)\|^2 \le \frac{1}{\lambda}\sum_{z\in\mathbf{Z}^d}\|\frac{\partial}{\partial\omega(z)}\sum_{x\in\mathbf{Z}^d} g(x)\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)\|^2 ,$$

where $\omega(\cdot) = \nabla\phi(\cdot)$. Translation invariance implies that

$$\sum_{z\in\mathbf{Z}^d}\|\frac{\partial}{\partial\omega(z)}\sum_{x\in\mathbf{Z}^d} g(x)F(\tau_x\omega(\cdot))\|^2 = \sum_{z\in\mathbf{Z}^d}\|g(x)H(x-z,\omega(\cdot)\|^2$$

for some function $H : \mathbf{Z}^d \times \Omega \to \mathbf{C}$. When $F(\cdot) = \partial_\xi\Phi(\xi,\eta,\cdot)$ then $H(y,\omega) = \nabla_y G(y,\omega(\cdot)$ where $G(y,\omega)$ is the Green's function for an elliptic PDE in y with coefficients which depend on ω. Hence (Meyers 1963) the uniform ellipticity and Calderon-Zygmund imply that for all q close to 2 there is a constant C independent of ω such that $\nabla_y G(y,\omega) \in L^q(\mathbf{Z}^d)$ and $\|\nabla_y G(\cdot,\omega)\|_q \le C$. Young's inequality then implies for $q < 2$ that (f) holds with $p = 2q/(3q-2) > 1$.

Correlation and averaged Green's functions

Let $(\Omega, \mathcal{F}, P)$ be a massless EFT. Then (Giacomin-Olla-Spohn 2001)

$$\langle\, \phi(x)\phi(0)\,\rangle \;=\; \int_0^\infty G_{\mathbf{a}}(x,t)\, dt,$$

where $G_{\mathbf{a}}(x,t)$ is the averaged Green's function for the parabolic PDE

$$\frac{\partial u(x,t,\omega)}{\partial t} + \nabla^* \mathbf{a}(\tau_{x,t}\omega)\nabla u(x,t,\omega) \;=\; 0, \quad u(x,0,\omega) = h(x),$$

where $\omega = \nabla\phi(x,t),\ x \in \mathbf{Z}^d, t \in \mathbf{R}$, and $\mathbf{a}(\omega) = V''(\nabla\phi(0,0))$.
Theorem 4 (Conlon-Spencer): Let $\tilde{\mathbf{a}} : \mathbf{R}^d \to \mathbf{R}^{d(d+1)/2}$ be a C^1 function on $\mathbf{R}^d$ with values in the space of symmetric $d \times d$ matrices which satisfies $\lambda I_d \le \tilde{\mathbf{a}}(\cdot) \le \Lambda I_d$ and $\|D\tilde{\mathbf{a}}(\cdot)\|_\infty < \infty$, and set $\mathbf{a}(\omega) = \mathbf{a}(\nabla\phi(\cdot)) = \tilde{\mathbf{a}}(\nabla\phi(0,0))$. Then there exists $\alpha, \gamma > 0$ depending only on $d, \lambda/\Lambda$ and C depending only on $\tilde{\mathbf{a}}(\cdot)$ such that for $d \ge 3$,

$$|G_{\mathbf{a}}(x,t) - G_{\mathbf{a}_{\rm hom}}(x,t)| \le \frac{C}{[\Lambda t + 1]^{(d+1+\alpha)/2}} \exp\left[-\gamma \min\left\{|x|,\ \frac{|x|^2}{\Lambda t + 1}\right\}\right].$$

Parabolic PDE on the probability space

For a function $\psi : \Omega \to \mathbf{C}$ define the derivative of $\psi(\cdot)$ in the time direction ∂ by

$$\partial\psi(\omega) = \lim_{t\to 0}[\psi(\tau_{0,t}\omega) - \psi(\omega)]/t \, .$$

Let $\mathbf{b}(\cdot) \in L^2(\Omega, \mathbf{C}^d)$ and $\Phi(\xi, \eta, \omega)$ the solution to the PDE

$$\partial\Phi(\xi,\eta,\omega) + \eta\Phi(\xi,\eta,\omega) + \partial_\xi^*\mathbf{a}(\omega)\partial_\xi\Phi(\xi,\eta,\omega) \ = \ \partial_\xi^*\mathbf{b}(\omega) \, .$$

Then as in the elliptic case $\|P\partial_\xi\Phi(\xi,\eta,\cdot)\| \le \|\mathbf{b}(\cdot)\|/\lambda$. Hence if $g(\cdot,\cdot) \in L^p(\mathbf{Z}^d \times \mathbf{R})$ there is an inequality

$$(g) \quad \|P\sum_{x\in\mathbf{Z}^d}\int_{\mathbf{R}} dt \; g(x,t) \; \partial_\xi\Phi(\xi,\eta,\tau_{x,t}\cdot)\| \ \le \ C_p\|g\|_p \, ,$$

which holds for $p = 1$ with $C_1 = 1/\lambda$. To prove Theorem 4 we need to prove (g) for some $p > 1$. To do this we use the Poincaré inequality on the space of time dependent fields $\phi(x,t), \; x \in \mathbf{Z}^d, t \in \mathbf{R}$.

The Malliavin Calculus

Let $W(t),\ t \in \mathbf{R}$, be the white noise Gaussian process, so if $B(t),\ t \geq 0$, is Brownian motion then

$$B(t) \ = \ \int_0^t W(s)\, ds \ .$$

Let $(\Omega, \mathcal{F}_T, P)$ be the probability space generated by $B(t),\ 0 \leq t \leq T$. Then $L^2(\Omega, \mathcal{F}_T, P)$ is unitarily equivalent to the space of functions $\xi : L^2([0,T]) \to \mathbf{R}$ with a Gaussian measure on $L^2([0,T])$. Let $[\cdot,\cdot]$ be the standard inner product on $L^2([0,T])$. The equivalence is:

$$\psi(\cdot) \in L^2([0,T]) \quad \text{corresponds to} \ \ \psi(t) \leftrightarrow W(t),\ 0 \leq t \leq T.$$

Measure on $L^2([0,T])$: If $\psi_j,\ j = 1,2,..$ is an orthonormal basis for $L^2([0,T])$ then the variables $\psi \to [\psi, \psi_j]$ are i.i.d. standard normal. For $h \in L^2([0,T])$ define the directional derivative $D_h\xi(\psi(\cdot))$ by

$$D_h\xi(\psi(\cdot)) = \lim_{\varepsilon \to 0} [\xi(\psi(\cdot) + \varepsilon h(\cdot)) - \xi(\psi(\cdot))]/\varepsilon = [D\xi(\psi(\cdot)), h] \ .$$

Hence $D\xi(\psi(\cdot))$ is a function in $L^2([0,T])$ which we write as $D_t\xi(\psi(\cdot)),\ 0 \leq t \leq T$.

The Clark-Ocone Formula (1984)

For $\xi : L^2([0,T]) \to \mathbf{R}$ then

$$\text{CO}: \ \xi(\cdot) - \langle \xi \rangle = \int_0^T \sigma_t(\cdot) dB(t) \quad \text{where } \sigma_t(\cdot) = E[D_t\xi(\cdot)|\mathcal{F}_t] \ .$$

CO implies Poincaré inequality since

$$\text{var}[\xi] = E\left[\int_0^T \sigma_t(\cdot)^2 \, dt\right] \le E\left[\int_0^T [D_t\xi(\cdot)]^2 \, dt\right] \ .$$

CO implies HS formula: Let $\phi(t), \ t \ge 0$, be the solution to the SDE

$$d\phi(t) = -\frac{1}{2} V'(\phi(t))dt + dB(t) \ , \quad \phi(0) = 0.$$

The stochastic process $\phi(\cdot)$ has invariant measure $\exp[-V(\phi)]$ so the distribution of $\phi(T)$ converges as $T \to \infty$ to $\exp[-V(\phi)]$.

First Variation Equation

For $h \in L^2([0,T])$ then $D_h\phi(t)$ is the solution to the first variation equation

$$\frac{d}{dt}[D_h\phi(t)] \;=\; -\frac{1}{2}V''(\phi(t))[D_h\phi(t)] + h(t)\;, \quad D_h\phi(0) = 0.$$

$$\text{Hence } D_h\phi(T) = \int_0^T h(t)\exp\left[-\frac{1}{2}\int_t^T V''(\phi(s))\,ds\right]\,dt\;, \quad \text{and so}$$

$$D_t\phi(T) = \exp\left[-\frac{1}{2}\int_t^T V''(\phi(s))\,ds\right] \text{ for } 0 \le t \le T, \quad D_t\phi(T) = 0 \text{ for } t > T.$$

Now for a C^1 function $F : \mathbf{R} \to \mathbf{R}$ use $\xi(\phi(\cdot)) = F(\phi(T))$, observing that $D_t\xi(\phi(\cdot)) = F'(\phi(T))D_t\phi(T)$. Then CO plus Feynman-Kac formula imply HS on letting $T \to \infty$.

Poincaré inequalities (Gourcy-Wu 2006)

Assume $\xi(\phi(\cdot))$ is a function of $\phi(t),\ 0 \leq t \leq T$. Then we can define two kinds of derivatives of $\xi(\phi(\cdot))$:
(a)The Malliavin derivative $D\xi(\cdot)$ since $\xi(\cdot)$ is a function of white noise.
(b) The field derivative $d\xi(\phi(\cdot))$ which measures the infinitesimal change in $\xi(\phi(\cdot))$ with respect to variations of the field $\phi(\cdot)$. Thus

$$d_h\xi(\phi(\cdot)) = \lim_{\varepsilon\to 0}[\xi(\phi(\cdot)+\varepsilon h(\cdot)) - \xi(\phi(\cdot))]/\varepsilon \ = \ [d\xi(\phi(\cdot)), h]\ .$$

The chain rule implies that (a) and (b) are related by

$$D_t\xi(\phi(\cdot)) \ = \ \int_0^t d_s\xi(\phi(\cdot))D_t\phi(s)\ ds\ .$$

If $V''(\cdot) \geq \lambda > 0$ then

$$|D_t\xi(\phi(\cdot))| \ \leq \ \int_0^t |d_s\xi(\phi(\cdot))|e^{-\lambda(t-s)/2}\ ds\ .$$

Poincaré inequality for fields

We have that

$$\mathrm{var}[\xi(\phi(\cdot))] \;\leq\; E\left[\int_0^T [D_t\xi(\phi(\cdot))]^2\,dt\right]$$

$$\leq\; E\left[\int_0^T \left\{\int_0^t |d_s\xi(\phi(\cdot))| e^{-\lambda(t-s)/2}\,ds\right\}^2 dt\right]\;,$$

and so we obtain the Poincaré field inequality

$$(h)\quad \mathrm{var}[\xi(\phi(\cdot))] \;\leq\; \frac{4}{\lambda^2} E\left[\int_0^T [d_t\xi(\phi(\cdot))]^2\,dt\right]\;.$$

The inequality (h) can be derived using HS in the case when $\xi(\phi(\cdot))$ has the form

$$\xi(\phi(\cdot)) \;=\; \int_0^\infty g(t)b(\phi(t))\,dt\;.$$

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Thank you!