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(Università di Roma 'Tor Vergata')

Variational Problems with Percolation

Gradient Random Fields

May 31, 2011 BIRS, Banff

Discrete system: with discrete variables $u = \{u_i\}$ indexed on a lattice (e.g., $\Omega \cap \mathbf{Z}^d$)

Discrete energy: (e.g., pair interactions)

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From discrete to continuous energies

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B. Γ-convergence for Beginners, OUP 2002
B. Handbook of Γ-convergence (Handbook of Diff. Eqns, Elsevier, 2006)

Cubic lattice: variables parameterized on $\Omega \cap \mathbf{Z}^d$

Binary systems: variable taking only **two values**; wlog $u_i \in \{-1, 1\}$ (spins).

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Only two possible energies (up to affine change of variables):

$$E(u) = E_{\rm ferr}(u) = -\sum_{\rm NN} u_i u_j \qquad \text{(ferromagnetic energy)}$$

(with two trivial minimizers $u_i \equiv 1$ and $u_i \equiv -1$)

$$E(u) = E_{\rm anti}(u) = \sum_{\rm NN} u_i u_j \qquad \text{(antiferromagnetic energy)}$$

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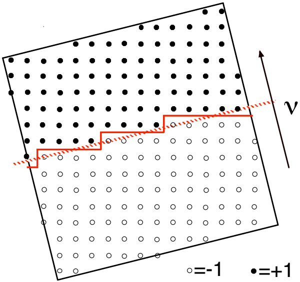
(with two minimizers $u_i \equiv \pm(-1)^i$)

Note: the change of variables $v_i = (-1)^i u_i$ is such that $E_{\rm anti}(v) = E_{\rm ferro}(u)$, so actually we have only one energy

Choice of the parameter: **(magnetization)** $u \in BV(\Omega; \{\pm 1\})$ continuous limit of piecewise-constant interpolations of $\{u_i\}$

Surface scaling: **(crystalline perimeter)**

$$E_\varepsilon(u) = \sum \varepsilon^{d-1}(1 - u_i u_j) \longrightarrow 2\int_{\Omega\cap\partial\{u=1\}} \|\nu\|\, d\mathcal{H}^{d-1}, \quad \text{with} \quad \|\nu\| = \sum_k |\nu_k|$$



ν = normal to the interface

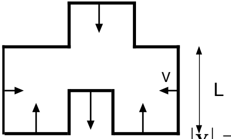
Continuous “flows” of the perimeter

Crystalline perimeter-driven motion of sets

$\Downarrow$

motion by crystalline mean curvature

(**Almgren-Taylor J. Diff. Geom. 1995** in 2D)



$$|\mathbf{v}| = \frac{\mathbf{2}}{\mathbf{L}}$$

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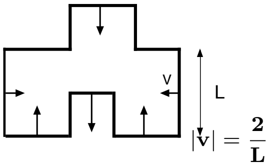

Motion is obtained by introducing a discrete time-step τ and initial set A_0, define a time-discrete motion by successive minimizations for fixed τ: A_{k+1} minimizes

$$\min\Big\{P(A) + \frac{1}{2\tau}\text{“dist}(A, A_k)\text{”}\Big\}$$

Define $A^\tau(t) = A_{[t/\tau]}$ (piecewise-constant interpolation of $\{A_k\}$) and pass to the limit as $\tau \to 0$ to get a continuous $A(t)$ (scheme by **Almgren-Taylor-Wang, SIAM J. Control Opt. 1983**)

Motion of discrete interfaces

Fix ε, τ and A_0. Then A_{k+1} minimizes (here, $A = \{u = 1\}$, $P_\varepsilon(A) = E_\varepsilon(u)$)

$$\min\Big\{P_\varepsilon(A) + \frac{1}{2\tau}\text{“dist}_\varepsilon(A, A_k)\text{”}\Big\}$$

Define $A^{\varepsilon,\tau}(t) = A_{[t/\tau]}$ and pass to the limit as $\tau \to 0$ to get a continuous $A(t)$.

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Pinning/depinning transition: (B-Gelli-Novaga ARMA 2009)

• For $\tau << \varepsilon$ the motion $A(t)$ is trivial (**pinning**):

$$A(t) = A_0$$

for all (sufficiently regular) bounded initial sets A_0;

• For $\varepsilon << \tau$ the sets $A(t)$ follow **motion by crystalline mean curvature**.

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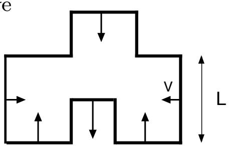
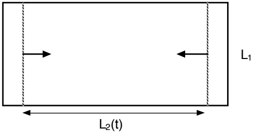
• At the **critical scale** $\tau = \alpha\varepsilon$ we have **‘quantized’ cristalline motion**



$$|\mathbf{v}| = \frac{\mathbf{1}}{\alpha}\Big\lfloor \frac{\mathbf{2}\alpha}{\mathbf{L}} \Big\rfloor$$

(i) (**critical pinning side-length**) If all $L > 2\alpha$ then the motion is trivial: $A(t) = A_0$;

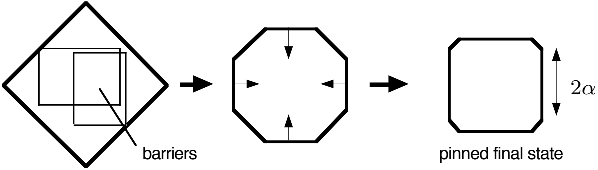
(ii) (**partial pinning and non strict inclusion principle**; e.g for rectangles) If $L_1 < 2\alpha$ and $L_2 > 2\alpha$ only one side is (initially) pinned



(iii) (**quantized velocity**)
$2\alpha/L(t) \notin \mathbb{N} \Rightarrow$ velocity integer multiple of $1/\alpha$;

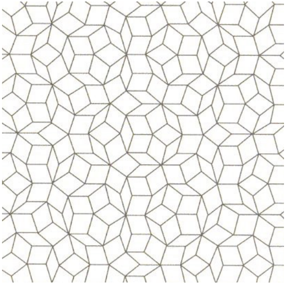
(iv) (**non-uniqueness**)
$2\alpha/L(t) \in \mathbb{N} \;\Rightarrow\;$ velocity not uniquely determined $\;\Rightarrow\;$ non-uniqueness

(v) (**non-convex pinned sets**)
(vi) (**pinning after initial motion**)



With the due changes the process can be repeated on more general periodic lattices (e.g. triangular, exagonal, FCC, BCC, etc.); even though we do not have in general a duality between ferro- and anti-ferromagnetic energies (**frustration**). For ferromagnetic energies we still have the **same continuous parameter** $u \in BV(\Omega; \{\pm 1\})$. The form of the *surface tension* changes accordingly.

Techniques must be refined to take care of **a-periodic lattices** (e.g. Penrose tilings or quasicrystals)



(B-Solci M^3AS 2011)

We may have more complex interactions:

$$-\sum_{i,j}\sigma_{ij}u_iu_j$$

Conditions of the type
• (**uniform minimal states**) $\sigma_{ij}\geq 0$
• (**coerciveness conditions**) $\sigma_{ij}\geq c>0$ for $|i-j|=1$
• (**decay conditions**) $\sum_j \sigma_{ij}\leq C<+\infty$ for all i
guarantee that (up to subsequences) the **continuous parameter** is still $u\in BV(\Omega;\{\pm1\})$ and

$$\sum_{ij}\varepsilon^{d-1}\sigma_{ij}(1-u_iu_j)\longrightarrow\int_{\Omega\cap\partial\{u=1\}}\varphi(x,\nu)\,d\mathcal{H}^{d-1}$$

i.e., the limit is still a (possibly inhomogeneous) interfacial energy.

The integrand φ is determined by a family of discrete (non-local) minimal-surface problems. In the $2D$ case and if only nearest-neighbours are considered ($\sigma_{ij}=0$ if $|i-j|>1$) equivalently it is given by an **asymptotic distance** on the lattice $\mathbb{Z}^2$ (where the distance between the nodes i and j is σ_{ij}) (**B-Piatnitsky 2010**)

COMMENTS III/ Other continuous parameters

When not only nearest neighbours are taken into account we do not have a correspondence between ferromagnetic and anti-ferromagnetic energies.

1) **Anti-ferromagnetic spin systems in 2D** **(B-Alicandro-Cicalese NHM 2006)**

$$E(u) = c_1 \sum_{\mathrm{NN}} u_i u_j + c_2 \sum_{\mathrm{NNN}} u_k u_l \qquad u_i \in \{\pm 1\}$$

(NNN = next-to-nearest neighbours)

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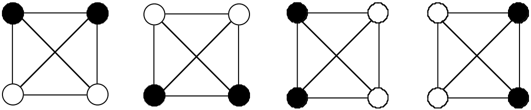
(NNN = next-to-nearest neighbours)
For suitable positive c_1 and c_2 the ground states are 2-periodic



(representation in the unit cell)

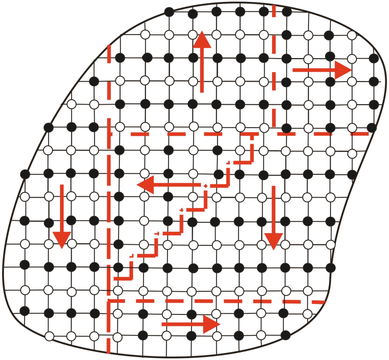
The correct order parameter is the **orientation** $v \in \{\pm e_1, \pm e_2\}$ of the ground state.

Surface-scaling limit

$$F(v) = \int_{S(v)} \psi(v^+ - v^-, \nu)\, d\mathcal{H}^1$$

$S(v)$ = discontinuity lines; ν = normal to $S(v)$
ψ given by an optimal-profile problem

Microscopic picture of a limit state with finite energy



Ferromagnetic-anti-ferromagnetic spin systems

We can consider e.g. two-dimensional systems with NN, NNN, NNNN (next-to-next-...) interactions, $u_i \in \{\pm 1\}$ and

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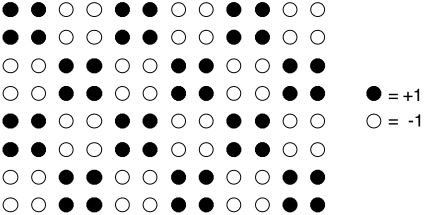
For suitable c_1 and c_2 again we have a non-trivial 4-periodic ground state



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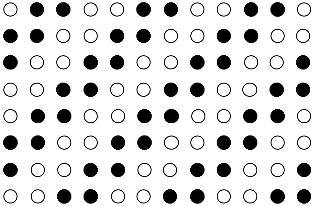

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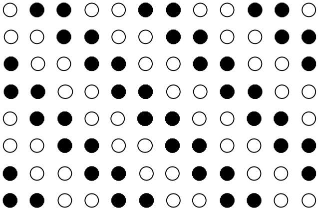

(counting translations 16 different ground states)
and a description for the surface-scaling Γ-limit combining the two previous examples

Let $d = 2$. Introduce a random variable depending on an ergodic stationary discrete random process on the bonds of $\mathbf{Z}^2$. The simplest energy depends on its realizations ω:

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- φ exists and a.s. is independent of ω
- we can substitute 0 and $T\nu^\perp$ with arbitrary x and $x + T\nu^\perp$ $(x = O(T))$
- oscillations of the minimal path from the segment $[x, x + T\nu^\perp]$ are small.

A Percolation Result for "Rigid spins" / Damage

Rigid spin systems. We may consider ω a realization of an **i.i.d. random variable** in $\mathbb{Z}^2$, and the corresponding energy (surface scaling)

$$E^\omega_\varepsilon(u) = \sum_{\rm NN} \varepsilon \sigma^\omega_{ij}(1 - u_i u_j) \quad \text{with} \quad \sigma^\omega_{ij} = \begin{cases} 1 & \text{with probability } p \\ +\infty & \text{with probability } 1-p \end{cases}$$

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Percolation Theorem (B-Piatnitski 2008)

In the surface scaling, the Γ-limit F_p of E^ω_ε is a.s.

(1) $F_p(u) = +\infty$ if $u \neq 1$ or $u \neq -1$ identically, for $p < 1/2$

(2) $F_p(u) = \displaystyle\int_{\Omega\cap\partial\{u=1\}} \varphi_p(\nu)\, d\mathcal{H}^1$ for $p > 1/2$ $\quad (u \in BV(\Omega;\{\pm1\}))$

The limit is deterministic and $\varphi_p(\nu)$ is given by an *asymptotic distance on the 'weak cluster'* for $p > 1/2$.

NOTE: this is the limit case when $\sigma^\omega_{ij} = \begin{cases} 1 & \text{with probability } p \\ T & \text{with probability } 1-p \end{cases}$ for $T \to +\infty$

Non-coercive spin systems. We may consider ω a realization of an **i.i.d. random variable** in $\mathbb{Z}^2$, and the corresponding energy

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In the surface scaling, the Γ-limit F_p of E^ω_ε is a.s.

(1) $F_p(u) = 0$ on all $u \in L^1(\Omega; [-1,1])$ for $p \le 1/2$

(2) $F_p(u) = \displaystyle\int_{\Omega\cap\partial\{u=1\}} \varphi_p(\nu)\, d\mathcal{H}^1$ for $p > 1/2$

The limit is deterministic and $\varphi_p(\nu)$ is given by a *first-passage percolation* formula for $p > 1/2$.

NOTE: the parameter $u \in BV(\Omega; \{\pm 1\})$ is the "dominant phase" (no control if $\sigma_{ij} = 0$)

Ferromagnetic/antiferromagnetic interactions: an **open problem** is when

$$E^\omega(u) = -\sum_{\rm NN} \sigma^\omega_{ij} u_i u_j \qquad \text{with} \qquad \sigma^\omega_{ij} = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

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Deterministic 'toy' problem (for the case $p \sim 0$): discrete 'perforated domain' with well-separated 'holes' where $\sigma_{ij} = -1$ (**B-Piatnitski 2010**). In this case

- need **stronger separation conditions** between the perforations
- the surface scaling is more complex and not explicit
- the Γ-limit may be still described by an interfacial energy $\displaystyle\int_{\Omega\cap\partial\{u=1\}} \varphi(\nu)\, d\mathcal{H}^1$

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Ferromagnetic/antiferromagnetic interactions: an **open problem** is when

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Question: How does p influence the geometry (and number) of ground states? What happens when $p \to 1/2$?