LECTURE 4. POINTS AVOIDED BY RANDOM WALK

The fourth and final lecture of this minicourse is devoted to the proof of Theorem 1.7 describing the scaling limit of the set of points avoided by the simple random walk. For reasons explained then, we will work with the time parametrization by the local time at the boundary vertex.

4.1 Setting the scales.

Let us start by explaining the formula for the normalizing sequence $\{\widehat{K}_N\}_{N\geqslant 1}$ from (1.33). Working for a moment in the general setting of a Markov chain on $V\cup\{\varrho\}$, recall the definition (1.31) of the local time L_t parametrized by the local time at ϱ. We then have:

Lemma 4.1 *For each $x\in V$ and all $t\geqslant 0$,*

$$P^\varrho\bigl(L_t(x)=0\bigr)=\mathrm{e}^{-\frac{t}{G^V(x,x)}} \tag{4.1}$$

Proof. Let $x\in V$. Observe that, by the (a.s.-unique) time s when $\ell_s(\varrho)=t$, the chain accumulated a Poisson($\pi(\varrho)t$) number of excursions into V. The probability that such an excursion visits x is $P^\varrho(H_x<\widehat{H}_\varrho)$, which by a Poisson thinning argument means that the total number of excursions that visit x by the time when the time at ϱ equals $\pi(\varrho)t$ has the law of

$$\text{Poisson}\Bigl(\pi(\varrho)P^\varrho(H_x<\widehat{H}_\varrho)t\Bigr) \tag{4.2}$$

If $L_t(x)=0$, no such excursion visited x and so

$$P^\varrho\bigl(L_t(x)=0\bigr)=\mathrm{e}^{-\pi(\varrho)P^\varrho(H_x<\widehat{H}_\varrho)t} \tag{4.3}$$

We now observe that, by (2.32–2.33), the exponent equals $t/G^V(x,x)$. $\square$

For $V:=D_N$ we have $G^{D_N}(x,x)=g\log N+O(1)$ for x sufficiently far away from the boundary and so (for $t=O((\log N)^2)$),

$$P^\varrho\bigl(\exists x\in D_N\colon L_t(x)=0\bigr)\leqslant E^\varrho\Biggl(\sum_{x\in D_N}1_{\{L_t(x)=0\}}\Biggr)\asymp|D_N|\mathrm{e}^{-\frac{t}{g\log N}} \tag{4.4}$$

Setting $t:=2g\theta(\log N)^2$, we conclude:

Corollary 4.2 *For $t_N\sim 2g\theta(\log N)^2$ with $\theta>1$, we have*

$$P^\varrho\bigl(\forall x\in D_N\colon L_t(x)>0\bigr)\underset{N\to\infty}{\longrightarrow}1. \tag{4.5}$$

For $\theta<1$ we at least get that the expected number of avoided points grows as a power of N. As we will show by proving Theorem 1.7, the actual number of avoided points runs on the same scale.

As discussed in the second lecture, our key tool will be the Second Ray-Knight Theorem (Theorem 2.7) that states that there exists a coupling of L_t and two copies h and $\tilde h$ of the DGFF on V such that

$$L_t\perp\!\!\!\perp h \quad\wedge\quad L_t+\frac12h^2=\frac12\bigl(\tilde h+\sqrt{2t}\bigr)^2 \tag{4.6}$$

We will use this roughly as follows: Suppose x is such that $L_t(x) = 0$. Then, if h_x happens to be order unity, also the field on the right is of order unity which means that

$$\tilde{h}_x = -\sqrt{2t} + O(1) \tag{4.7}$$

For the choice $t = 2g\theta(\log N)^2$ we have $\sqrt{2t} = 2\sqrt{g}\sqrt{\theta}\log N$, which means that x is a $\sqrt{\theta}$-thick point of $\tilde{h}$!

Of course, assuming that h_x is order unity needs to be justified because h_x is (at typical x) normal with variance $\log N$ and so it is order unity only with probability proportional to $(\log N)^{-1/2}$. Since h is independent and the points where L_t vanishes are somewhat scattered, we may think of the above argument though Poisson thinning: Diluting the points where $L_t(x) = 0$ (more or less) independently with probability of order $(\log N)^{-1/2}$ gives us, roughly, the $\sqrt{\theta}$-points of $\tilde{h}$.

4.2 Light points.

In order to implement the above strategy quantitatively, a moment's thought reveals that tracking only the points where the local time vanishes is not sufficient. Instead, we will need to track the set where the local time is as well of order unity that we sometimes refer to as the random walk *light points*. We introduce the notation for the corresponding measure

$$\vartheta_N := \frac{1}{\widehat{K}_N} \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{L_t(x)} \tag{4.8}$$

Our first goal is to show that the family of measures $\{\vartheta_N\}_{N \geqslant 1}$ is tight. For this we need to upgrade Lemma 4.1 to the form:

Lemma 4.3 *For each $x \in V$, $t \geqslant 0$ and $b \geqslant 0$,*

$$P^{\varrho}\big(L_t(x) \leqslant b\big) \leqslant \mathrm{e}^{-\frac{t}{G^V(x,x)} \exp\{-\frac{b}{G^V(x,x)}\}} \leqslant \mathrm{e}^{-\frac{t}{G^V(x,x)} + b\frac{t}{G^V(x,x)^2}} \tag{4.9}$$

Proof. The argument from the proof of Lemma 4.1 gives us the representation

$$\pi(x) L_t(x) \overset{\text{law}}{=} \sum_{k=1}^{N_t} \sum_{j=1}^{Z_k} T_{k,j} \tag{4.10}$$

where

- $\{T_{k,j}\}_{k,j \geqslant 1}$ are i.i.d. Exponential with parameter 1
- $\{Z_k\}_{k \geqslant 1}$ are i.i.d. Geometric with parameter $P^x(H_\varrho < \widehat{H}_x)$
- N_t is Poisson with parameter $\pi(\varrho) P^{\varrho}(H_x < \widehat{H}_\varrho) t$.

with all the random variables independent of each other. Indeed, all we need to realize that, when an excursion from ϱ hits x, the number of visits to x on this excursion will be Geometric with parameter $P^x(H_\varrho < \widehat{H}_x)$ and each visits leaves an independent Exponential(1)-time to the time spent at x.

Abbreviate $q := P^x(H_\varrho < \widehat{H}_x)$. A thinning argument for exponential random variables gives

$$\sum_{j=1}^{Z_k} T_{k,j} \overset{\text{law}}{=} \text{Exponential}(q^{-1}) \tag{4.11}$$

Observe also that reversibility (2.32) gives

$$\pi(\varrho)P^\varrho(H_x < \widehat{H}_\varrho) = \pi(x)P^x(H_\varrho < \widehat{H}_x) = \pi(x)q \tag{4.12}$$

and so $N_t = \text{Poisson}(\pi(x)q)$. Hence we get

$$\begin{aligned} P^\varrho\big(L_t(x) \leqslant b\big) &\leqslant P\left(\forall k = 1, \dots, N_t \colon \sum_{j=1}^{Z_k} T_{k,j} \leqslant \pi(x)b\right) \\ &= \sum_{n=0}^{\infty} \frac{[t\pi(x)q]^n}{n!}\big[1 - \mathrm{e}^{-\pi(x)bq}\big]^n \mathrm{e}^{-t\pi(x)q} = \mathrm{e}^{-t\pi(x)q\exp\{b\pi(x)q\}} \end{aligned} \tag{4.13}$$

To get the first bound in the claim, we now observe that $\pi(x)q = G^V(x,x)^{-1}$. The second bound follows from the inequality $\mathrm{e}^{-s} \geqslant 1 - s$ for all $s \geqslant 0$. $\square$

Hereby we now conclude:

Corollary 4.4 *Suppose $t_N \sim 2g\theta(\log N)^2$ for $\theta \in (0,1)$. Then for all $b \geqslant 0$ there exists $c = c(b) < \infty$ such that for all $A \subseteq \mathbb{R}^2$,*

$$E^\varrho\, \vartheta\big(A \times [0,b]\big) \leqslant c\frac{|A_N|}{N^2} \tag{4.14}$$

where $A_N := \{x \in D_N \colon x/N \in A\}$.

Proof. Let $b \geqslant 0$. Lemma 4.3 gives us

$$E^\varrho\, \vartheta\big(A \times [0,b]\big) \leqslant \frac{1}{\widehat{K}_N}\sum_{x \in A_N} \min\Big\{\mathrm{e}^{-\frac{t_N}{G(x,x)} + b\frac{t_N}{G(x,x)^2}},\, \mathrm{e}^{-\frac{t_N}{G(x,x)}\exp\{-\frac{b}{G(x,x)}\}}\Big\}, \tag{4.15}$$

where we write $G(x,x)$ instead of $G^{D_N}(x,x)$ for brevity. If x is such that $G(x,x) \geqslant \mathrm{e}^{-b}g\log N$, then the uniform bound $G(x,x) \leqslant g\log N + c$ gives

$$\frac{t_N}{G(x,x)} - b\frac{t_N}{G(x,x)^2} \geqslant \frac{t_N}{g\log N + c} - b\mathrm{e}^{2b}\frac{t_N}{(g\log N)^2} \tag{4.16}$$

which is at least $\frac{t_N}{g\log N} - c'$ for some constant c' depending only on b. On the other hand, if x satisfies $G(x,x) \leqslant \mathrm{e}^{-b}g\log N$ then the fact that $G(x,x) \geqslant 1$ implies

$$\frac{t_N}{G(x,x)}\exp\Big\{-\frac{b}{G(x,x)}\Big\} \geqslant \mathrm{e}^b\frac{t_N}{g\log N}\,\mathrm{e}^{-b} = \frac{t_N}{g\log N} \tag{4.17}$$

Hence the minimum in (4.14) is at most a constant times $\mathrm{e}^{-\frac{t_N}{g\log N}} = \widehat{K}_N/N^2$. $\square$

4.3 Extended process.

The tightness of $\{\vartheta_N\}_{N\geqslant 1}$ permits us to extract subsequential weak limits. Our goal is to characterize these limits with the help of the coupling (4.6) but this in turn requires that, along with small values of L_t we also track small values of h. For this we introduce

$$\vartheta_N^{\text{ext}} := \frac{\sqrt{\log N}}{\widehat{K}_N} \sum_{x\in D_N} \delta_{x/N}\otimes\delta_{L_t(x)}\otimes\delta_{h_x} \tag{4.18}$$

Here the additional $\sqrt{\log N}$ in the normalization reflects the fact that, forcing a point with small value of $L_t(x)$ to have a small value of h_x costs $O((\log N)^{-1/2})$ in probability. A key technical lemma to prove then is:

Lemma 4.5 *Suppose $\{N_k\}_{k\geqslant 1}$ is a strictly increasing sequence such that $\vartheta_{N_k}\overset{\text{law}}{\longrightarrow}\vartheta$. Then*

$$\vartheta_{N_k}^{\text{ext}} \underset{k\to\infty}{\overset{\text{law}}{\longrightarrow}} \vartheta\otimes\text{Leb} \tag{4.19}$$

where Leb *is the Lebesgue measure on* $\mathbb{R}$.

Proof (modulo a technical step). We first note that the convergence holds in expectation. Indeed, writing $\mathbb{E}$ for the expectation with respect to h only, for any $f = f(x,\ell,h)$ non-negative and continuous with compact support,

$$\begin{aligned}\mathbb{E}\big(\langle\vartheta_N^{\text{ext}},f\rangle\big) &= \frac{\sqrt{\log N}}{\widehat{K}_N}\sum_{x\in D_N}\mathbb{E}f\big(x/N,L_t(x),h_x\big)\\ &= \frac{\sqrt{\log N}}{\widehat{K}_N}\sum_{x\in D_N}\int\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{G(x,x)}}\mathrm{e}^{-\frac{h^2}{2G(x,x)}}f\big(x/N,L_t(x),h\big)\mathrm{d}h\end{aligned} \tag{4.20}$$

The restriction on compact support means that h is bounded in a compact interval and x is away from the boundary of D_N. This means that

$$\frac{1}{\sqrt{G(x,x)}}\mathrm{e}^{-\frac{h^2}{2G(x,x)}} = \frac{1}{\sqrt{g\log N}} + O\Big(\frac{1}{\log N}\Big) \tag{4.21}$$

uniformly in h of interest. Since $2\pi g = 1$, we thus get

$$\mathbb{E}\big(\langle\vartheta_N^{\text{ext}},f\rangle\big) = \bigg(1+O\Big(\frac{1}{\sqrt{\log N}}\Big)\bigg)\langle\vartheta_N\otimes\text{Leb},f\rangle \tag{4.22}$$

which, in light of tightness of $\{\vartheta_N\}_{N\geqslant 1}$, is $\langle\vartheta_N\otimes\text{Leb} f\rangle + o(1)$.

In order to use this to prove the statement, observe that the conditional Jensen inequality immediate upgrades the above to the form

$$E^\varrho\otimes\mathbb{E}\big(\mathrm{e}^{-\langle\vartheta_N^{\text{ext}},f\rangle}\big) \geqslant \mathrm{e}^{o(1)}E^\varrho\big(\mathrm{e}^{-\langle\vartheta_N\otimes\text{Leb},f\rangle}\big) \tag{4.23}$$

Since convergence of Laplace transforms implies convergence in law, we thus need to show that an opposite inequality holds as $N\to\infty$. This is done roughly as follows: Given $s>0$, consider the expectation

$$E^\varrho\otimes\mathbb{E}\big(\langle\vartheta_N^{\text{ext}},f\rangle\mathrm{e}^{-s\langle\vartheta_N^{\text{ext}},f\rangle}\big). \tag{4.24}$$

This is related to the above by

$$E^{\varrho}\otimes\mathbb{E}\bigl(\mathrm{e}^{-\langle\vartheta_N^{\mathrm{ext}},f\rangle}\bigr)=1-\int_0^s E^{\varrho}\otimes\mathbb{E}\bigl(\langle\vartheta_N^{\mathrm{ext}},f\rangle\mathrm{e}^{-s\langle\vartheta_N^{\mathrm{ext}},f\rangle}\bigr)\mathrm{d}s \tag{4.25}$$

Now the additive form of $\langle\vartheta_N^{\mathrm{ext}},f\rangle$ implies

$$\begin{aligned}&E^{\varrho}\otimes\mathbb{E}\bigl(\langle\vartheta_N^{\mathrm{ext}},f\rangle\mathrm{e}^{-s\langle\vartheta_N^{\mathrm{ext}},f\rangle}\bigr)\\&\qquad=\frac{\sqrt{\log N}}{\widehat{K}_N}\sum_{x\in D_N}E^{\varrho}\otimes\mathbb{E}\Bigl(f\bigl(x/N,L_t(x),h_x\bigr)\mathrm{e}^{-s\langle\vartheta_N^{\mathrm{ext}},f\rangle}\Bigr)\end{aligned} \tag{4.26}$$

Since $f\geqslant 0$ and the argument of f under expectation depends only on h_x, the conditional Jensen inequality gives

$$\begin{aligned}&E^{\varrho}\otimes\mathbb{E}\Bigl(f\bigl(x/N,L_t(x),h_x\bigr)\,\mathrm{e}^{-s\langle\vartheta_N^{\mathrm{ext}},f\rangle}\Bigr)\\&\qquad\qquad\geqslant E^{\varrho}\otimes\mathbb{E}\Bigl(f\bigl(x/N,L_t(x),h_x\bigr)\mathrm{e}^{-s\mathbb{E}(\langle\vartheta_N^{\mathrm{ext}},f\rangle\,|\,h_x)}\Bigr)\end{aligned} \tag{4.27}$$

The point is now to show that the conditioning on h_x can be ignored and the conditional expectation can be replaced by $\langle\vartheta_N\otimes\mathrm{Leb},f\rangle+o(1)$. This is done through a truncation argument for which we refer the reader to Lemma 7.1 in the paper with Y. Abe.

Once the conditioning is taken care of, we again apply the calculation (4.20–4.22) to the remaining occurrence of h_x in the expression under the sum in (4.26). Hence we get

$$E^{\varrho}\otimes\mathbb{E}\bigl(\langle\vartheta_N^{\mathrm{ext}},f\rangle\mathrm{e}^{-s\langle\vartheta_N^{\mathrm{ext}},f\rangle}\bigr)\geqslant o(1)+\mathrm{e}^{o(1)}E^{\varrho}\bigl(\langle\vartheta_N\otimes\mathrm{Leb},f\rangle\mathrm{e}^{-s\langle\vartheta_N\otimes\mathrm{Leb},f\rangle}\bigr) \tag{4.28}$$

where both $o(1)$ tend to zero as $N\to\infty$ uniformly in $s\in[0,1]$. Plugging this in (4.25) then gives

$$E^{\varrho}\otimes\mathbb{E}\bigl(\mathrm{e}^{-\langle\vartheta_N^{\mathrm{ext}},f\rangle}\bigr)\leqslant o(1)+\mathrm{e}^{o(1)}E^{\varrho}\bigl(\mathrm{e}^{-\langle\vartheta_N\otimes\mathrm{Leb},f\rangle}\bigr) \tag{4.29}$$

This, along with (4.23), completes the proof. □

4.4 Distributional identity.

With the convergence (4.19) in hand, we are ready for the application of the coupling (4.6) which links every subsequential weak limit of random measures $\{\vartheta_{N\geqslant 1}$ to the measures describing the thick points of the DGFF:

Lemma 4.6 *Given $f\colon\overline{D}\times\mathbb{R}_+\to\mathbb{R}$ with compact support, denote*

$$f^{*\mathrm{Leb}}(x,\ell):=\int_{\mathbb{R}}\mathrm{d}h\,f\bigl(x,\ell+\tfrac{h^2}{2}\bigr). \tag{4.30}$$

Then every subsequential weak limit ϑ of random measures $\{\vartheta_{N\geqslant 1}$ satisfies

$$\bigl\langle\vartheta,f^{*\mathrm{Leb}}\bigr\rangle\overset{\mathrm{law}}{=}\int Z^D_{\sqrt{\theta}}(\mathrm{d}x)\otimes\mathrm{e}^{\alpha\sqrt{\theta}h}\mathrm{d}h\,f\bigl(x,\tfrac{h^2}{2}\bigr) \tag{4.31}$$

simultaneously for all f as above.

Proof. Consider the coupling of L_t, h and $\tilde{h}$ such that (4.6) holds. Denote

$$K_N := \frac{N^2}{\sqrt{\log N}} \mathrm{e}^{-\frac{(\sqrt{2t_N})^2}{2g\log N}} \tag{4.32}$$

and observe that

$$\frac{\sqrt{\log N}}{\widehat{K}_N} = \frac{1}{K_N} \tag{4.33}$$

Given f as above, abbreviate

$$f^{\text{ext}}(x,\ell,h) := f\bigl(x, \ell + \tfrac{h^2}{2}\bigr) \tag{4.34}$$

The coupling then gives

$$\begin{aligned} \frac{1}{K_N}\sum_{x\in D_N} f\bigl(x/N, \tfrac{1}{2}(\tilde{h}_x + \sqrt{2t_N})^2\bigr) &= \frac{\sqrt{\log N}}{\widehat{K}_N}\sum_{x\in D_N} f\bigl(x/N, L_t(x) + \tfrac{1}{2}h_x^2\bigr) \\ &= \bigl\langle \vartheta_N^{\text{ext}}, f^{\text{ext}}\bigr\rangle \end{aligned} \tag{4.35}$$

Lemma 4.5 then tells us that, along the subsequence $\{N_k\}_{k\geqslant 1}$ that takes ϑ_{N_k} to ϑ, the left-hand side tends weakly to

$$\bigl\langle \vartheta\otimes\text{Leb}, f^{\text{ext}}\bigr\rangle = \int \vartheta(\mathrm{d}x\mathrm{d}\ell)\mathrm{d}h f^{\text{ext}}\bigl(x,\ell+\tfrac{h^2}{2}\bigr) = \bigl\langle\vartheta, f^{*\text{Leb}}\bigr\rangle \tag{4.36}$$

On the other hand, since $\sqrt{2t_N} \sim 2\sqrt{g}\sqrt{\theta}\log N$, Theeorem 1.5 tells us that the left-hand side of (4.35) tends weakly to

$$\int Z^D_{\sqrt{\theta}}(\mathrm{d}x)\otimes \mathrm{e}^{\alpha\sqrt{\theta}h}\mathrm{d}h\, f\bigl(x,\tfrac{h^2}{2}\bigr) \tag{4.37}$$

This now gives the claim. □

We now claim that this gives:

Corollary 4.7 *Suppose μ is a deterministic measure on $\mathbb{R}_+$ with the Laplace transform*

$$\int_{\mathbb{R}+} \mu(\mathrm{d}\ell)\mathrm{e}^{-s\ell} = \exp\Bigl\{\frac{\alpha^2\theta}{2s}\Bigr\}, \quad s>0. \tag{4.38}$$

Then every subsequential weak limit ϑ of random measures $\{\vartheta_{N\geqslant 1}$ takes the form

$$\vartheta(\mathrm{d}x\mathrm{d}\ell) = Z^D_{\sqrt{\theta}}(\mathrm{d}x)\otimes\mu(\mathrm{d}\ell) \tag{4.39}$$

Proof. Given an open set $A\subseteq\mathbb{R}^2$, abbreviate $\zeta_A(B) := \vartheta(A\times B)$. Given $s>0$, abbreviate $g_s(\ell) := \mathrm{e}^{-s\ell}$. Take a sequence $\{f_n\}_{n\geqslant 1}$ of continuous compactly supported functions that increase to $f := 1_A\otimes g$ and note that $f_n^{*\text{Leb}}$ then increases to

$$f^{*\text{Leb}}(x,\ell) = 1_A(x)\mathrm{e}^{-s\ell}\sqrt{\frac{2\pi}{s}} \tag{4.40}$$

Realizing the equality in law in (4.31) as almost sure equality, applying the identity along the above sequence with the help of the Monotone Convergence Theorem then shows

$$\sqrt{\frac{2\pi}{s}}\int\zeta_A(\mathrm{d}\ell)\mathrm{e}^{-s\ell} = Z^D_{\sqrt{\theta}}(A)\int\mathrm{d}h\,\mathrm{e}^{\alpha\sqrt{\theta}h}\mathrm{e}^{-s\frac{1}{2}h^2} \quad \text{a.s.} \tag{4.41}$$

The Gaussian integral on the right equals $\mathrm{e}^{\frac{\alpha^2\theta}{2s}}\sqrt{\frac{2\pi}{s}}$ which tells us that the measure

$$\nu_s(A) := \exp\Big\{-\frac{\alpha^2\theta}{2s}\Big\}\int_{A\times\mathbb{R}_+}\vartheta(\mathrm{d}x\mathrm{d}\ell)\mathrm{e}^{-s\ell} \tag{4.42}$$

equals to $Z^D_{\sqrt{\theta}}(A)$ a.s. on all open $A\subseteq\mathbb{R}^2$, regardless of $s>0$. To overcome the fact that the implicit null set may depend on A and s, note that the Borel sets in $\mathbb{R}^2$ are generated by a countably many open sets. Hence we get that, on a set of full probability, $\nu_s = Z^D_{\sqrt{\theta}}$ for all rational $s>0$. But then also

$$\int_{A\times\mathbb{R}_+}\vartheta(\mathrm{d}x\mathrm{d}\ell)\mathrm{e}^{-s\ell} = Z^D_{\sqrt{\theta}}(A)\int_{\mathbb{R}}\mu(\mathrm{d}\ell)\mathrm{e}^{-s\ell} \tag{4.43}$$

holds for all Borel $A\subseteq\mathbb{R}^2$ and all $s>0$ on an event of full probability measure, where we also used that both sides are continuous whenever finite. The fact that the Laplace transform determines the measure then gives the claim. □

We have basically proved:

Theorem 4.8 *For all $\theta\in(0,1)$ and any $\{t_N\}_{N\geqslant1}$ with $t_N\sim 2g\theta(\log N)^2$,*

$$\vartheta_N \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} Z^D_{\sqrt{\theta}}(\mathrm{d}x)\otimes\mu(\mathrm{d}\ell) \tag{4.44}$$

where μ is the measure

$$\mu(\mathrm{d}\ell) := \delta_0(\mathrm{d}\ell) + \bigg(\sum_{n\geqslant0}\frac{1}{n!(n+1)!}\Big(\frac{\alpha^2\theta}{2}\Big)^{n+1}\ell^n\bigg)1_{[0,\infty)}(\ell)\,\mathrm{d}\ell \tag{4.45}$$

Proof. We just need to check that μ has the Laplace transform (4.38) which is a straightforward calculation. □

4.5 Proof of Theorem 1.7.

Intuitively, the measure μ above gives us access to the "distribution" of $O(1)$-values of the local time so the setting of Theorem 1.7 should correspond to taking just the atom at 0 from μ. However, to make this precise we have to invoke approximation by continuous, compactly supported functions which requires checking that no part of that atom came from infinitesimal values somehow accumulating to zero in the limit. This is done in:

Lemma 4.9 *For any $\delta>0$ there exists $c>0$ such that*

$$\frac{1}{\widehat{K}_N}E^{\varrho}\bigg(\sum_{\substack{x\in D_N\\ d_\infty(x,D_N^{\mathrm{c}})>\delta N}}1_{\{0<L_{t_N}(x)\leqslant\epsilon\}}\bigg)\leqslant c\epsilon \tag{4.46}$$

holds for all $N\geqslant1$.

Proof. Invoking one more time the representation (4.10) we have

$$P^{\varrho}\big(0<L_t(x)\leqslant\epsilon\big)\leqslant P\bigg(N_t\geqslant1\,\wedge\,\forall k=1,\dots,N_t\colon\sum_{k=1}^{Z_K}T_{k,j}\leqslant\epsilon\pi(x)\bigg) \tag{4.47}$$

Proceeding as in (4.13) this gives

$$P^{\varrho}\bigl(0 < L_t(x) \leqslant \epsilon\bigr) \leqslant \mathrm{e}^{-\frac{t}{G(x,x)}\exp\{-\frac{\epsilon}{G(x,x)}\}} - \mathrm{e}^{-\frac{t}{G(x,x)}} \leqslant \mathrm{e}^{-\frac{t}{G(x,x)}}\bigl[\mathrm{e}^{\epsilon\frac{t}{G(x,x)^2}} - 1\bigr] \tag{4.48}$$

Now set $V := D_N$ and $t_N = O((\log N)^2)$. For any $\delta > 0$ small, once x at least δN from the boundary, we have

$$P^{\varrho}\bigl(0 < L_t(x) \leqslant \epsilon\bigr) \leqslant c\frac{\widehat{K}_N}{N^2}\epsilon \tag{4.49}$$

The claim follows by summing this over $x \in D_N$ with $d_\infty(x, D_N^{\mathrm{c}}) > \delta N$. □

We now finally give:

Proof of Theorem 1.7. Let $g\colon D \to \mathbb{R}$ be continuous with compact support and, for each $n \geqslant 1$, set $f_n(x,\ell) := g(x)(1-n\ell)_+$. Denote

$$\kappa_N := \frac{1}{\widehat{K}_N}\sum_{x\in D_N} 1_{\{L_{t_N}(x)=0\}}\delta_{x/N} \tag{4.50}$$

Then

$$\bigl|\langle\vartheta_N, f_n\rangle - \langle\kappa_N, g\rangle\bigr| \leqslant \|g\|\frac{1}{\widehat{K}_N}E^{\varrho}\Biggl(\sum_{\substack{x\in D_N\\ x/N\in\mathrm{supp}(g)}} 1_{\{0<L_{t_N}(x)\leqslant 1/n\}}\Biggr) \tag{4.51}$$

Since $d_\infty(\mathrm{supp}(g), D^{\mathrm{c}}) > \delta$, Lemma 4.9 tells us that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\bigl|\langle\vartheta_N, f_n\rangle - \langle\kappa_N, g\rangle\bigr| = 0 \tag{4.52}$$

Invoking Theorem 4.8, this gives

$$\langle\kappa_N, g\rangle \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \bigl\langle Z^D_{\sqrt{\theta}}, g\bigr\rangle \lim_{n\to\infty}\int \mu(\mathrm{d}\ell) f_n(\ell) \tag{4.53}$$

The limit on the right equals 1 and so $\kappa_N \overset{\text{law}}{\longrightarrow} Z^D_{\sqrt{\theta}}$ as measures on $D \times \mathbb{R}_+$. The convergence is extended to $\overline{D} \times \mathbb{R}_+$ with the help of tightness proved in Corollary 4.4 and the fact that ϑ_N naturally dominates κ_D. □