LECTURE 3. THICK POINTS OF THE DGFF

The main goal of this lecture is to give the proof of Theorem 1.5. For lack of time, not all of the details will be spelled out; the point is to convey the main ideas and explain the key technical steps. The reader is referred to the PIMS notes of the author for deeper treatment and, if even that is not sufficient, to the original paper with O. Louidor.

3.1 Gibbs-Markov property of DGFF.

We start by an important fact about the Gaussian Free Field that we call the *Gibbs-Markov property*. This is nothing but the "domain-Markov property" introduced in N. Berestycki's lectures; the reason for attaching Gibbs' name to this concept is that, for DGFF, this property arises from the fact that the law of the DGFF is a Gibbs measure for a nearest-neighbor Hamiltonian. Here is the precise statement:

Lemma 3.1 (Gibbs-Markov property) *Let $U, V \subseteq \mathbb{Z}^2$ be non-empty finite sets with $U \subsetneq V$ and let h^V be the DGFF in V. Define*

$$\varphi_x^{V,U} := \begin{cases} \sum_{z\in\partial U} H^U(x,z)h_z^V, & \text{if } x \in U, \\ h_x^V, & \text{if } x \notin U. \end{cases} \tag{3.1}$$

Then

$$h^V - \varphi^{V,U} \text{ and } \varphi^{V,U} \text{ are independent} \tag{3.2}$$

with

$$h^V - \varphi^{V,U} \overset{\text{law}}{=} \text{ DGFF in } U \tag{3.3}$$

Every sample path of $\varphi^{V,U}$ is discrete harmonic on U.

Proof. If we set $H^U(x,z) = \delta_{xz}$ when $x \notin U$, we can write (3.1) concisely as

$$\varphi_x^{V,U} = \sum_{z\in V\smallsetminus U} H^U(x,z)h_z^V \tag{3.4}$$

Hereby we get

$$\begin{aligned} f(x,y) &:= \text{Cov}\big(h_x^V - \varphi_x^{V,U}, \varphi_y^{V,U}\big) \\ &= \sum_{z,z'\in V\smallsetminus U} \big[G^V(x,z') - G^V(z,z')\big] H^U(x,z)H^U(y,z') \end{aligned} \tag{3.5}$$

Here are some facts about f. First, $f(x,y) = 0$ whenever $x \in V \smallsetminus U$. Second, $x \mapsto f(x,y)$ is discrete harmonic in $y \in U$ and, for each $y \in V \smallsetminus U$, equals

$$G^V(x,y) - \sum_{z\in V\smallsetminus U} G^V(z,y)H^U(x,z) \tag{3.6}$$

which is discrete harmonic in $x \in U$. The uniqueness of discrete harmonic extension forces $f(x,y) = 0$ whenever $y \in V \smallsetminus U$. From discrete harmonicity of $y \mapsto f(x,y)$ we then get that $f(x,y) = 0$ for all $x, y \in V$; i.e.,

$$\text{Cov}\big(h_x^V - \varphi_x^{V,U}, \varphi_y^{V,U}\big), \quad x, y \in V \tag{3.7}$$

meaning that $h^V - \varphi^{V,U}$ and $\varphi^{V,U}$ are uncorrelated. As they are both multivariate Gaussian, they are independent, proving (3.2).

In order to prove (3.3) we use (3.4) to observe that

$$\mathrm{Cov}\big(h_x^V - \varphi_x^{V,U},\, h_y^V - \varphi_y^{V,U}\big) = G^V(x,y) + \text{harmonic in } x,y \in U \tag{3.8}$$

The potential-kernel representation (2.8) then shows that, for each $y \in U$,

$$\mathfrak{a}(x-y) + \mathrm{Cov}\big(h_x^V - \varphi_x^{V,U},\, h_y^V - \varphi_y^{V,U}\big) \tag{3.9}$$

is harmonic in $x \in U$ and equal to $\mathfrak{a}(z-y)$ at all $z \in \partial U$. The argument in the proof of the identity (2.8) then gives

$$\mathrm{Cov}\big(h_x^V - \varphi_x^{V,U},\, h_y^V - \varphi_y^{V,U}\big) = G^U(x,y) \tag{3.10}$$

proving (3.3). The discrete harmonicity of $x \mapsto \varphi_x^{V,U}$ is a consequence of the same property of $x \mapsto H^U(x,z)$ for $z \notin U$. $\square$

Note that (3.1) means that $\varphi^{V,U}$ is a discrete-harmonic extension into U of the values of h^V outside U. In order to make referencing to the Gibbs-Markov property easier, we will often write it in the form

$$h^V \overset{\text{law}}{=} h^U + \varphi^{V,U} \quad \text{where} \quad h^U \perp\!\!\!\perp \varphi^{V,U} \tag{3.11}$$

where h^V and h^U are DGFFs in V and U and $\varphi^{V,U}$ has the law as specified above.

One setting in which we will need the Gibbs-Markov property is when $U := V \smallsetminus \{x\}$. Then (3.11) reads

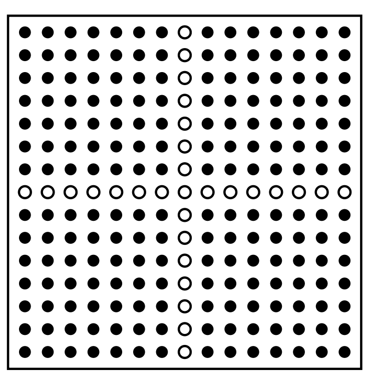
$$h^V \overset{\text{law}}{=} \mathfrak{g}_x(\cdot)h_x^V + h^U \quad \text{where} \quad h^U \perp\!\!\!\perp h_x^V \tag{3.12}$$

and $\mathfrak{g}_x\colon V \to [0,1]$ is a (deterministic) function that is discrete harmonic on $V \smallsetminus \{x\}$ with $\mathfrak{g}_x(x) = 1$ and $\mathfrak{g}_x = 0$ on V^{c}. As is easy to check, we have

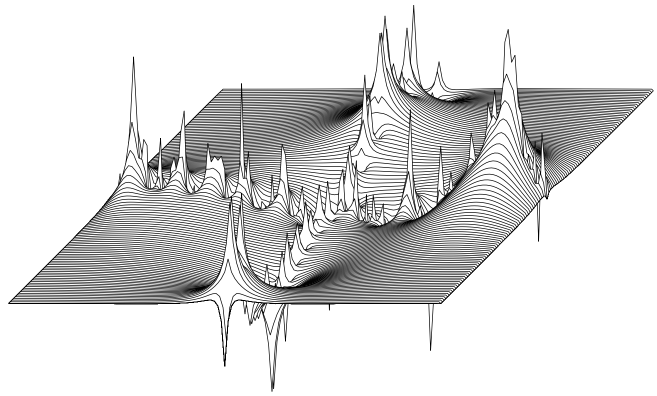
$$\mathfrak{g}_x(y) = \frac{G^V(x,y)}{G^V(x,x)} \tag{3.13}$$

for all $y \in V$.

Another instance where the Gibbs-Markov property will be used is when V is the square $\{1,\dots,2N-1\}^2$ and U is the union of four translates of the square $\{1,\dots,N-1\}^2$ by vectors $(0,0)$, $(N,0)$, $(0,N)$ and (N,N). The set $V \smallsetminus U$ is a "cross" of two lines of vertices as depicted in



On the "cross" $\varphi^{V,U}$ has the law of h^V and is thus quite rough there. However, thanks to discrete harmonicity, $\varphi^{V,U}$ is quite smooth, at least as soon as we lock sufficiently far from the boundary. A sample of $\varphi^{V,U}$ is shown in



An important fact associated with this setting in domains $U \subseteq V$ that are scaled-up version of two continuum domains by N, the field $\varphi^{V,U}$ is well approximated (and converges to) a smooth process. Explicitly, we have:

Lemma 3.2 *Let $\{D_N\}_{N\geqslant 1}$ and $\{\widetilde{D}_N\}_{N\geqslant 1}$ be admissible approximation of two admissible domains $\widetilde{D} \subseteq D \subseteq \mathbb{R}^2$. For each $N \geqslant 1$ there exists a coupling of $\varphi^{D_N,\widetilde{D}_N}$ and a Gaussian process $\{\Phi^{D,\widetilde{D}}(x)\colon x \in \widetilde{D}\}$ with law determined by*

$$\forall x \in \widetilde{D}\colon \quad \mathbb{E}\Phi^{D,\widetilde{D}}(x) = 0 \tag{3.14}$$

and

$$\forall x, y \in \widetilde{D}\colon \quad \mathbb{E}\big(\Phi^{D,\widetilde{D}}(x)\Phi^{D,\widetilde{D}}(y)\big) = \widehat{G}^D(x,y) - \widehat{G}^{\widetilde{D}}(x,y), \tag{3.15}$$

where $\widehat{G}^D$ is the continuum Green function in D defined in (2.6)*, such that for each $\delta > 0$,*

$$\sup_{\substack{x\in\widetilde{D}\\ d(x,\widetilde{D}^c)>\delta}} \big|\varphi^{D_N,\widetilde{D}_N}_{\lfloor xN\rfloor} - \Phi^{D,\widetilde{D}}(x)\big| \ \underset{N\to\infty}{\overset{P}{\longrightarrow}}\ 0 \tag{3.16}$$

Moreover, a.e. sample path of $\Phi^{D,\widetilde{D}}$ is harmonic on $\widetilde{D}$.

Note that the singular parts of the continuum Green function cancel in the expression $\widehat{G}^D(x,y) - \widehat{G}^{\widetilde{D}}(x,y)$. That this is a covariance follows from it being the limit of the covariances of $\varphi^{D_N,\widetilde{D}_N}$. This gives convergence in law in the sense of finite-dimensional distributions. To get convergence in local-supremum norm, one has to control the oscillation of the two processes.

3.2 Subsequential limits.

We now move to the main goal of this section. Fix a sequence $\{a_N\}_{N\geqslant 1}$ such that λ defined by the limit in (1.25) belongs to $(0,1)$. We will only carry out the proof under the assumption that $\lambda < 1/\sqrt{2}$ because this does not require truncations in second-moment calculations we perform below.

Let $\{D_N\}_{N\geqslant 1}$ be admissible approximations of an admissible domain $D\subseteq\mathbb{R}^2$. Abbrevite the measures of interest as

$$\eta_N := \frac{1}{K_N}\sum_{x\in D_N}\delta_{x/N}\otimes\delta_{h_x^{D_N}-a_N} \tag{3.17}$$

and recall that $K_N = N^{2(1-\lambda^2)+o(1)}$. We also introduce the notation

$$\Gamma_N(b) := \bigl\{x\in D_N\colon h_x^{D_N}\geqslant a_N+b\bigr\} \tag{3.18}$$

and, using the shorthand, $\langle\mu,f\rangle := \int f\mathrm{d}\mu$, note that, given any $A\subseteq D$ open and abbreviating $A_N := \{x\in\mathbb{Z}^2\colon x/N\in A\}$, we have

$$\frac{1}{K_N}\bigl|\Gamma_N(b)\cap A_N\bigr| = \bigl\langle\eta_N, 1_A\otimes 1_{[b,\infty)}\bigr\rangle \tag{3.19}$$

Our aim is now to compute two moments of the size of the level set $\Gamma_N(b)$. We start with a bound on the first moment:

Lemma 3.3 *There exists $c>0$ such that for all $N\geqslant 1$ and all $b\in[-a_N/2,a_N]$,*

$$\forall A\subseteq D_N\colon\quad \mathbb{E}\bigl|\Gamma_N(b)\cap A\bigr|\leqslant c\frac{|A|}{N^2}\,\mathrm{e}^{-\frac{a_N}{g\log N}b}K_N \tag{3.20}$$

Proof. Recall that $G(x,x) := G^{D_N}(x,x)\leqslant g\log N+\tilde c$ uniformly in $x\in D_N$. Using the standard Gaussian estimate

$$X=\mathcal{N}(0,\sigma^2)\quad\Rightarrow\quad\forall t\geqslant 0\colon\; P(X\geqslant t)\leqslant\sigma t^{-1}\mathrm{e}^{-\frac{t^2}{2\sigma^2}} \tag{3.21}$$

we get

$$\begin{aligned}\mathbb{P}\bigl(h^{D_N}\geqslant a_N+b\bigr)&\leqslant\frac{\sqrt{G(x,x)}}{a_N+b}\,\mathrm{e}^{-\frac{(a_N+b)^2}{2G(x,x)}}\\&\leqslant\frac{\sqrt{g\log N+\tilde c}}{a_N/2}\,\mathrm{e}^{-\frac{(a_N+b)^2}{2[g\log N+\tilde c]}}\leqslant\frac{c}{\sqrt{\log N}}\,\mathrm{e}^{-\frac{a_N^2}{2g\log N}}\,\mathrm{e}^{-\frac{a_N}{g\log N}b}\end{aligned} \tag{3.22}$$

where we used that $(g\log N+\tilde c)^{-1} = (g\log N)^{-1}+O(\log N)^{-2})$. Writing the prefactor of the last exponential as cK_N/N^2, the claim follows by summing over $x\in A$. $\square$

The purpose of the upper bound is that it allows us to control the contribution from the part of D_N close to the boundary where the Green function is not close to the upper bound we used in the proof. This, along with the precise asymptotic in Theorem 2.1(2), is the main ingredient for the proof of:

Lemma 3.4 *Let $A\subseteq D$ be open and set $A_N := \{x\in\mathbb{Z}^2\colon x/N\in A\}$. Then for all $b\in\mathbb{R}$,*

$$\frac{1}{K_N}\mathbb{E}\bigl|\Gamma_N(b)\cap A_N\bigr|\;\underset{N\to\infty}{\longrightarrow}\;\hat c(\alpha\lambda)^{-1}\mathrm{e}^{-\alpha\lambda b}\int_A r^D(x)^{2\lambda^2}\mathrm{d}x \tag{3.23}$$

where

$$\hat c := \frac{\mathrm{e}^{-2c_0\lambda^2/g}}{\sqrt{8\pi}}\alpha \tag{3.24}$$

for c_0 the constant in (2.3) *and r^D is as in* (2.4).

Leaving the proof to an exercise, we now move to the second moment. It is here where the restriction on λ comes from.

Lemma 3.5 *Suppose $\lambda \in (0, 1/\sqrt{2})$. There exists $c > 0$ such that for all $N \geqslant 1$,*

$$\mathbb{E}\Bigl(\bigl|\Gamma_N(b)\bigr|^2\Bigr) \leqslant cK_N^2 \tag{3.25}$$

Proof. Let us set $b := 0$ for simplicity (or absorb the term into a_N). Then

$$\mathbb{E}\Bigl(\bigl|\Gamma_N(b)\bigr|^2\Bigr) = \sum_{x,y \in D_N} \mathbb{P}\bigl(h_x^{D_N} \geqslant a_N,\, h_y^{D_N} \geqslant a_N\bigr) \tag{3.26}$$

To bound the summand uniformly in x or y regardless how far these are from the boundary D_N, we replace h^{D_N} by the field in the enlarged domain

$$\widetilde{D}_N := \{x \in \mathbb{Z}^2 \colon d_\infty(x, D_N) \leqslant N\} \tag{3.27}$$

noting that then

$$\frac{1}{4}\mathbb{P}\bigl(h_x^{D_N} \geqslant a_N,\, h_y^{D_N} \geqslant a_N\bigr) \leqslant \mathbb{P}\bigl(h_x^{\widetilde{D}_N} \geqslant a_N,\, h_y^{\widetilde{D}_N} \geqslant a_N\bigr) \tag{3.28}$$

holds by the Gibbs-Markov property.

Next we split the sum according to whether $d_\infty(x,y) \leqslant \sqrt{K_N}$ or not. The first part we bound using Lemma 3.3 as

$$\sum_{\substack{x,y \in D_N \\ d_\infty(x,y) \leqslant \sqrt{K_N}}} \mathbb{P}(h_x^{\widetilde{D}_N} \geqslant a_N,\, h_y^{\widetilde{D}_N} \geqslant a_N) \leqslant (2\sqrt{K_N} + 1)^2 \sum_{x \in D_N} \mathbb{P}(h_x^{\widetilde{D}_N} \geqslant a_N) \leqslant cK_N^2 \tag{3.29}$$

In the second part we distinguish whether $h_x^{\widetilde{D}_N}$ exceeds $2a_N$ or not. Using that

$$\mathbb{P}(h_x^{\widetilde{D}_N} \geqslant 2a_N) \leqslant \frac{c}{\sqrt{\log N}} \mathrm{e}^{-2\frac{a_N^2}{g \log N}} = c\Bigl(\frac{K_N}{N^2}\Bigr)^2 \sqrt{\log N}\, \mathrm{e}^{-\frac{a_N^2}{g \log N}} \leqslant c\Bigl(\frac{K_N}{N^2}\Bigr)^2 \tag{3.30}$$

once N is sufficiently large, the part where $h_x^{\widetilde{D}_N} \geqslant 2a_N$ contributes at most

$$\sum_{x,y \in D_N} \mathbb{P}\bigl(h_x^{\widetilde{D}_N} > 2a_N,\, h_y^{\widetilde{D}_N} \geqslant a_N\bigr) \leqslant c\Bigl(\frac{|D_N|}{N^2}\Bigr)^2 K_N^2 \tag{3.31}$$

where the right-hand side is again at most a constant times K_N^2.

We are thus left to bound the expression

$$\sum_{\substack{x,y \in D_N \\ d_\infty(x,y) > \sqrt{K_N}}} \mathbb{P}\bigl(2a_N \geqslant h_x^{\widetilde{D}_N} \geqslant a_N,\, h_y^{\widetilde{D}_N} \geqslant a_N\bigr) \tag{3.32}$$

Here we note that the Gibbs-Markov property allows us to condition on $h_x^{\widetilde{D}_N} \geqslant a_N$ by way of the decomposition (3.12–3.13) that in the present setting reads

$$h_y^{\widetilde{D}_N} \overset{\text{law}}{=} \mathfrak{g}_x(y) h_x^{\widetilde{D}_N} + \hat{h}_y^{\widetilde{D}_N \smallsetminus \{x\}} \tag{3.33}$$

where $\mathfrak{g}_x\colon \mathbb{Z}^2 \to [0,1]$ is the unique discrete-harmonic function in $D_N \smallsetminus \{x\}$ extending the boundary values $\mathfrak{g}_x(x) = 1$ and $\mathfrak{g}_x = 0$ on $D_N^{\rm c}$ and $\hat h_y^{\widetilde D_N\smallsetminus\{x\}}$ is the DGFF in $D_N \smallsetminus \{x\}$ that is sampled independently of $h_x^{\widetilde D_N}$. Using this we can write

$$\begin{aligned}\mathbb{P}(2a_N \geqslant h_x^{\widetilde D_N} \geqslant a_N,\, h_y^{\widetilde D_N} \geqslant a_N)\\ = \int_0^{a_N} \mathbb{P}(h_x^{\widetilde D_N} - a_N \in {\rm d}s)\mathbb{P}\Bigl(\hat h_y^{\widetilde D_N\smallsetminus\{x\}} \geqslant a_N\bigl[1-\mathfrak{g}_x(y)\bigr] - s\mathfrak{g}_x(y)\Bigr)\end{aligned} \tag{3.34}$$

In order to bound the integrand, observe that $d_\infty(x,y) > \sqrt{K_N} = N^{1-\lambda^2+o(1)}$ along with the fact that x, y are "deep" inside D_N imply

$$\begin{aligned}\mathfrak{g}_x(y) = \frac{G^{\widetilde D_N}(x,y)}{G^{\widetilde D_N}(x,x)} &\leqslant \frac{\log\frac{N}{\|x-y\|}+c}{\log N - c}\\ &\leqslant 1-(1-\lambda^2)+o(1) = \lambda^2+o(1)\end{aligned} \tag{3.35}$$

Since $\lambda < 1/\sqrt{2}$, it follows that, given any $\epsilon \in (0, 1-2\lambda^2)$,

$$\epsilon a_N \leqslant a_N\bigl[1-\mathfrak{g}_x(y)\bigr] - s\mathfrak{g}_x(y) \leqslant a_N \tag{3.36}$$

holds for all for $s \in [0, a_N]$ once N is sufficiently large, uniformly in x and y contributing to the sum (3.32). The standard Gaussian estimate (3.21) along with the bound

$$\bigl(a_N[1-\mathfrak{g}_x(y)] - s\mathfrak{g}_x(y)\bigr)^2 \geqslant a_N^2 - 2(a_N+s)\mathfrak{g}_x(y) \tag{3.37}$$

then show

$$\begin{aligned}\mathbb{P}\Bigl(\hat h_y^{\widetilde D_N\smallsetminus\{x\}} \geqslant a_N\bigl[1-\mathfrak{g}_x(y)\bigr] - s\mathfrak{g}_x(y)\Bigr)\\ \leqslant \frac{\sqrt{G(y,y)}}{\epsilon a_N}\,{\rm e}^{-\frac{(a_N[1-\mathfrak{g}_x(y)]-s\mathfrak{g}_x(y))^2}{2G(y,y)}} \leqslant c\frac{K_N}{N^2}\,{\rm e}^{\frac{a_N^2}{g\log N}\mathfrak{g}_x(y)+\frac{a_N}{g\log N}\mathfrak{g}_x(y)s}\end{aligned} \tag{3.38}$$

where $G(y,y)$ abbreviates $G^{\widetilde D_N\smallsetminus\{x\}}(y,y)$. (We also used that the upper bound $G(y,y) \leqslant g\log N + c$ applies regardless of how far y is to the boundary of $\widetilde D_N \smallsetminus \{x\}$.)

Now observe that the first inequality in (3.35) gives

$${\rm e}^{\frac{a_N^2}{g\log N}\mathfrak{g}_x(y)} \leqslant c\biggl(\frac{N}{\|x-y\|}\biggr)^{4\lambda^2+o(1)} \tag{3.39}$$

Using in the explicit form of the probability density of $h_x^{\widetilde D_N}$ we also get

$$\mathbb{P}(h_x^{\widetilde D_N} - a_N \in {\rm d}s) \leqslant c\frac{K_N}{N^2}{\rm e}^{-\frac{a_N}{g\log N}s}{\rm d}s \tag{3.40}$$

With the help of these we bring (3.34) to the form

$$\begin{aligned}\mathbb{P}(2a_N \geqslant h_x^{\widetilde D_N} \geqslant a_N,\, h_y^{\widetilde D_N} \geqslant a_N)\\ \leqslant c\biggl(\frac{K_N}{N^2}\biggr)^2\biggl(\frac{N}{\|x-y\|}\biggr)^{4\lambda^2+o(1)}\int_0^{a_N}{\rm e}^{-\frac{a_N}{g\log N}[1-\mathfrak{g}_x(y)]s}{\rm d}s\end{aligned} \tag{3.41}$$

In light of $\mathfrak{g}_x(y) \leqslant \lambda^2 + o(1)$, the integral converges uniformly in all y of concern. As a consequence, we get

$$\begin{aligned}\sum_{\substack{x,y \in D_N \\ d_\infty(x,y) > \sqrt{K_N}}} &\mathbb{P}\big(2a_N \geqslant h_x^{\widetilde{D}_N} \geqslant a_N,\, h_y^{\widetilde{D}_N} \geqslant a_N\big) \\ &\leqslant c\Big(\frac{K_N}{N^2}\Big)^2 \sum_{\substack{x,y \in D_N \\ d_\infty(x,y) > \sqrt{K_N}}} \Big(\frac{N}{\|x-y\|}\Big)^{4\lambda^2 + o(1)}\end{aligned} \tag{3.42}$$

Using that $4\lambda^2 < 2$, the sum is dominated by pairs x and y such that $\|x - y\|$ is order N and so is order $(N^2)^2$. (Alternatively, dominate the sum by an integral.) The expression is thus bounded by a constant times K_N^2, as desired. □

3.3 Factorization and uniqueness.

As a consequence of Lemmas 3.3–3.5 we get:

Corollary 3.6 *Suppose $\lambda < 1/\sqrt{2}$. Then $\{\eta_N\}_{N \geqslant 1}$ form a tight sequence of measures on $\overline{D} \times (\mathbb{R} \cup \{+\infty\})$ and every subsequential weak limit η obeys*

$$\mathbb{E}\,\eta\big(A \times [b, \infty)\big) = \hat{c}(\alpha\lambda)^{-1} \mathrm{e}^{-\alpha\lambda b} \int_A r^D(x)^{2\lambda^2} \mathrm{d}x \tag{3.43}$$

for any relatively open $A \subseteq D$ and any $b \in \mathbb{R}$.

A particular consequence of (3.43) is that η is non-vanishing on each non-empty open set with positive probability. Similar as Lemma 3.4 refines the bound from Lemma 3.3, the second moment calculation from Lemma 3.5 can be refined to get:

Lemma 3.7 *Suppose $\lambda < 1/\sqrt{2}$ and, given $A \subseteq D$ open, set $A_N := \{x \in \mathbb{Z}^2 \colon x/N \in A\}$. Then for all $b \in \mathbb{R}$,*

$$\frac{1}{K_N^2} \mathbb{E}\bigg(\Big[\big|\Gamma_N(b) \cap A_N\big| - \mathrm{e}^{-\alpha\lambda b}\big|\Gamma_N(0) \cap A_N\big|\Big|^2\bigg) \underset{N\to\infty}{\longrightarrow} 0 \tag{3.44}$$

Proof (idea). We write the expectation as the sum over $x, y \in A_N$ of

$$\begin{aligned}\mathbb{P}\big(h_x^{D_N} \geqslant a_N + b, h_y^{D_N} \geqslant a_N + b\big) &- \mathrm{e}^{-\alpha\lambda b}\mathbb{P}\big(h_x^{D_N} \geqslant a_N + b, h_y^{D_N} \geqslant a_N\big) \\ - \mathrm{e}^{-\alpha\lambda b}\mathbb{P}\big(h_x^{D_N} \geqslant a_N, h_y^{D_N} \geqslant a_N + b\big) &+ \mathrm{e}^{-2\alpha\lambda b}\mathbb{P}\big(h_x^{D_N} \geqslant a_N, h_y^{D_N} \geqslant a_N\big)\end{aligned} \tag{3.45}$$

The proof of Lemma 3.5 (with $\sqrt{K_N}$ cut-off replaced by $\delta\sqrt{K_N}$) tells us that it suffices to control the pairs $d_\infty(x, y) \geqslant \delta N$. Here we suffices to show that for any $b_1, b_2 \in \{0, b\}$,

$$\begin{aligned}\mathbb{P}\big(h_x^{D_N} &\geqslant a_N + b, h_y^{D_N} \geqslant a_N + b\big) \\ &= \big(\mathrm{e}^{-\alpha\lambda(b_1+b_2)} + o(1)\big)\mathbb{P}\big(h_x^{D_N} \geqslant a_N, h_y^{D_N} \geqslant a_N\big)\end{aligned} \tag{3.46}$$

which is checked by a similar calculation as that in the proof of Lemma 3.5. □

This now allows us to upgrade the conclusion of Corollary 3.6 as:

Corollary 3.8 (Factorization) *Suppose $\lambda < 1/\sqrt{2}$. Then every subsequential weak limit η of measures $\{\eta_N\}_{N\geqslant 1}$ factors as*

$$\eta = Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\mathrm{d}h \tag{3.47}$$

where Z^D is a random Borel measure such that

$$\mathbb{E}Z^D(A) = \hat{c}\int_A r^D(x)^{2\lambda^2}\mathrm{d}x \tag{3.48}$$

holds for any relatively open $A \subseteq D$. In particular, $Z^D(A) = 0$ a.s. for any $A \subseteq \mathbb{R}^d$ with vanishing Lebesgue measure.

Proof. Lemma 3.7 implies

$$\eta\big(A \times [b,\infty)\big) = \mathrm{e}^{-\alpha\lambda b}\eta\big(A \times [0,\infty)\big) \tag{3.49}$$

Setting

$$Z^D(A) := (\alpha\lambda)^{-1}\eta\big(A \times [0,\infty)\big) \tag{3.50}$$

the right hand side of (3.49) coincides with the integral of the measure on the right of (3.47) over $A \times [b,\infty)$. Varying A and b then identifies the form of the measure uniquely. The identity (3.48) then follows from (3.43). □

With the measure taking the desired product form, the following question remain: Is the subsequential weak limit unique in law? And, if so, is there a way to characterize Z^D? In order to answer these, we need to first show how the Gibbs-Markov property manifests itself for the limit object:

Lemma 3.9 *Let $D, \widetilde{D}$ be admissible domains with $\widetilde{D} \subseteq D$ yet with $D \smallsetminus \widetilde{D}$ of vanishing Lebesgue measure. Then for Z^D and $Z^{\widetilde{D}}$ constructed along the same subsequence,*

$$Z^D(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha\lambda\Phi^{D,\widetilde{D}}(x)}Z^{\widetilde{D}}(\mathrm{d}x), \quad \Phi^{D,\widetilde{D}} \perp\!\!\!\perp Z^{\widetilde{D}} \tag{3.51}$$

where on the right $\Phi^{D,\widetilde{D}}$ is the Gaussian process from Lemma 3.2.

Proof. Let η_N^D, resp, $\eta_N^{\widetilde{D}}$ denote the finite N processes in D_N, resp., $\widetilde{D}_N$. The proof is based on the observation that the Gibbs-Markov property cast as a.s. equality translates into

$$\big\langle \eta_N^D, f(\cdot,\cdot)\big\rangle = \big\langle \eta_N^{\widetilde{D}}, f(\cdot,\cdot + \varphi_{\lfloor\cdot N\rfloor}^{D_N,\widetilde{D}_N})\big\rangle \tag{3.52}$$

whenever f is compactly supported in $\widetilde{D} \times \mathbb{R}$. Passing to a joint distributional limit along the same subsequence shows that the limiting processes η^D and $\eta^{\widetilde{D}}$ obey

$$\langle \eta^D, f\rangle = \big\langle \eta^{\widetilde{D}}, f(\cdot,\cdot + \Phi^{D,\widetilde{D}})\big\rangle \tag{3.53}$$

A change of coordinates then shows $Z^D(\mathrm{d}x) = \mathrm{e}^{\alpha\lambda\Phi^{D,\widetilde{D}}}Z^{\widetilde{D}}(\mathrm{d}x)$ as measures on $\widetilde{D}$. Since neither sides charges $D \smallsetminus \widetilde{D}$, the equality applies even as measures on D. □

We are finally ready to give:

Proof of Theorem 1.5 for $\lambda < 1/\sqrt{2}$, modulo a technical step. We will argue that the law of Z^D is uniquely determined by the expectation (3.48) and the Gibbs-Markov property (3.51).

We start with D being a square $S := (0,1) \times (0,1)$. Use $\{S_i^n : i = 1, \dots, 4^n\}$ to label the boxes of the form $k2^{-n} + \ell 2^{-n} + (0, 2^{-n}) \times (0, 2^{-n})$ for $k, \ell = 0, \dots, 2^n - 1$ and denote

$$S^n := \bigcup_{i=1}^{4^n} S_i^n \tag{3.54}$$

The Gibbs-Markov property then gives

$$Z^S(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha\lambda\Phi^{S,S^n}(x)} Z^{S^n}(\mathrm{d}x), \quad Phi^{S,S^n} \perp\!\!\!\perp Z^{S^n} \tag{3.55}$$

Since the DGFF is independent over connected components of the underlying domain, we can write

$$\mathrm{e}^{\alpha\lambda\Phi^{S,S^n}(x)} Z^{S^n}(\mathrm{d}x) = \sum_{i=1}^{4^n} 1_{S_i^n}(x) \mathrm{e}^{\alpha\lambda\Phi^{S,S^n}(x)} Z^{S_i^n}(\mathrm{d}x) \tag{3.56}$$

Introduce the measure in which $Z^{S_i^n}$ is replaced by its expectation,

$$Y_n^S(\mathrm{d}x) := \hat{c} \sum_{i=1}^{4^n} 1_{S_i^n}(x) \mathrm{e}^{\alpha\lambda\Phi^{S,S^n}(x)} r^{S_i^n}(x)^{2\lambda^2} \mathrm{d}x \tag{3.57}$$

For any $f \colon S \to [0, \infty)$ continuous with compact support, taking expectation with respect to the law of the measures $\{Z^{S_i^n} : i = 1, \dots, 4^n\}$ via (3.48) and the conditional Jensen inequality then gives

$$\mathbb{E}\big(\mathrm{e}^{-\langle Z^D, f\rangle}\big) \geqslant \mathbb{E}\big(\mathrm{e}^{-\langle Y_n^D, f\rangle}\big) \tag{3.58}$$

The point is to prove the reverse inequality, at least in the limit as $n \to \infty$. Here we invoke:

Exercise 3.10 (Reverse Jensen inequality) *Prove that if $X_1, \dots, X_n$ are independent non-negative random variables, then*

$$E\bigg(\exp\Big\{-\sum_{i=1}^n X_i\Big\}\bigg) \leqslant \exp\bigg\{-\mathrm{e}^{-\epsilon} \sum_{i=1}^n E(X_i | X_i \leqslant \epsilon)\bigg\} \tag{3.59}$$

holds for all $\epsilon > 0$.

Given $\delta \in (0, 1/2)$ we apply this to

$$X_i := \big\langle Z^{S_i^n},\, 1_{S_i^{n,\delta}} f\big\rangle, \tag{3.60}$$

where $S_i^{n,\delta}$ is obtained by same translate as S_i^n but of the box $(\delta 2^{-n}, (1-\delta)2^{-n})$. Since

$$\langle Z^D, f\rangle \geqslant \sum_{i=1}^{4^n} X_i \tag{3.61}$$

the inequality (3.59) gives

$$\mathbb{E}\big(\mathrm{e}^{-\langle Z^D, f\rangle}\big) \leqslant \mathbb{E}\bigg(\exp\Big\{-\mathrm{e}^{-\epsilon} \sum_{i=1}^{4^n} E(X_i | X_i \leqslant \epsilon)\Big\}\bigg) \tag{3.62}$$

In order to show that these approximations are negligible, we then have to show that the errors incurred by conditioning are negligible,

$$\forall\epsilon > 0\colon \quad \sum_{i=1}^{4^n} E(X_i | X_i > \epsilon) \underset{n\to\infty}{\overset{P}{\longrightarrow}} 0 \tag{3.63}$$

Second, denoting $S_\delta^n := \bigcup_{i=1}^{4^n} S_i^{n,\delta}$, we have to show that the restriction of the integral underlying $\langle Y_n^S, f\rangle$ from S to S_δ^n is negligible,

$$\forall\epsilon > 0\colon \quad \lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\bigl(Y_n^S(S \smallsetminus S_\delta^n) > \epsilon\bigr) = 0 \tag{3.64}$$

These are proved by calculations that still do require $\lambda < 1/\sqrt{2}$. We refer to the PIMS lecture notes (page 217) of the author for full details. □

We note that the above proof shows that

$$Y_n^S \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} Z^S \tag{3.65}$$

This gives Z^S a representation of Gaussian multiplicative chaos. Moreover,

$$\mathrm{Var}\bigl(\Phi^{D,\widetilde{D}}(x)\bigr) = g \log\left(\frac{r^D(x)}{r^{\widetilde{D}}(x)}\right) \tag{3.66}$$

along with the fact (implied by $\mathrm{Cov}(\Phi^{D,\widetilde{D}}) = \widehat{G}^D - \widehat{G}^{\widetilde{D}}$) that the increment fields

$$\{\Phi^{S,S^{n+1}} - \Phi^{S,S^n} : n \geqslant 0\} \tag{3.67}$$

can be realized as independent on the same probability space shows that $\langle Y_n^S, f\rangle$ is for any continuous compactly-supported $f \geqslant 0$ a positive martingale and the convergence (3.65) thus takes place a.s. We remark that, since $\widehat{G}^D - \widehat{G}^{\widetilde{D}}$ are all non-negative functions, a criterion of Kahane characterizes the law of Z^S uniquely. The Gibbs-Markov property and some limits extend this to Z^D for all admissible D.