Lecture 2

Green function asymptotic and connection to DGFF

The purpose of the second lecture, which is also a kind of tutorial, is to develop two important technical ingredients that enter our later proofs. The first of these is the asymptotic of the Green function which is responsible for many of the underlying phenomena. The second ingredient concerns the connection of the local time to the DGFF which will later allow us to derive Theorem 1.7 from Theorem 1.5.

2.1 Asymptotic of the Green function.

Suppose that D is an admissible domain. For any $x \in D$, let Π^D denote the *harmonic measure* from x. This can be defined as the exit distribution from D of the standard Brownian motion B started at x, i.e., for any Borel $A \subseteq \mathbb{R}^2$,

$$\Pi^D(x,A) := P^x\bigl(B_{\tau_{D^c}} \in A\bigr) \quad \text{where} \quad \tau_{D^c} := \inf\bigl\{t \geqslant 0\colon B_t \notin D\bigr\} \tag{2.1}$$

Let $\{D_N\}_{N\geqslant 1}$ be a sequence of admissible approximations of D. Write $\lfloor xN\rfloor$ for the unique $z \in \mathbb{Z}^d$ such that $x - z \in [0,1)^2$. We wish to prove:

Theorem 2.1 *Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. We then have:*

(1) *There exist a constant $c \in (0,\infty)$ such that for all $N \geqslant 1$ and all $x,y \in D_N$*

$$G^{D_N}(x,y) \leqslant g\log\biggl(\frac{N}{1+\|x-y\|}\biggr) + c \tag{2.2}$$

where g is as in (1.23)*.*

(2) *For all $x \in D$,*

$$G^{D_N}\bigl(\lfloor xN\rfloor, \lfloor xN\rfloor\bigr) = g\log N + c_0 + g\log r^D(x) + o(1) \tag{2.3}$$

where $c_0 := \frac14(2\gamma + \log 8)$ for γ denoting the Euler constant,

$$r^D(x) := \exp\biggl\{\int_{\partial D} \Pi^D(x,\mathrm{d}z)\log\|z-x\|\biggr\} \tag{2.4}$$

and $o(1) \to 0$ as $N \to \infty$ locally uniformly in $x \in D$.

(3) *For all $x,y \in D$ with $x \neq y$,*

$$G^{D_N}\bigl(\lfloor xN\rfloor, \lfloor yN\rfloor\bigr) = -g\log\|x-y\| + g\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log\|z-y\| + o(1) \tag{2.5}$$

where $o(1) \to 0$ as $N \to \infty$ locally uniformly in $(x,y) \in D\times D \smallsetminus \{(z,z)\colon z \in D\}$.

Before we delve into the proof, let us make three remarks. First, the asymptotic (2.3) and (2.5) require that the points x and y stay away from ∂D and from each other. This is because the Green function vanishes near the boundary and has a logarithmic singularity on the "diagonal."

Second, the somewhat strange way of writing the x dependent term on the right of (2.3) is motivated by the interpretation of r^D. Indeed, for D simply connected this quantity coincides with the *conformal radius* of D from x, which is a measure of the linear size of D that is invariant under conformal maps. The fact that $x \mapsto \log\|x-z\|$ is harmonic in $x \neq z$ is used crucially in verifying this property.

Third, the limit function on the right of (2.5), namely,

$$\widehat{G}^D(x,y) := -g\log\|x-y\| + g\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log\|z-y\| \tag{2.6}$$

is the so called *continuum Green function* in D with Dirichlet boundary condition. Since also this function is symmetric and positive semidefinite, it is a covariance, albeit only for a generalized Gaussian process called the *Continuum Gaussian Free Field* (CGFF).

We remark that the CGFF is defined only by projections on suitable test functions (see, e.g., Sheffield's review [60]) due to the fact that $\widehat{G}^D(x,y) \to \infty$ as $y \to x$ which makes pointwise value meaningless. This makes working with CGFF somewhat technically involved. Still, the singularity is only logarithmic and so thinking of the field as a random function usually gives a very good intuition.

2.2 Proof of Theorem 2.1.

The proof of Theorem 2.1 is based on a convenient representation of the Green function using the so called *potential kernel*. In our normalization, this is a function $\mathfrak{a}\colon \mathbb{Z}^2 \to \mathbb{R}$ defined by

$$\mathfrak{a}(x) := \frac{1}{4}\int_{(-\pi,\pi)^2}\frac{\mathrm{d}k}{(2\pi)^2}\,\frac{1-\cos(k\cdot x)}{\sin(k_1/2)^2+\sin(k_2/2)^2} \tag{2.7}$$

where the integral converges because the numerator in the integrand vanishes quadratically (in k) in the limit as $k \to 0$. With this we get:

Lemma 2.2 *For all finite $V \subseteq \mathbb{Z}^2$ and all $x, y \in V$,*

$$G^V(x,y) = -\mathfrak{a}(x-y) + \sum_{z\in\partial V} H^V(x,z)\mathfrak{a}(z-y) \tag{2.8}$$

where $H^V(x,z)$ is the probability that X started at x exists V at $z \in \partial V$.

Proof. Denote the discrete Laplacian acting on $f\colon \mathbb{Z}^2 \to \mathbb{R}$ with compact support as

$$\Delta f(x) := \sum_{y\sim x}\bigl[f(y)-f(x)\bigr] \tag{2.9}$$

where $y \sim x$ denotes that (x,y) is an edge in $\mathbb{Z}^2$. Using the Markov property of X it is then checked that, for each $h \in V$, we have

$$\begin{cases} \Delta G^V(\cdot,y) = -\delta_y(\cdot) & \text{on } V \\ \;\; G^V(\cdot,y) = 0 & \text{on } \mathbb{Z}^2 \smallsetminus V \end{cases} \tag{2.10}$$

What makes the potential kernel particularly useful in this proof is that it solves a similar problem; namely,

$$\begin{cases} \Delta\mathfrak{a}(\cdot) = \delta_0(\cdot) \quad \text{on } \mathbb{Z}^2 \\ \mathfrak{a}(0) = 0 \end{cases} \tag{2.11}$$

as is explicitly checked from (2.7). Combining (2.10–2.11) we conclude that

$$x \mapsto G^V(x,y) + \mathfrak{a}(x-y) \text{ is discrete harmonic on } V \tag{2.12}$$

Relying on the fact that the discrete harmonic function is a martingale for the underlying random walk, we thus get

$$G^V(x,y) + \mathfrak{a}(x-y) = \sum_{z\in\partial V} H^V(x,z)\bigl[G^V(z,y) + \mathfrak{a}(z-y)\bigr] \tag{2.13}$$

Noting that $G^V(z,y) = 0$ for $z \notin V$, this reduces to (2.8). $\square$

In order to make use of the formula (2.8) we need two lemmas whose proof we will leave to an exercise and/or literature study. The first of these concerns the asymptotic growth of the potential kernel, which is also where the constants g and c_0 in Theorem 2.1 enter the fray:

Lemma 2.3 *For $x \neq 0$ we have $\mathfrak{a}(x) \geqslant 0$. Moreover,*

$$\mathfrak{a}(x) = g\log\|x\| + c_0 + O\bigl(\|x\|^{-2}\bigr) \tag{2.14}$$

This was apparently first proved by A. Stöhr [61] in 1950. An article by G. Kozma and E. Schreiber [45] from 2004 links the constant g and c_0 to geometric properties of the underlying lattice, which allows then to verify the formula for other lattices as well. A very probabilistic approach to the theory of the potential kernel can be found in Section 4.4 of the monograph by G. Lawler and V. Limić [46].

Exercise 2.4 *Prove* (2.14) *by way of asymptotic analysis of the integral* (2.7).

With Lemma 2.3 in hand, we are able to give:

Proof of (1) in Theorem 2.1. Since $V \mapsto G^V(x,y)$ is non-decreasing with respect to the set inclusion, it suffices to prove this for x and y such that $d_\infty(y, D_N^{\mathrm{c}}) \geqslant N$. Assuming $x \neq y$ and plugging the asymptotic (2.14) shows

$$\begin{aligned} G^{D_N}(x,y) = &-\Bigl[g\log\|x-y\| + c_0 + O\bigl(\|x-y\|^{-2}\bigr)\Bigr] \\ &+ \sum_{z\in\partial D_N} H^{D_N}(x,z)\Bigl[g\log\|z-y\| + c_0 + O\bigl(\|z-y\|^{-2}\bigr)\Bigr] \end{aligned} \tag{2.15}$$

Using that $H^{D_N}(x,\cdot)$ is a probability mass function, the constant c_0 cancels in both terms while the assumption that $d_\infty(y, D_N^{\mathrm{c}}) \geqslant N$ means that the last term in the square bracket is $O(N^{-2})$. Bounding $\|z-y\|$ by a constant times N, we then get the desired bound. In the case when $x = y$ we use that $\mathfrak{a}(0) = 0$ and bound the second term as above. $\square$

For the remaining two parts of Theorem 2.1 we also need:

Lemma 2.5 *For any $x \in D$,*

$$\sum_{z\in\partial D_N} H^{D_N}\bigl(\lfloor xN\rfloor,z\bigr)\delta_{z/N} \underset{N\to\infty}{\overset{\text{vaguely}}{\longrightarrow}} \Pi^D(x,\cdot) \tag{2.16}$$

The proof of this is somewhat technical due to the fact that we want to make this work for rather general D. The argument proceeds by coupling the random walk to Brownian motion so that their exit distributions remain close to each other. Details can be found in Appendix of a joint paper of the author with O. Louidor [18].

We are now ready for:

Proof of (2-3) in Theorem 2.1. Starting with (2), since $\mathfrak{a}(0) = 0$ we only need to use the asymptotic on the second term on the right of (2.8). This gives

$$\begin{aligned} G^{D_N}\bigl(\lfloor xN\rfloor,\lfloor xN\rfloor\bigr) &= \sum_{z\in\partial D_N} H^{D_N}\bigl(\lfloor xN\rfloor,z\bigr)\mathfrak{a}\bigl(z-\lfloor xN\rfloor\bigr) \\ &= g\log N + c_0 + g\sum_{z\in\partial D_N} H^{D_N}\bigl(\lfloor xN\rfloor,z\bigr)\log\biggl(\frac{\|z-\lfloor xN\rfloor\|}{N}\biggr) + O(N^{-2}) \end{aligned} \tag{2.17}$$

where the error term uses the fact that, for N large enough, $\|z-z-\lfloor xN\rfloor\| \geqslant \delta N$ for some x-dependent $\delta > 0$ uniformly in $z \in \partial D_N$. For similar reason the vague convergence in Lemma 2.5 applies to the function $z \mapsto \log\|z-x\|$ and, by way of an elementary approximation to get rid of the integer rounding, makes the sum to converge to $\log r^D(x)$, locally uniformly in $x \in D$.

For part (3) we assume $x \neq y$ and again use the asymptotic to get

$$\begin{aligned} G^{D_N}\bigl(\lfloor xN\rfloor,\lfloor yN\rfloor\bigr) &= -g\log\biggl(\frac{\|\lfloor xN\rfloor-\lfloor yN\rfloor\|}{N}\biggr) \\ &\quad + g\sum_{z\in\partial D_N} H^{D_N}\bigl(\lfloor xN\rfloor,z\bigr)\log\biggl(\frac{\|z-\lfloor yN\rfloor\|}{N}\biggr) + O(N^{-2}) \end{aligned} \tag{2.18}$$

where c_0 again dropped out using that $z \mapsto H^{D_N}(x,z)$ is a probability mass function. Passing to the limit using Lemma 2.5 then yields the claim. $\square$

2.3 Connection between the local time and the DGFF.

The second topic of our interest in this lecture is a connection between the local time and the DGFF. We will treat this in the general case of a Markov chain on $V \cup \{\varrho\}$ where ϱ is the distinguished vertex that was used to define the Green function G^V.

Recall that L_t is the local time parametrized by the local time at ϱ which in particular means $L_t(\varrho) = t$ a.s. As our first result, we state the following limit theorem:

Theorem 2.6 (DGFF limit) *For L_t sampled under P^ϱ,*

$$\frac{L_t - t}{\sqrt{2t}} \underset{t\to\infty}{\overset{\text{law}}{\longrightarrow}} h^V \tag{2.19}$$

and, in particular,

$$\sqrt{L_t} - \sqrt{t} \underset{t\to\infty}{\overset{\text{law}}{\longrightarrow}} \frac{1}{\sqrt{2}}h^V \tag{2.20}$$

where h^V is the DGFF on V.

This result tells us that, at large times, a properly shifted and scaled local time profile is close to a sample of the DGFF. The rewrite (2.20) explains why it is sometimes better with the square-root of L_t as no normalization is required. However, the connection runs far deeper and, in fact, applies at any fixed time t:

Theorem 2.7 (Second Ray-Knight Theorem) *For each $t \geqslant 0$, there exists a coupling of L_t and two copies h and $\tilde{h}$ of DGFF on V such that*

$$L_t \text{ and } h \text{ are independent} \tag{2.21}$$

and

$$\forall x \in V \cup \{\varrho\}\colon \quad L_t(x) + \frac{1}{2}h_x^2 \,=\, \frac{1}{2}\big(\tilde{h}_x + \sqrt{2t}\big)^2 \quad \text{a.s.} \tag{2.22}$$

This has been proved as equality in distribution by Eisenbaum, Kaspi, Marcus, Rosen and Shi [35] in 2000 with the coupling part added by Zhai [64] in 2018.

We remark that Theorem 2.7 belongs to a larger collection of results that link local time of stochastic processes to random fields. That such connection exists was first conceived of by K. Symanzik [62], and later developed by mathematical physicics (D. Brydges, J. Fröhlich and T. Spencer [25]) and probabilists (E.B. Dynkin [34]). While Theorem 2.7 is sometimes referred to as "Dynkin isomorphism," this is a misattribution as the "isomorphism" in [34, Theorem 1] works under a different setting than (2.22).

To explain the reliance on the specific time parameterization, observe that, under P^ϱ, the local time L_t on V is the sum of a random number of independent excursions that start by an exponential waiting time at ϱ, then exit into V and, after running around V for a while, terminate by hitting ϱ again. Denoting, for each $x \in V \cup \{\varrho\}$, the first return time of the chain to x by

$$\hat{H}_x := \inf\big\{t \geqslant 0\colon X_t = x \wedge \exists s \in [0,t)\colon X_s \neq x\big\} \tag{2.23}$$

we thus get:

Lemma 2.8 *Given $t > 0$, let $\{\ell_j\}_{j\geqslant 1}$ be i.i.d. copies of $\ell_{\hat{H}_\varrho}$ sampled under P^ϱ and let N_t denote an independent Poisson random variable with parameter $\pi(\varrho)t$. Then*

$$L_t \text{ under } P^\varrho \;\overset{\text{law}}{=}\; \sum_{j=1}^{N_t} \ell_j, \quad \text{on } V \tag{2.24}$$

Proof. The independence of the excursions is a consequence of the strong Markov property of X. That the number of excursions has Poisson law is the standard fact that the number of i.i.d. Exponentials with parameter one needed to accumulate the total value at least u is Poisson with parameter u. $\square$

Note that (2.24) fails at ϱ a.s. due to the fact that $L_t(\varrho)$ is non-random, which is the reason why we exclude ϱ from many statements below. With Lemma 2.8 in hand one can already prove the formula (2.19). While we will prove both theorems along the same lines, we leave the alternative argument to:

Exercise 2.9 *Using the notation in Lemma 2.8, prove that*

$$\forall x \in V \cup \{\varrho\}\colon \quad E^\varrho\big(\ell_1(x)\big) = 1 \tag{2.25}$$

and

$$\forall x,y \in V \cup \{\varrho\}\colon \quad \mathrm{Cov}_{P^\varrho}\big(\ell_1(x),\ell_1(y)\big) = G^V(x,y) \tag{2.26}$$

Then use the multivariate (random-index) Central Limit Theorem and the decomposition in Lemma 2.8 to prove Theorem 2.6.

2.4 Kac moment formula.

The proof of the above results is actually somewhat algebraic in nature. In order to present the details, introduce the standard inner product

$$\langle f,g\rangle := \sum_{x\in V\cup\{\varrho\}} f(x)g(x), \tag{2.27}$$

and let M_f be the operator of point-wise multiplication by f acting as

$$M_f g(x) := f(x)g(x), \quad x \in V \cup \{\varrho\} \tag{2.28}$$

The driving force of all subsequent derivations is then:

Lemma 2.10 (Kac moment formula) *For each $f\colon V \cup \{\varrho\} \to \mathbb{R}$ with $f(\varrho) = 0$,*

$$E^\varrho\big(\langle \ell_1, f\rangle^n\big) = n!\frac{1}{\pi(\varrho)}\big\langle f, (G^V M_f)^{n-1}\, 1\big\rangle, \quad n \geqslant 1 \tag{2.29}$$

where $(G^V M_f)g(x) = \sum_{y\in V\cup\{\varrho\}} G^V(x,y)f(y)g(y)$.

Proof. The Markov property and elementary symmetrization tells us

$$E^\varrho\big(\langle \ell_1, f\rangle^n\big) = n!\int_{0\leqslant t_1<\dots<t_n<H_\varrho} \mathrm{d}t_1\dots\mathrm{d}t_n\, \frac{f(X_{t_1})}{\pi(X_{t_1})}\dots\frac{f(X_{t_n})}{\pi(X_{t_n})} \tag{2.30}$$

Abbreviating

$$\mathsf{P}^t(x,y) := \mathsf{P}^x(X_t = y,\, \hat H_\varrho > t) \tag{2.31}$$

and changing variables to $s_k := t_k - t_{k-1}$ (where $t_0 := 0$), the Markov property of X allows us to rewrite the integral in (2.30) as

$$\sum_{x_1,\dots,x_n\in V\cup\{\varrho\}} \left(\prod_{i=1}^n \frac{f(x_i)}{\pi(x_i)}\right)\int_{s_1,\dots,s_n\geqslant 0} \mathrm{d}s_1\dots\mathrm{d}s_n\, \mathsf{P}^{s_1}(\varrho,x_1)\dots\mathsf{P}^{s_n}(x_{n-1},x_n) \tag{2.32}$$

Next we observe that

$$\int_0^\infty \mathrm{d}s\, \mathsf{P}^s(x,y) = E^x\Big(\int_0^{H_\varrho}\mathrm{d}s\, 1_{\{X_s=y\}}\Big) = \pi(y)G^V(x,y), \quad x,y \neq \varrho \tag{2.33}$$

and, using the strong Markov property at the first hitting time of y,

$$\int_0^\infty \mathrm{d}s\, \mathsf{P}^s(\varrho,y) = P^\varrho(H_y < \hat H_\varrho)\pi(y)G^V(y,y), \quad x \neq \varrho \tag{2.34}$$

To bring (2.34) to a better form, use reversibility to get

$$\pi(\varrho)P^\varrho(H_y < \hat H_\varrho) = \pi(y)P^y(H_\varrho < \hat H_y) \tag{2.35}$$

and then note that, by a decomposition of the form (2.33) and the fact that Exponential(1)-random variable has mean one, $\pi(y)G^V(y,y)$ equals one plus the expected time to first succeed in independent trials with success probability $P^y(H_\varrho < \hat{H}_y)$. This implies

$$\pi(y)G^V(y,y) = \frac{1}{P^y(H_\varrho < \hat{H}_y)} \tag{2.36}$$

which combining with (2.35) gives

$$\int_0^\infty \mathrm{d}s\ \mathsf{P}^s(\varrho,y) = \frac{\pi(y)}{\pi(\varrho)}, \quad y \neq \varrho \tag{2.37}$$

Note that this a different structure than (2.33).

For $f\colon V \cup \{\varrho\} \to \mathbb{R}$ with $f(\varrho) = 0$ we now restrict the sums in (2.32) to $x_i \in V$ and note that (2.33) and (2.37) give

$$E^\varrho\big(\langle \ell_1, f\rangle^n\big) = \frac{n!}{\pi(\varrho)} \sum_{x_1,\dots,x_n \neq \varrho} f(x_1)G(x_1,x_2)\dots G(x_{n-1},x_n)f(x_n). \tag{2.38}$$

The sum on the right is identified with $\langle f, (GM_f)^{n-1}1\rangle$. $\square$

A formula of the kind (2.29) was first derived by M. Kac [43] with a follow up by D.A. Darling and M. Kac [29]. A reader interested in more background and further results should consult P.J. Fitzimmons and J. Pitman [37].

The Kac moment formula gives us an explicit handle of the law of L_t:

Corollary 2.11 *For any $f\colon V \cup \{\varrho\} \to \mathbb{R}$ with $f(\varrho) = 0$ and $\max_{x\in V} |f(x)|$ small enough so that $\|G^V M_f\| < 1$,*

$$E^\varrho\big(\mathrm{e}^{\langle \ell_1, f\rangle}\big) = 1 + \frac{1}{\pi(\varrho)}\Big\langle f, (1 - G^V M_f)^{-1}1\Big\rangle. \tag{2.39}$$

In particular, for each $t \geqslant 0$,

$$E^\varrho\big(\mathrm{e}^{\langle L_t, f\rangle}\big) = \mathrm{e}^{\,t\langle f, (1-G^V M_f)^{-1}\,1\rangle} \tag{2.40}$$

Proof. Assume that f is so small that $\|GM_f\| < 1$. The identity (2.39) then follows by summing (2.29) on $n \geqslant 1$. With the help from (2.24) we then get

$$E^\varrho\big(\mathrm{e}^{\langle L_t, f\rangle}\big) = \exp\Big\{t\pi(\varrho)\big[E\big(\mathrm{e}^{\langle \ell_1, f\rangle}\big) - 1\big]\Big\} \tag{2.41}$$

and so (2.40) follows from (2.39). $\square$

This now permits us to conclude:

Proof of Theorem 2.6. Assuming f small enough, rewrite (2.40) as

$$E^\varrho\big(\mathrm{e}^{\langle (L_t - t), f\rangle}\big) = \mathrm{e}^{\,t\langle f, (1-G^V M_f)^{-1}\,G^V f\rangle} \tag{2.42}$$

Now rescale f by $\sqrt{2t}$ and notice that, as $t \to \infty$, the right-hand side tends to $\mathrm{e}^{\frac{1}{2}\langle f, G^V f\rangle}$, which is the Laplace transform of $\langle f, h^V\rangle$. The Curtiss theorem then gives the claim. $\square$

2.5 Proof of the Second Ray-Knight Theorem.

Since V is fixed throughout, we will ease the notation by writing G for G^V throughout this subsection. In order to prepare for the proof of Theorem 2.7, we recall:

Lemma 2.12 (Gaussian integration by parts) *Let $X = (X_1, \dots, X_n)$ be a multivariate Gaussian with mean zero and covariance matrix C. Then for any $g \in C^1(\mathbb{R}^n)$ with subgaussian growth of ∇g and any linear $f \colon \mathbb{R}^n \to \mathbb{R}$,*

$$\operatorname{Cov}\big(f(X), g(X)\big) = \sum_{i,j=1,\dots,n} C(i,j) E\biggl(\frac{\partial f}{\partial x_i}(X)\frac{\partial g}{\partial x_j}(X)\biggr) \tag{2.43}$$

Proof. For $X_1, \dots, X_n$ i.i.d. $\mathcal{N}(0,1)$, this is checked readily from $x\mathrm{e}^{-\frac12 x^2} = -\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-\frac12 x^2}$ and integration by parts. The general case is handled by writing $X = AZ$ where Z is a vector of i.i.d. $\mathcal{N}(0,1)$ and A is a matrix such that $\operatorname{Cov}(X) = AA^{\mathrm{T}}$. □

Using this we first note:

Lemma 2.13 *For all $f, g \colon V \cup \{\varrho\} \to \mathbb{R}$ with f small enough and each $s \in \mathbb{R}$,*

$$\mathbb{E}\Bigl(\bigl\langle h+s, g\bigr\rangle \mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr) = s\bigl\langle 1, (1-M_f G)^{-1} g\bigr\rangle\, \mathbb{E}\Bigl(\mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr) \tag{2.44}$$

Proof. Gaussian integration by parts shows

$$\begin{aligned}\mathbb{E}\Bigl(\bigl\langle h+s, g\bigr\rangle \mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr) &= s\langle 1, g\rangle\, \mathbb{E}\Bigl(\mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr) + \mathbb{E}\Bigl(\bigl\langle h, g\bigr\rangle \mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr)\\ &= s\langle 1, g\rangle \mathbb{E}\Bigl(\mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr) + \mathbb{E}\Bigl(\bigl\langle M_f(h+s), Gg\bigr\rangle \mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr)\end{aligned} \tag{2.45}$$

Putting the last term on the right together with the term on the left we get

$$\mathbb{E}\Bigl(\bigl\langle h+s, g - M_f G g\bigr\rangle \mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr) = s\langle 1, g\rangle\, \mathbb{E}\Bigl(\mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr) \tag{2.46}$$

The claim follows by relabeling g for $(1 - M_f G)^{-1} g$. □

Hence we get:

Corollary 2.14 *For any $f \colon V \cup \{\varrho\} \to \mathbb{R}$ sufficiently small and any $s \geqslant t$,*

$$\mathbb{E}\Bigl(\mathrm{e}^{\frac12\langle (h+s)^2, f\rangle}\Bigr) = \mathrm{e}^{\frac12(s^2-t^2)\langle 1, (1-M_f G)^{-1} f\rangle}\, \mathbb{E}\Bigl(\mathrm{e}^{\frac12\langle (h+t)^2, f\rangle}\Bigr) \tag{2.47}$$

Proof. Using the previous lemma, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\mathbb{E}\Bigl(\mathrm{e}^{\frac12\langle (h+r)^2, f\rangle}\Bigr) &= \mathbb{E}\Bigl(\langle h+r, f\rangle \mathrm{e}^{\frac12\langle (h+r)^2, f\rangle}\Bigr)\\ &= r\bigl\langle 1, (1-M_f G)^{-1} f\bigr\rangle\, \mathbb{E}\Bigl(\mathrm{e}^{\frac12\langle (h+r)^2, f\rangle}\Bigr)\end{aligned} \tag{2.48}$$

The differential equation is readily solved to get the result. □

Proof of Theorem 2.7. Noting that

$$\mathrm{e}^{\frac12(s^2-t^2)\langle 1, (1-M_f G)^{-1} f\rangle} \tag{2.49}$$

is the exponential moment of $\langle L_r, f\rangle$ for $r := \frac{1}{2}(s^2 - t^2)$, we rewrite (2.47) as

$$\mathbb{E}\Bigl(\mathrm{e}^{\frac{1}{2}\langle (h+s)^2, f\rangle}\Bigr) = E^{\varrho}\bigl(\mathrm{e}^{\langle L_r, f\rangle}\bigr)\,\mathbb{E}\Bigl(\mathrm{e}^{\frac{1}{2}\langle (h+t)^2, f\rangle}\Bigr) \tag{2.50}$$

As this holds for all f small, solving for s as a function of t and r and applying the fact that the Laplace transform determines the underlying law shows

$$L_r \perp\!\!\!\perp h \quad\Rightarrow\quad L_r + \frac{1}{2}(h+t)^2 \overset{\text{law}}{=} \frac{1}{2}\bigl(h + \sqrt{t^2+2r}\,\bigr)^2 \tag{2.51}$$

Setting $t := 0$ then gives (2.22) as equality in distribution. In order to construct the coupling, given independent L_t and h, sample $\tilde{h}$ from

$$\mathbb{P}\Bigl(h^V \in \cdot \,\Bigm|\, \tfrac{1}{2}\bigl(h^V + \sqrt{2t}\bigr)^2 = \phi\Bigr)\Bigr|_{\phi := L_t + \frac{1}{2}h^2} \tag{2.52}$$

where the conditioning is well defined by the fact that the probability density of h^V is a continuous function. The identity (2.22) then holds a.s. $\square$

We finish with the following remark: Note that relabeling t for $\sqrt{2t}$ in (2.51) gives (2.22) in the form

$$L_r \perp\!\!\!\perp h \quad\Rightarrow\quad L_r + \frac{1}{2}\bigl(h^V + \sqrt{2t}\bigr)^2 \overset{\text{law}}{=} \frac{1}{2}\Bigl(h^V + \sqrt{2(r+t)}\,\Bigr)^2 \tag{2.53}$$

which can alternatively be derived by iterating (2.22) while using the independence of increments of $t \mapsto L_t$. However, since the construction of the signs of $\tilde{h} + \sqrt{2t}$, which is what sampling from the conditional measure (2.52) is really about, is non-constructive, a question remains whether an almost-sure coupling can be constructed simultaneously for all times. We thus pose:

Question 2.15 Is there a coupling of the local time $\{L_t \colon t \geqslant 0\}$ (sampled under P^{ϱ}) and an $\mathbb{R}^{V\cup\{\varrho\}}$-valued càdlàg process $\{h(t)\colon t \geqslant 0\}$ such that

(1) $\forall t \geqslant 0\colon\ h(t) \overset{\text{law}}{=} h^V$,
(2) $\forall t \geqslant 0\colon\ \{h(s)\colon s \leqslant t\}$ and $\{L_{t+u} - L_t \colon u \geqslant 0\}$ are independent,
(3) for all $r, t \geqslant 0$,

$$\forall r,t \geqslant 0\colon\quad \frac{1}{2}\bigl(h(r) + \sqrt{2r}\bigr)^2 - L_r = \frac{1}{2}\bigl(h(t) + \sqrt{2t}\,\bigr)^2 - L_t, \quad \text{a.s.} \tag{2.54}$$

hold true?

To see that (2.54) is consistent with (2.53) note that, for $t \geqslant r$ we have $L_t - L_r \overset{\text{law}}{=} L_{t-r}$ and so bringing L_t to the left-hand side results in an identity that at least holds in distribution. The reason why we ask for such a coupling is two-fold. First, we find this to be an interesting possibility. Second, having the coupling would make some of the technical arguments in, e.g., [4] much easier.