Applications of Girsanov's Thm

$$\tilde{M}_t = \exp\left\{ \int_0^t Y_s \, dB_s - \frac{1}{2} \int_0^t Y_s^2 \, ds \right\}$$

Thm (Novikov's condition) Let $M \in \mathcal{M}_{loc}^{cont}$ be s.t. $M_0 = 0$ and $E\left(e^{\frac{1}{2}\langle M \rangle_t}\right) < \infty$.
Then $E\left(e^{M_t - \frac{1}{2}\langle M \rangle_t}\right) = 1$.

we proved this with $\frac{1}{2}+\varepsilon$ instead

Caveats in Girsanov:

- finite time horizon (statement is for fixed $t > 0$)
- the measure may look very different from Wiener law when $t \to \infty$ is considered.

Thm (Removal of drift term for SDE). Let $a : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be Borel s.t. for $B = $ SBM with $B_0 = 0$ and some $x \in \mathbb{R}$:

$$\forall t \geq 0: \quad E\left(\exp\left\{ \frac{1}{2} \int_0^t a(s, x + B_s)^2 \, ds \right\} \right) < \infty$$

Then SDE $\quad dX_t = a(t, X_t)\,dt + dB_t, \quad X_0 = x$

admits a (global) weak solution.

$$V_t = X_t - B_t$$

$$\frac{dV_t}{dt} = a(t, V_t + B_t)$$

Pf $\Omega := C([0,\infty))$, $\mathcal{F} = \mathcal{B}(C[0,\infty))$, $P = $ Wiener measure, $B_t(\omega) = \omega(t)$.

Denote $\mathcal{F}_t^B = \sigma(B_s : s \leq t)$. Set

$$M_t := \exp\left\{ \int_0^t a(s, x + B_s)\, dB_s - \frac{1}{2} \int_0^t a(s, x + B_s)^2 \, ds \right\}$$

Novikov + Girsanov $\Rightarrow M \in \mathcal{M}^{\text{cont}}$.

For each $t \geq 0$, define P_t on $(\Omega, \mathcal{F}_t^B)$ by $P_t(A) := E(\mathbb{1}_A M_t)$, $\quad A \in \mathcal{F}_t^B$.

$EM_t = 1 \Rightarrow P_t = $ prob.

M = martingale: $\forall s \le t\ \forall A \in \mathcal{F}_s^B$: $P_t(A) = E(1_A M_t) = E(1_A E(M_t | \mathcal{F}_s))$

So $\{P_t\}$ is Kolmogorov consistent. $= E(1_A M_s) = P_s(A)$.

So $\exists \tilde{P}$ on $\bigcup_{t \ge 0} \mathcal{F}_t^B$ additive and $\forall t \ge 0$: $\tilde{P}|_{\mathcal{F}_t} = P_t$.

Underlying $(\Omega, \mathcal{F})$ is standard Borel so Kolmogorov extension thm shows that $\tilde{P}$ extends to a measure on $\mathcal{F}_\infty^B = \sigma\left(\bigcup_{t \ge 0} \mathcal{F}_t^B\right)$.

On $(\Omega, \mathcal{F}_\infty^B, \tilde{P})$: Girsanov tells me that

$$\tilde{B}_t = B_t - \int_0^t a(s, x + B_s)\, ds$$

is standard Brownian motion. Setting $X_t = x + B_t$, this translates

$$X_t = x + \int_0^t a(s, X_s)\, ds + \tilde{B}_t$$

So X solves $dX_t = a(t, X_t)\, dt + d\tilde{B}_t$ with $X_0 = x$. $\boxtimes$

Brownian motion conditioned to stay positive

law P^x.

Let B = SBM started at $x>0$. Set $\tau_0 := \inf\{t\geq 0 : B_t = 0\}$.

Want $P^x(\cdot \mid \tau_0 = \infty)$ ... problem because $P(\tau_0=\infty)=0$ by recurrence.

Instead, pick $T>0$, set $Q^x_T(A) := P^x(A \mid \tau_0 > T)$ and take limit $T\to\infty$.

Thm $\forall x>0\ \forall t\geq 0\ \forall A \in \mathcal{F}^B_t$: $Q^x(A) := \lim_{T\to\infty} Q^x_T(A)$ exists and extends to a prob. measure on $(\Omega, \mathcal{F}^B_\infty)$. Moreover,

$$\{B_t : t\geq 0\} \text{ under } Q^x \overset{\text{law}}{=} \text{3-dimensional Bessel process started at } x.$$

Note: 3-dimensional Bessel process

$$dX_t = \frac{d-1}{2X_t}dt + dB_t \quad \text{for } d=3,$$ does not hit zero a.s.

Def Existence of limit

Let $A \in \mathcal{F}_t^B$, define $h_t(x) = P^x(\tau_0 > t)$.

Markov property:

h-transform.

$$Q_T^x(A) = E^x\left(\mathbb{1}_{A \cap \{\tau_0 > t\}} \frac{h_{T-t}(B_t)}{h_T(x)} \right)$$

Need $\lim\limits_{T \to \infty} \dfrac{h_{T-t}(B_t)}{h_T(x)}$ under expectation

Lemma $\forall x > 0 \ \forall t > 0$: $h_t(x) = P^x(B_t > 0) - P^x(B_t < 0)$.

Moreover:
$$e^{-\frac{x^2}{2t}} \sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{t}} \le h_t(x) \le \sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{t}} + \frac{x^2}{t}$$

and so $\lim\limits_{t \to \infty} \sqrt{t}\, h_t(x) = \sqrt{\frac{2}{\pi}}\, x$

The bounds show:

$$\frac{h_{T-x}(B_t)}{h_T(x)} \underset{x\le\sqrt{T}}{\le} \frac{\sqrt{e}}{x}\frac{\sqrt{T}}{\sqrt{T-t}}\left(B_t + 2\frac{B_t^2}{\sqrt{T-t}}\right) \underset{\substack{2t\le T \\ T\ge 2}}{\le} C(x)\left(B_t + B_t^2\right)$$

The RHS $\in L^1$ so Dominated convergence:

$$\lim_{T\to\infty} Q_T(A) = E^x\left(1_{A\cap\{\tau_0>t\}}\frac{B_t}{B_0}\right)$$

Itô: $\quad \dfrac{B_t}{B_0} \overset{\tau_0>t}{=} \exp\left\{\log B_t - \log B_0\right\}$

$$\underset{a>0}{=} \exp\left\{\int_0^t \frac{1}{B_s}1_{\tau_a>s}\,dB_s - \frac{1}{2}\int_0^t \frac{1}{B_s^2}1_{\tau_a>s}\,ds\right\} = M_t^{(a)} \quad \text{on } \{\tau_a>t\}.$$

Girsanov:

$$\tilde{B}_t := B_t - \int_0^t \frac{1}{B_s}1_{\tau_a>s}\,ds \quad \text{is SBM.}$$

So B obeys SDE: $dB_t = \dfrac{1}{B_t}1_{\tau_a>t}\,dt + d\tilde{B}_t$. $\ldots\ \{B_{\tau_a\wedge t} : t\ge 0\}$ = 3 dim Bessel process stopped at hitting $a>0$.

$\boxtimes$