Girsanov's Thm:

Thm (Girsanov '1960) Let $B = SBM$ w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$, $Y \in \mathcal{V}_B^{loc}$.

Set
$$M_t := \exp\left\{ \int_0^t Y_s\, dB_s - \frac{1}{2}\int_0^t Y_s^2\, ds \right\}$$

Then $EM_t \leq 1$. Moreover, if $T>0$ is s.t. $EM_T = 1$, then $\{M_{t\wedge T} : t \geq 0\}$ is a martingale, $\tilde{P}(A) = E(\mathbb{1}_A M_T)$ is a prob. measure on $(\Omega, \mathcal{F}_T)$ and

$$\tilde{B}_t := B_t - \int_0^t Y_s\, ds \qquad t \in [0,T]$$

(T = deterministic time)

has a law of SBM under $\tilde{P}$.

Pf Assume $EM_T = 1$. Then $\{M_{t\wedge T} : t\geq 0\}$ is a martingale:

Itô formula: $dM_t = Y_t M_t\, dB_t \Rightarrow M \in \mathcal{M}_{loc}^{cont}$.

Let $\{\tau_n\}_{n\in\mathbb{N}}$ be stopping times s.t. $\{M_{t\wedge\tau_n} : t\geq 0\}$ is a martingale $\wedge$ $\tau_n \to \infty$ a.s.

Then $\forall s \leq t$: $\quad E(M_{t\wedge\tau_n} \mid \mathcal{F}_s) = M_{s\wedge\tau_n}$

Conditional Fatou: $E(M_t \mid \mathcal{F}_s) \leq M_s$ . . . . . . M is supermartingale.

So $\quad 1 = EM_0 \geq EM_t \underset{t\leq T}{\geq} EM_T = 1$. Then $E(E(M_t\mid\mathcal{F}_s)) = EM_s = 1$

implies $E(M_t \mid \mathcal{F}_s) = M_s$ a.s.

So we get $\{M_{T\wedge t} : t \geq 0\}$ is a martingale; $\hat{P}$ is prob. measure.

$\{\hat{B}_t : t \in [0,T]\}$ is SBM under $\hat{P}$

Pick $Z \in \mathcal{V}_B$ s.t. $\exists C > 0 : \sup_{t \geq 0} \| Z_t \| \leq C$. Sol

$$N_t := \exp\left\{ i \int_0^t Z_s \, d\tilde{B}_s + \frac{1}{2} \int_0^t Z_s^2 \, ds \right\}$$

$$= \exp\left\{ i \int_0^t Z_s \, dB_s - i \int_0^t Z_s Y_s \, ds + \frac{1}{2} \int_0^t Z_s^2 \, ds \right\}.$$

<u>claim</u> $\{N_t M_t : t \geq 0\} \in \mathcal{M}^{cont}_{loc}$ (Calculation in notes).

Since $\{M_{t\wedge T} : t \geq 0\}$ is Martingale, it is UI.

As N_t is bounded, $\{N_{t\wedge T} M_{t\wedge T} : t \geq 0\}$ is UI.

So $\{N_{t\wedge T} M_{t\wedge T} : t \geq 0\}$ is a martingale.

$$1 = E(N_0 M_0) = E(N_T M_T) = \hat{E}(N_T). \quad \text{Choose } Z_s = \sum_{j=1}^n \lambda_j \mathbb{1}_{(t_{j-1}, t_j]}(s)$$

$0 = t_0 < \dots < t_n = T$

Then $\hat{E}(N_T) = 1$ translates into

$$\tilde{E}\left(\exp\left\{ i \sum_{j=1}^n \lambda_j (\tilde{B}_{t_j} - \tilde{B}_{t_{j-1}}) \right\} \right) = \exp\left\{ - \sum_{j=1}^n \frac{\lambda_j^2}{2} (t_j - t_{j-1}) \right\}. \Rightarrow$$

$\tilde{B}|_{[0,T]}$ has FDD's of SBM.

continuity $\Rightarrow$ $\tilde{B}$ is SBM. $\boxtimes$

Example where $EM_T < 1$

Lemma Let $d > 2$, $X = d$-dim. Bessel process, i.e. solution to $dX_t = \frac{d-1}{2X_t} dt + dB_t$ with $X_0 = 1$.
Then $M_t := X_t^{2-d}$ is a local martingale of the form:

$$M_t = \exp\left\{ -(d-2) \int_0^t \frac{1}{X_s} dB_s - \frac{(d-2)^2}{2} \int_0^t \frac{1}{X_s^2} ds \right\}.$$

which is well defined (i.e. $M_t \in (0,\infty)$ a.s) yet $\forall t > 0$: $EM_t < 1$.

Pf: Assume $T > 0$ s.t. $EM_T = 1$. Girsanov: $\tilde{B}_t = B_t + (d-2)\int_0^t \frac{1}{X_s} ds$
is SBM under $\tilde{P}$. Then

$$dX_t = \left(\frac{d-1}{2X_t} - \frac{d-2}{X_t} \right) dt + d\tilde{B}_t = \frac{\overset{= d'-1}{(3-d)}}{2X_t} dt + d\tilde{B}_t.$$

So under $\tilde{P}$, X is $4-d$-dimensional Bessel process.
So $\tilde{P}(\inf_{t \le T} X_t = 0) > 0$ yet $P(\inf_{t \le T} X_t = 0) = 0$ so $\tilde{P} \not\ll P$
a contradiction!

Lemma Let $M \in \mathcal{M}^{cont}_{loc}$ be s.t. $M_0 = 0$ and $\exists \varepsilon > 0$: $E\left(e^{(\frac{1}{2}+\varepsilon)\langle M\rangle_t}\right) < \infty$. Then $E\left(e^{M_t - \frac{1}{2}\langle M\rangle_t}\right) = 1$

Pf: $X_t := e^{M_t - \frac{1}{2}\langle M\rangle_t}$, so $X \in \mathcal{M}^{cont}_{loc}$ by Itô. $\exists \{\tau_n\}$ = stopping times turning X is a martingale.

Let $\lambda > 1$. Then

$$E\left(X^{\lambda}_{t\wedge\tau_n}\right) = E\left(e^{\lambda M_{t\wedge\tau_n} - \frac{\lambda}{2}\langle M\rangle_{t\wedge\tau_n}}\right)$$

$$\underset{p>1}{=} E\left(e^{\lambda M_{t\wedge\tau_n} - \frac{\lambda^2 p}{2}\langle M\rangle_{t\wedge\tau_n} + \frac{1}{2}\lambda(\lambda p - 1)\langle M\rangle_{t\wedge\tau_n}}\right)$$

$$\overset{\text{Hölder}}{\le} \underbrace{E\left(e^{\lambda p M_{t\wedge\tau_n} - \frac{\lambda^2 p^2}{2}\langle M\rangle_{t\wedge\tau_n}}\right)^{1/p}}_{=1 \quad (M_0 = 0)} E\left(e^{\frac{1}{2}\lambda(\lambda p-1) q \langle M\rangle_{t\wedge\tau_n}}\right)^{1/q}$$

So $\sup_{n\ge 1} E\left(X^{\lambda}_{t\wedge\tau_n}\right) \le \left[E\left(e^{\frac{1}{2}\lambda(\lambda p - 1) q \langle M\rangle_t}\right)\right]^{1/q}$.

Observe: $\lambda(\lambda p - 1) q = \lambda p \frac{\lambda p - 1}{p - 1} \xrightarrow[\lambda, p \downarrow 1 \text{ suitably}]{} 1$ ... $\exists \lambda > 1\ \exists p > 1$: $\frac{1}{2}\lambda(\lambda p - 1) q < \frac{1}{2} + \varepsilon$

Then $\{X_{t\wedge\tau_n} : n \ge 0\}$ is UI and we have $E(X_t) = \lim_{n\to\infty} E(X_{t\wedge\tau_n}) = E(X_0) = 1$ $\boxtimes$